On the Structure Theorem for Quasi-Hopf Bimodules

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The main aim

Fix $\Bbbk$ a field. We assume to work in the category $\mathfrak{M} := \mathrm{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces.

Fact

H bialgebra $\Rightarrow$ the category $\mathfrak{M}_H$ of H-modules is monoidal. The category $\mathfrak{M}_H^H$ of Hopf modules is the category of comodules on the H-module coalgebra H: $(\mathfrak{M}_H)^H$.

Our aim is to extend the following result to the framework of quasi-bialgebras.

Theorem

T.F.A.E. for a bialgebra H:

1. *the functor $(-) \otimes H : \mathfrak{M} \to \mathfrak{M}_H^H$ is an equivalence of categories with quasi-inverse $(-)^{\mathrm{co}H} : \mathfrak{M}_H^H \to \mathfrak{M}$, where $M^{\mathrm{co}H} := \{m \in M \mid \rho(m) = m \otimes 1\}$;*
2. *H is a Hopf algebra, i.e. it admits an antipode $s : H \to H$.*

Sketch of proof.

The assignment $[\, m \mapsto \tau_M(m_0) \otimes m_1 \,]$, where $\tau_M : M \to M^{\mathrm{co}H}, [\, m \mapsto m_0 \cdot s(m_1) \,]$, defines the inverse for the counit $\vartheta_M : M^{\mathrm{co}H} \otimes H \to M, [\, m \otimes h \mapsto m \cdot h \,]$. The unit is always invertible. □

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Monoidal categories

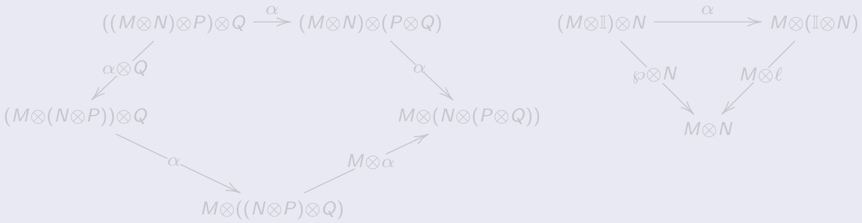
Definition (Benabou/Mac Lane, 1963)

A monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \ell, \wp)$ is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (tensor product), an object $\mathbb{I}$ (unit) and 3 natural isomorphisms:

$$\alpha_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P) \qquad \text{(associativity constraint)}$$

$$\ell_M : \mathbb{I} \otimes M \to M, \qquad \wp_N : N \otimes \mathbb{I} \to N \qquad \text{(unit constraints)}$$

such that the following diagrams commute (pentagon and triangle axioms):



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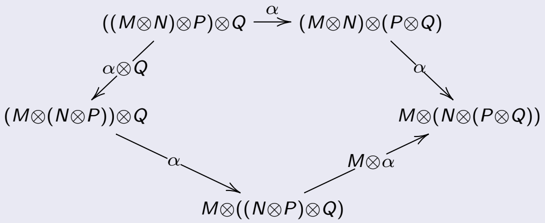

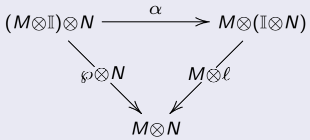

Example

Recall that a bialgebra is an algebra (H, m, u) endowed with two algebra maps $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to \Bbbk$ such that the following diagrams commute

$$\begin{array}{ccc} H \otimes H \otimes H & \xleftarrow{H\otimes\Delta} & H \otimes H \\ {\scriptstyle \Delta\otimes H}\uparrow & & \uparrow{\scriptstyle \Delta} \\ H \otimes H & \xleftarrow[\Delta]{} & H \end{array} \qquad \begin{array}{ccccc} \Bbbk \otimes H & \xleftarrow{\varepsilon\otimes H} & H \otimes H & \xrightarrow{H\otimes\varepsilon} & H \otimes \Bbbk \\ & {\scriptstyle\cong}\nwarrow & {\scriptstyle \Delta}\uparrow & \nearrow{\scriptstyle\cong} & \\ & & H & & \end{array}$$

Example

The category $(\mathfrak{M}_H, \otimes, \Bbbk)$ of (right) H-modules is monoidal. The H-module structure on the tensor product is given by the diagonal action:

$$(M \otimes N) \otimes H \to M \otimes N\colon\ (m \otimes n) \otimes h \mapsto (m \cdot h_1) \otimes (n \cdot h_2)$$

and the one on the base field via the trivial action:

$$\Bbbk \otimes H \to \Bbbk\colon k \otimes h \mapsto k\varepsilon(h).$$

Quasi-bialgebras

Definition (Drinfel'd, [Dr, 1989])

A quasi-bialgebra is a datum $(A, m, u, \Delta, \varepsilon, \Phi)$ where:

1. (A, m, u) is an associative and unital algebra;
2. $\Delta : A \to A \otimes A$ (comultiplication) and $\varepsilon : A \to \Bbbk$ (counit) are algebra maps;
3. $\Phi \in A \otimes A \otimes A$ is an invertible element (reassociator) that satisfies:

$$(A \otimes A \otimes \Delta)(\Phi) \cdot (\Delta \otimes A \otimes A)(\Phi) = (1 \otimes \Phi) \cdot (A \otimes \Delta \otimes A)(\Phi) \cdot (\Phi \otimes 1),$$
$$(A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1.$$

Moreover, ε is a counit for Δ and Δ is quasi-coassociative, i.e.

$$\Phi \cdot ((\Delta \otimes A) \circ \Delta) = ((A \otimes \Delta) \circ \Delta) \cdot \Phi.$$

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If A is a quasi-bialgebra then ${}_A\mathfrak{M}_A$ is a monoidal category:

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$$a \cdot (m \otimes n) \cdot b = (a_1 \cdot m \cdot b_1) \otimes (a_2 \cdot n \cdot b_2);$$
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$${}_A\alpha_A((m \otimes n) \otimes p) = \Phi \cdot (m \otimes (n \otimes p)) \cdot \Phi^{-1}.$$

Proposition/Definition (Hausser and Nill, [HN, 1999])

$((A, m, m), \Delta, \varepsilon)$ is a coassociative A-bimodule coalgebra. Its category of (right) quasi-Hopf bimodules is the category of A-comodules in ${}_A\mathfrak{M}_A$: ${}_A\mathfrak{M}_A^A := \left({}_A\mathfrak{M}_A\right)^A$.

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An adjunction between ${}_A\mathfrak{M}$ and ${}_A\mathfrak{M}_A^A$

Henceforth, let us fix a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ and denote by $A^+ := \ker(\varepsilon)$ its augmentation ideal.
The subsequent result is contained in the proof of Theorem 3.1 in

[Sc] P. Schauenburg, *Two characterizations of finite quasi-Hopf algebras.* J. Algebra **273** (2004), no. 2, 538-550.

Theorem

Set $\overline{M} := \frac{M}{MA^+} \in {}_A\mathfrak{M}$. We have that the functor $R := (-) \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is right adjoint to the functor $L := \overline{(-)} : {}_A\mathfrak{M}_A^A \to {}_A\mathfrak{M}$. Unit and counit are given by:

$$\eta_M : M \to \overline{M} \otimes A, \left[m \mapsto \overline{m_0} \otimes m_1\right] \quad \text{and} \quad \epsilon_N : \overline{N \otimes A} \to N, \left[\overline{n \otimes a} \mapsto n\,\varepsilon(a)\right]$$

respectively. Moreover ϵ is always a natural isomorphism.

Main question: When is R an equivalence of categories?

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Answering the main question (I)

Consider the quasi-Hopf bimodule $A \mathbin{\widehat{\otimes}} A$ with underling vector space $A \otimes A$ and structures given explicitly by:

$$a \cdot (x \otimes y) = x \otimes ay, \qquad (x \otimes y) \cdot a = xa_1 \otimes ya_2,$$
$$\rho(x \otimes y) = \big((x \otimes y_1) \otimes y_2\big) \cdot \Phi$$

The component of the unit associated to $A \mathbin{\widehat{\otimes}} A$ satisfies:

$$\widehat{\eta}_A := \eta_{A \widehat{\otimes} A} \colon A \mathbin{\widehat{\otimes}} A \to \overline{A \mathbin{\widehat{\otimes}} A} \otimes A, \left[a \otimes b \mapsto \overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3\right]$$

Definition

A preantipode for a quasi-bialgebra (A, Φ) is a linear map $S \colon A \to A$ that satisfies:

(P1) $b_1 S(ab_2) = S(a)\varepsilon(b)$, $\forall\, a, b \in A$; $\overset{a=1}{\rightsquigarrow}$ $b_1 S(b_2) = S(1)\varepsilon(b)$

(P2) $S(a_1 b)a_2 = \varepsilon(a)S(b)$, $\forall\, a, b \in A$;

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Answering the main question (II)

Theorem (Structure Theorem for quasi-Hopf bimodules)

Let $(A, m, u, \Delta, \varepsilon, \Phi)$ *be a quasi-bialgebra. T.F.A.E.:*

(i) (L, R, η, ϵ) *is an equivalence of categories;*

(ii) $\widehat{\eta}_A \colon A \widehat{\otimes} A \to \overline{A \widehat{\otimes} A} \otimes A$ *is an isomorphism;*

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(iv) *for every* $M \in {}_A\mathfrak{M}_A^A$ *there exists a linear map* $\widetilde{\tau}_M \colon \overline{M} \to M$ *such that*

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Proof.

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Hopf case

Let $(H, m, u, \Delta, \varepsilon)$ be an ordinary bialgebra.

- (H, s) is a Hopf algebra with antipode s if and only if $(H, m, u, \Delta, \varepsilon, \Phi, s)$ is a quasi-bialgebra with preantipode s and reassociator $\Phi = 1 \otimes 1 \otimes 1$. One checks that the two maps τ_M coincide for all $M \in \mathfrak{M}_H^H$ and then the inverse to the original counit is given by:

$$\vartheta_M^{-1} \colon m \mapsto (\widetilde{\tau}_M \otimes H)\left(\eta_M(m)\right) = \tau_M(m_0) \otimes m_1.$$

- If every H-Hopf module satisfies the Fundamental Theorem, then one can verify that for every $M \in {}_H\mathfrak{M}_H^H$

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In this context, the Structure Theorem for quasi-Hopf bimodules reduces to the classical Fundamental Theorem of Hopf modules.

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Revisiting classical results (II)

Definition (Drinfel'd, 1989)

We say that a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra if it is endowed with an algebra anti-homomorphism $s \colon A \to A$ and two distinguished elements α and β such that:

$$s(a_1)\alpha a_2 = \alpha\,\varepsilon(a) \qquad a_1\beta s(a_2) = \beta\,\varepsilon(a)$$

$$\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = 1 \qquad s(\phi^1)\alpha\phi^2\beta s(\phi^3) = 1$$

The triple (s, α, β) is called quasi-antipode.

Quasi-Hopf case

1. Every quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ admits a preantipode:

$$S(\cdot) := \beta s(\cdot)\alpha.$$

2. If s is invertible, then Hausser and Nill's $M^{\mathrm{co}H}$ is isomorphic as left module with $\overline{M}$ and their projection corresponds to our map τ_M.

It is then possible to obtain Hausser and Nill's result from our Structure Theorem.

Revisiting classical results (II)

Definition (Drinfel'd, 1989)

We say that a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra if it is endowed with an algebra anti-homomorphism $s\colon A \to A$ and two distinguished elements α and β such that:

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From preantipodes to quasi-antipodes (I)

It is sometimes possible to produce a quasi-antipode given a preantipode. E.g. we have implicitly seen the case of ordinary bialgebras.

Proposition

If $(A, m, u, \Delta, \varepsilon, \Phi, S)$ is a commutative quasi-bialgebra with preantipode, then A is an Hopf algebra with antipode $s(a) = \Phi^1 S(a\Phi^2)\Phi^3$ and $(A, m, u, \Delta, \varepsilon, \Phi, s, 1, S(1))$ is a quasi-Hopf algebra.

Theorem (Theorem 3.1 in [Sc])

For a finite dimensional quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$, T.F.A.E.:

1. *A is a quasi-Hopf algebra.*
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If ξ is invertible then $\left(\left(a \xrightarrow{S} 1^1S(a1^2)\right), 1, S(1)\right)$, where $\overline{1^1 \otimes 1^2} = \xi^{-1}(1)$, defines a quasi-antipode (without any hypothesis on the dimension of A).

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If $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a finite dimensional quasi-Hopf algebra and α is invertible, then we can recover explicitly the quasi-antipode from the preantipode.

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Concluding example

Example (Preliminaries 2.3 in [EG])

Let $C_2 = \langle g \rangle$ be the cyclic group of order 2 and let $H(2) := \Bbbk C_2$ be its group algebra ($\mathrm{char}(\Bbbk) \neq 2$):

$$m(p \otimes q) = p \cdot q, \quad u(1_\Bbbk) = 1_{C_2}, \quad \Delta(p) = p \otimes p, \quad \varepsilon(p) = 1_\Bbbk \quad (\forall\, p, q \in C_2\,).$$

Let us consider the non trivial reassociator:

$$\Phi := (1 \otimes 1 \otimes 1) - 2(\lambda \otimes \lambda \otimes \lambda) \quad \text{where} \quad \lambda := \frac{1}{2}(1 - g).$$

One can verify that $(H(2), m, u, \Delta, \varepsilon, \Phi, \mathrm{Id}_{H(2)}, g, 1)$ is a quasi-Hopf algebra. Therefore $S\colon H(2) \to H(2), [\, z \mapsto z \cdot g\,]$ provides a preantipode for $H(2)$ and

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We firmly believe that such an example should exist. The dual notion of a preantipode for a coquasi-bialgebra has been introduced and studied in

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