FUNCTORIAL CONSTRUCTIONS FOR NON-ASSOCIATIVE ALGEBRAS WITH APPLICATIONS TO QUASI-BIALGEBRAS

ALESSANDRO ARDIZZONI, LAIACHI EL KAOUTIT, AND PAOLO SARACCO

Abstract. The aim of this paper is to establish a contravariant adjunction between the category of quasi-bialgebras and a suitable full subcategory of dual quasi-bialgebras, adapting the notion of finite dual to this framework. Various functorial constructions involving non-associative algebras and non-coassociative coalgebras are then carried out. Several examples illustrating our methods are expounded as well.

1. Introduction

Algebras and coalgebras are dual notions, in the sense that the latter ones are obtained from the first ones by reversing the structure arrows, that is using the opposite base category. Furthermore, over vector spaces there is a contravariant adjunction (or duality) between the category of algebras and that of coalgebras, whose functors are described as follows. In one direction, to each coalgebra one associates, in a functorial way, its convolution algebra. In the other direction, to each algebra one associates, in a similar way, its topological dual (i.e. finite dual) coalgebra. This adjunction descends in fact to the category of bialgebras (and in particular to Hopf algebras), and also establishes a contravariant adjunction between the category of Lie algebras and the category of Lie coalgebras. All these adjunctions and other ones are captured by the following diagram

$$\begin{array}{ccc}
\mathsf{Alg}_{\Bbbk} & \underset{(-)^*}{\overset{(-)^\circ}{\rightleftarrows}} & \mathsf{Coalg}_{\Bbbk} \\
\uparrow & & \uparrow \\
\mathsf{Bialg}_{\Bbbk} & \underset{(-)^\circ}{\overset{(-)^\circ}{\rightleftarrows}} & \mathsf{Bialg}_{\Bbbk} \\
\mathcal{U} \uparrow \downarrow \mathcal{P} & & \mathcal{P}^c \downarrow \uparrow \mathcal{U}^c \\
\mathsf{Lie}_{\Bbbk} & \underset{(-)^*}{\overset{(-)^\bullet}{\rightleftarrows}} & \mathsf{LieCo}_{\Bbbk}
\end{array} \tag{1}$$

where $\Bbbk$ denotes the base field, and the notations for the involved categories as well as the ones used in the sequel, are summarized in Table 1 below.

It is noteworthy to mention that in diagram (1), the functor $\mathcal{P} : \mathsf{Bialg}_{\Bbbk} \to \mathsf{Lie}_{\Bbbk}$ associates to each bialgebra its Lie algebra of primitive elements, and its left adjoint $\mathcal{U} : \mathsf{Lie}_{\Bbbk} \to \mathsf{Bialg}_{\Bbbk}$ is the universal enveloping algebra functor. Note that in characteristic zero there is a natural isomorphism $\mathcal{P}\mathcal{U} \cong \mathrm{id}_{\mathsf{Lie}_{\Bbbk}}$, see [MM, Theorem 5.18]. The functor $\mathcal{P}^c$ is the one given by the vector space of indecomposables, see e.g. [Mi2, Definition 1.9] where this functor is denoted by Q, and $\mathcal{U}^c$ is a functor defined by Michaelis. The adjunction $\mathcal{P}^c \dashv \mathcal{U}^c$ is established in [Mi2, Theorem 3.11] for Hopf algebras instead of bialgebras (the same proof can be adapted to our case, as the antipode is not used therein). The bottom contravariant adjunction is established in [Mi2, Theorem 3.7]. For the top horizontal adjunction see e.g. [Sw, Theorem 6.0.5]. Concerning the middle horizontal adjunction, the finite dual yields an endofunctor of $\mathsf{Bialg}_{\Bbbk}$ in view of [Sw, Section 6.2]. Moreover, this comes out to be adjoint to itself as in case of Hopf algebras (cf. e.g. [Ab, page 87]).

Date: March 20, 2022.

2010 *Mathematics Subject Classification.* Primary 17A01, 17B62; Secondary 16T99, 18D25.

Key words and phrases. Non-(co)associative (co)algebras; Quasi-bialgebras; Contravariant adjunctions; Finite duals.

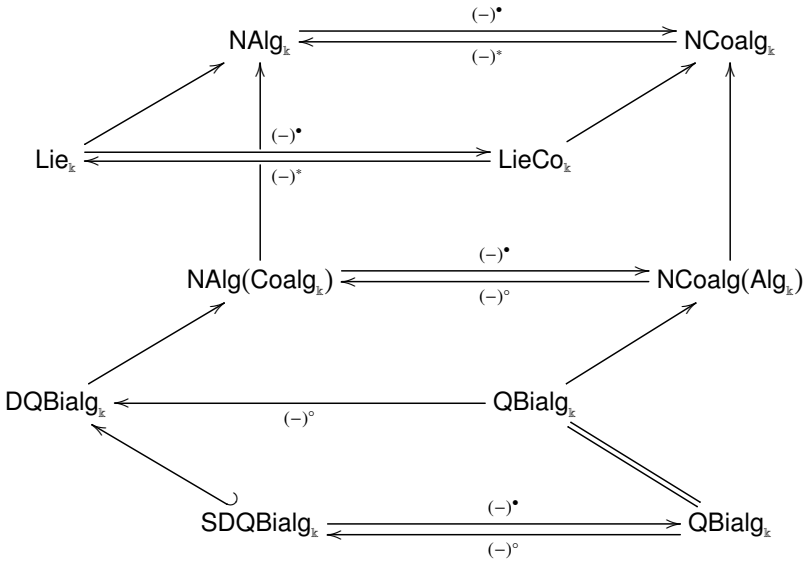
This paper was written while A. Ardizzoni and P. Saracco were members of the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INdAM). The research of L. El Kaoutit is supported by the Spanish Ministerio de Economía y Competitividad and the European Union, grant MTM2013-41992-P..

TABLE 1. Notations for the handled categories.

Abbreviation		The category of
$\mathsf{Vect}_{\Bbbk}$	·····	vector spaces
$\mathsf{Alg}_{\Bbbk}$	·····	associative algebras
$\mathsf{Coalg}_{\Bbbk}$	·····	coassociative coalgebras
$\mathsf{Bialg}_{\Bbbk}$	·····	bialgebras
$\mathsf{Lie}_{\Bbbk}$	·····	Lie algebras
$\mathsf{LieCo}_{\Bbbk}$	·····	Lie coalgebras
$\mathsf{NAlg}_{\Bbbk}$	·····	non-associative algebras
$\mathsf{NCoalg}_{\Bbbk}$	·····	non-coassociative coalgebras
$\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$	·····	non-associative algebras inside the monoidal category of coalgebras
$\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$	·····	non-coassociative coalgebras inside the monoidal category of algebras
$\mathsf{QBialg}_{\Bbbk}$	·····	quasi-bialgebras
$\mathsf{DQBialg}_{\Bbbk}$	·····	dual quasi-bialgebras
$\mathsf{SDQBialg}_{\Bbbk}$	·····	split dual quasi-bialgebras
		(i.e. the 3-cocycle splits as a finite sum of tensor products of linear maps)

Quasi-bialgebras are generalization of ordinary bialgebras, in which the constraint of coassociativity at the coalgebra level is weakened. Dual quasi-bialgebras are in a certain sense a dual notion, which can also be seen as a generalization of bialgebras, by affecting this time the associativity constraint.

The main aim of this paper is to investigate the second horizontal adjunction of diagram (1) in the context of quasi and dual quasi-bialgebras. Explicitly, we establish a contravariant adjunction between the category of quasi-bialgebras and the one of split dual quasi-bialgebras (a certain full subcategory of the category of dual quasi-bialgebras) here introduced. To do so, we investigate how some of the adjunctions represented in diagram (1) extend to the wider framework of non-(co)associative (co)algebras, as in diagram (2). We just point out here that the upper adjunction in diagram (2) already appeared in [ACM, page 4700].



(2)

By a *non-associative algebra* we mean a unital but *not necessarily associative algebra over* $\Bbbk$, i.e. a vector space A endowed with two linear maps $m : A \otimes A \longrightarrow A$, $a \otimes b \longmapsto ab$ (the multiplication) and $u : \Bbbk \longrightarrow A$, $k \longmapsto k1_A$ (the unit) such that $a1_A = a = 1_A a$, for every $a \in A$. A similar terminology is used for coalgebras.

2. The construction of the finite dual of a non-associative algebra and examples.

In this section we give the main construction of the paper. More explicitly, starting from a non-associative algebra, we are able to construct a non-coassociative coalgebra, which in the associative case coincides with the so called *finite dual coalgebra*, see [Ab, Mo, Sw] and [AGW] for coalgebras over commutative rings. This is the largest coalgebra inside the linear dual of the underlying vector space of the initial algebra. To illustrate our techniques, we include two basic examples concerning alternative and (special) Jordan algebras.

Given a vector space V, we denote by $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ its linear dual. The unadorned tensor product $\otimes$ stands for $\otimes_{\Bbbk}$. The identity morphism of a vector space V will be denoted by id_V or by V itself.

2.1. **Good subspace of linear dual vector space.** Given two vector spaces V and W, we can consider the canonical natural injection

$$\varphi_{V,W} : V^* \otimes W^* \longrightarrow (V \otimes W)^*, \quad \Big(f \otimes g \longmapsto [v \otimes w \mapsto f(v)g(w)]\Big), \tag{3}$$

which is clearly a natural isomorphism over finite-dimensional vector spaces.

Let (A, m, u) be a non-associative algebra, see e.g. [Bo, page 428]. Mimicking [Mi1, page 13], a subspace $V \subseteq A^*$ is called *good* in case $m^*(V) \subseteq \varphi_{A,A}(V \otimes V)$, where $m^* : A^* \to (A \otimes A)^*$ is the dual of the multiplication map m. For instance, let I be an ideal of A, that is a vector subspace of A stable under both left and right A-actions: for every $a \in A$, we have $aI \subseteq I$, $Ia \subseteq I$, see [Bo, page 430]. Assume that A/I is finite dimensional as a vector space. Set $V = (A/I)^*$ which we identify with a subspace of A^*. One can show that V is a good subspace of A^*.

Let $\mathcal{G}$ denote the set of all good subspaces of A^* and set

$$A^\bullet := \sum_{V \in \mathcal{G}} V. \tag{4}$$

By the same proof of [Mi1, Proposition, page 13], one gets that $A^\bullet$ is a good subspace of A^* and hence it is the maximal good subspace of A^*.

Given two non-associative algebras A and B and a linear map $f : A \to B$ such that $f^*(B^\bullet) \subseteq A^\bullet$, then we can consider the linear map $f^\bullet : B^\bullet \to A^\bullet$, $h \mapsto f^*(h)$, which is uniquely determined by the commutativity of the following diagram:

$$\begin{array}{ccc} B^\bullet & \overset{f^\bullet}{\dashrightarrow} & A^\bullet \\ {\scriptstyle j_B}\downarrow & & \downarrow{\scriptstyle j_A} \\ B^* & \xrightarrow{\quad f^* \quad} & A^* \end{array} \tag{5}$$

where the vertical arrows are the canonical injections.

If we consider a good subspace $V \subseteq A^*$, then we may define a unique map $\Delta_V : V \to V \otimes V$ such that

$$\varphi_{A,A}(\Delta_V(f)) = m^*(f), \text{ for every } f \in V.$$

In particular, for every $f \in A^\bullet$, $a, b \in A$, $\Delta_{A^\bullet}(f) = \sum f_1 \otimes f_2$ is uniquely determined by

$$f(ab) = m^*(f)(a \otimes b) = \varphi_{A,A}(\Delta_{A^\bullet}(f))(a \otimes b) = \sum f_1(a) f_2(b). \tag{6}$$

2.2. **The coalgebra structure of $A^\bullet$ and examples.** Parts of the subsequent lemma find their analogues for associative algebras in [Sw, Lemma 6.0.1] and for Lie algebras in [Mi1, pages 14-15].

Lemma 2.1. *For every pair of non-associative algebras (A, m, u) and (B, m', u') and for any morphism $f : A \longrightarrow B$, denote with $f^* : B^* \longrightarrow A^*$ the dual map. Then the dual map $m^* : A^* \to (A \otimes A)^*$ induces a map $\Delta_{A^\bullet} := m^* : A^\bullet \to A^\bullet \otimes A^\bullet$ and the dual map $u^* : A^* \to \Bbbk^* \cong \Bbbk : f \mapsto f(1)$ restricts to a map $\varepsilon_{A^\bullet} := u^* : A^\bullet \to \Bbbk$ such that $(A^\bullet, \Delta_{A^\bullet}, \varepsilon_{A^\bullet})$ becomes a non-coassociative coalgebra.*

Proof. First observe that $\Delta_{A^\bullet}$ exists by definition of the finite dual and satisfies (6). Therefore, let us show that $\varepsilon_{A^\bullet}$ is a counit for $\Delta_{A^\bullet}$. Pick an element $f \in A^\bullet$. For every $a \in A$ we have that:

$$\begin{aligned} (((\varepsilon_{A^\bullet} \otimes A^\bullet) \circ \Delta_{A^\bullet})(f))(a) &= \Big((\varepsilon_{A^\bullet} \otimes A^\bullet)\Big(\sum f_1 \otimes f_2\Big)\Big)(a) = \Big(\sum f_1(1) f_2\Big)(a) \\ &= \sum f_1(1) f_2(a) \overset{(6)}{=} f(a), \\ (((A^\bullet \otimes \varepsilon_{A^\bullet}) \circ \Delta_{A^\bullet})(f))(a) &= \Big((A^\bullet \otimes \varepsilon_{A^\bullet})\Big(\sum f_1 \otimes f_2\Big)\Big)(a) = \Big(\sum f_2(1) f_1\Big)(a) \end{aligned}$$

$$= \sum f_2(1)f_1(a) \overset{(6)}{=} f(a).$$

whence $(A^\bullet, \Delta_{A^\bullet}, \varepsilon_{A^\bullet})$ is a non-coassociative coalgebra in $\mathsf{Vect}_{\Bbbk}$. □

Remark 2.2. Let A be an object in $\mathsf{NAlg}_{\Bbbk}$ and set

$$A^\circ = \Big\{ g \in A^* \mid \ker(g) \text{ contains a finite-codimensional ideal of } A \Big\}. \tag{7}$$

Here an ideal I of A is of finite codimension, means that A/I is a finite-dimensional vector space. For any $f \in A^\circ$, there exists a finite-codimensional ideal I such that $f(I) = 0$. Then f belongs to the space $(A/I)^*$, which is identified with a good subspace of A^* as in subsection 2.1. By equation (4), this means that $f \in A^\bullet$. We have so proved that $A^\circ \subseteq A^\bullet$. This fact can also be seen as a consequence of [ACM, Theorem (2.6)] which asserts that $A^\circ = \mathrm{Loc}(A^\bullet)$, where the latter denotes the sum of all locally finite subcoalgebras of $A^\bullet$ (recall that a non-coassociative coalgebra C is named *locally finite* if and only if any $x \in C$ lies in some finite-dimensional subcoalgebra $D \subset C$).

For any A in $\mathsf{Alg}_{\Bbbk}$ the finite dual $A^\bullet$ coincides with A°. By the foregoing $A^\circ \subseteq A^\bullet$. Conversely, identifying $A^* \otimes A^*$ with $\varphi_{A,A}(A^* \otimes A^*)$, if $V \subseteq A^*$ is any good subspace then for every $v \in V$, $m^*(v) \in A^* \otimes A^*$. Therefore, in view of [Sw, Proposition 6.0.3], $v \in A^\circ$ and hence $V \subseteq A^\circ$. Thus $A^\bullet \subseteq A^\circ$ so that $A^\bullet = A^\circ$.

We now provide two examples of finite dual of a non-associative algebra.

Example 2.3. *Coalternative coalgebras.* Assume $\Bbbk$ has a characteristic $\neq 2$. Let A be an *alternative algebra*, that is a not necessarily associative algebra over $\Bbbk$, which satisfies the following identity

$$x(yx) \;=\; (xy)x, \quad \text{for every } x, y \in A.$$

Replacing x by $x + z$ one sees that the last equality is equivalent to the identity

$$x(yz) + z(yx) \;=\; (xy)z + (zy)x, \quad \text{for every } x, y, z \in A.$$

Denote by $\tau : V \otimes W \to W \otimes V$ the natural flip map, and set $\tau_1 = \tau \otimes id$ and $\tau_2 = id \otimes \tau$. Consider the finite dual coalgebra $C = A^\bullet$ as in Lemma 2.1. Then the comultiplication of C satisfies the identity

$$\Big(id + (\tau_1 \circ \tau_2 \circ \tau_1)\Big) \circ \Big((\Delta \otimes C) - (C \otimes \Delta)\Big) \circ \Delta \;=\; 0, \tag{8}$$

which, over elements, says that for any function $f \in C$ we have

$$\sum f_{1,1} \otimes f_{1,2} \otimes f_2 + \sum f_2 \otimes f_{1,2} \otimes f_{1,1} \;=\; \sum f_1 \otimes f_{2,1} \otimes f_{2,2} + \sum f_{2,2} \otimes f_{2,1} \otimes f_1. \tag{9}$$

A coalgebra C which satisfies the identity (8) is called a *coalternative coalgebra*.

Example 2.4. *Jordan coalgebra.* Assume $\Bbbk$ has a characteristic $\neq \{2, 3\}$. Let A be *a (special) Jordan algebra*, that is a not necessarily associative algebra over $\Bbbk$, which satisfies the following identities

$$xy \;=\; yx, \quad x^2(yx) \;=\; (x^2y)x, \quad \text{for every } x, y \in A.$$

The second equality above comes out to be equivalent to

$$((xy)z)t + ((xt)z)y + ((ty)z)x = (xy)(zt) + (xt)(zy) + (ty)(zx).$$

Denote by $\tau : V \otimes W \to W \otimes V$ the natural flip map, and set $\tau_1 = \tau \otimes id \otimes id$, $\tau_2 = id \otimes \tau \otimes id$ and $\tau_3 = id \otimes id \otimes \tau$. Following [ACM, Example (3) page 4709], we can consider the finite dual coalgebra $C = A^\bullet$ as in Lemma 2.1. Then the comultiplication of C is cocommutative and satisfies the identity

$$\Big[id + (\tau_3 \circ \tau_2 \circ \tau_3) \;+\; (\tau_3 \circ \tau_2 \circ \tau_1 \circ \tau_2 \circ \tau_3)\Big] \circ \Big[(\Delta \otimes C \otimes C) - (C \otimes C \otimes \Delta)\Big] \circ (\Delta \otimes C) \circ \Delta \;=\; 0, \tag{10}$$

which, over elements, says that for any function $f \in C$ we have

$$\begin{aligned} \sum f_{1,1,1} \otimes f_{1,1,2} \otimes f_{1,2} \otimes f_2 + \sum f_{1,1,1} \otimes f_2 \otimes f_{1,2} \otimes f_{1,1,2} + \sum f_2 \otimes f_{1,1,2} \otimes f_{1,2} \otimes f_{1,1,1} \\ = \sum f_{1,1} \otimes f_{1,2} \otimes f_{2,1} \otimes f_{2,2} + \sum f_{1,1} \otimes f_{2,2} \otimes f_{2,1} \otimes f_{1,2} + \sum f_{2,2} \otimes f_{1,2} \otimes f_{2,1} \otimes f_{1,1}. \end{aligned} \tag{11}$$

A cocommutative coalgebra C which satisfies the identity (10) is called a *Jordan coalgebra*.

3. Contravariant adjunction between non-associative algebras and non-coassociative coalgebras.

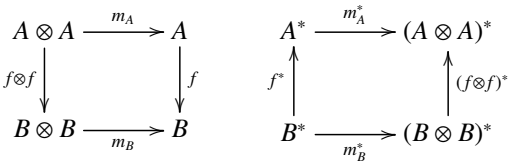
In [ACM, page 4700] it is claimed that the contravariant functor $(-)^\bullet$ is the right adjoint of the functor $(-)^*$ from the category $\mathsf{NCoalg}_{\Bbbk}$ of non-coassociative coalgebras to the category $\mathsf{NAlg}_{\Bbbk}$ of non-associative algebras (we just point out that their (co)algebras have no (co)unit). Such an adjunction extends the usual contravariant adjunction between algebras and coalgebras, that is the first horizontal adjunction in diagram (1). For the sake of completeness, and as reference for the sequel, we decided to detail the relevant proofs. We first check that the construction $A \longmapsto A^\bullet$ of Section 2 defines a contravariant functor from the category $\mathsf{NAlg}_{\Bbbk}$ to the category $\mathsf{NCoalg}_{\Bbbk}$. Next we show that it is adjoint to the functor $(-)^*$ defined as in the classical case by using the convolution product. In addition, we prove that $(A \otimes B)^\bullet \cong A^\bullet \otimes B^\bullet$, for every A, B in $\mathsf{NAlg}_{\Bbbk}$.

3.1. **The functorial construction.** Keep the notations of Section 2.

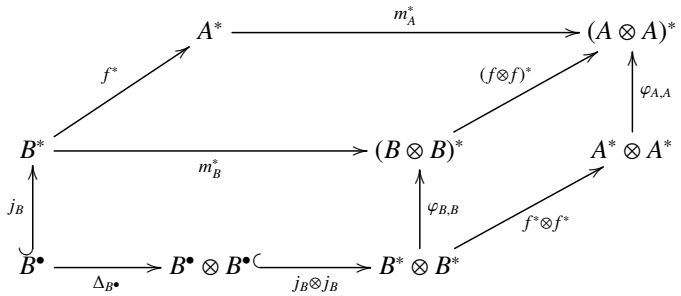
Lemma 3.1. *Let $f : A \to B$ be a morphism of non-associative algebras and $f^* : B^* \longrightarrow A^*$ its linear dual map. Then $f^*(B^\bullet) \subseteq A^\bullet$, whence f^* induces a map $f^\bullet : B^\bullet \to A^\bullet$, which comes out to be a morphism in $\mathsf{NCoalg}_{\Bbbk}$. Moreover, the assignments $A \longmapsto A^\bullet$ and $f \longmapsto f^\bullet$ establish a functor*

$$(-)^\bullet : \mathsf{NAlg}_{\Bbbk} \longrightarrow \mathsf{NCoalg}_{\Bbbk}{}^{\mathrm{op}}.$$

Proof. Since f is multiplicative, the left-hand side diagram below commutes, so that, by functoriality of $(-)^*$, the right-hand side one commutes, too.



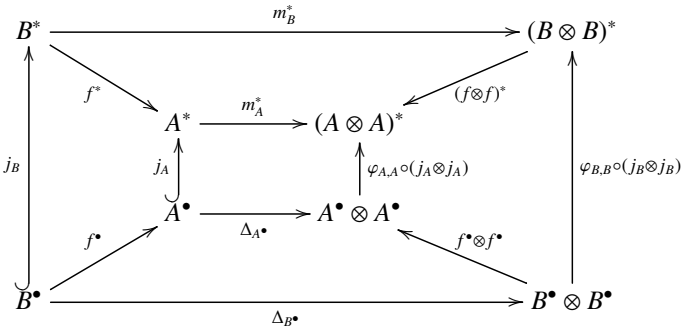
The latter diagram is part of the following bigger one:



which still commutes by definition of $\Delta_{B^\bullet}$ and by naturality of $\varphi_{-,-}$. In particular, for every $g \in B^\bullet$

$$m_A^*\left(f^*(g)\right) = \varphi_{A,A}\left(\left(f^* \otimes f^*\right)\left(\Delta_{B^\bullet}(g)\right)\right) \in \varphi_{A,A}\left(f^*\left(B^\bullet\right) \otimes f^*\left(B^\bullet\right)\right)$$

so that $f^*\left(B^\bullet\right)$ is a good subspace of A^*. Furthermore, the commutativity of all the other quads in the subsequent diagram implies the commutativity of the one at the bottom, which encodes the comultiplicativity of $f^\bullet$:



Moreover, $f^\bullet$ is counital, since $\varepsilon_{A^\bullet} \circ f^\bullet = (u_A)^\bullet \circ f^\bullet = (f \circ u_A)^\bullet = (u_B)^\bullet = \varepsilon_{B^\bullet}$. By Lemma 2.1 it follows that $(-)^\bullet$ actually defines a contravariant functor from $\mathsf{NAlg}_{\Bbbk}$ to $\mathsf{NCoalg}_{\Bbbk}$. □

3.2. **The contravariant adjunction.** Recall that the assignment $(-)^* : \mathsf{Coalg}_{\Bbbk} \longrightarrow \mathsf{Alg}_{\Bbbk}$ defines a contravariant functor between the category of coassociative $\Bbbk$-coalgebras and the category of associative $\Bbbk$-algebras (cf., e.g., [Sw, Theorem 6.0.5]) that easily extends to a contravariant functor $(-)^* : \mathsf{NCoalg}_{\Bbbk} \longrightarrow \mathsf{NAlg}_{\Bbbk}$. In this way, we will show that this functor is in fact an adjoint to the functor $(-)^\bullet$ of subsection 3.1. By adapting this construction to the categories $\mathsf{Lie}_{\Bbbk}$ and $\mathsf{LieCo}_{\Bbbk}$, one recovers Michaelis' result [Mi1, Theorem on page 15].

Proposition 3.2. *Let (A, m, u) be a non-associative algebra and (C, Δ, ε) a non-coassociative coalgebra. We have a natural isomorphism*

$$\Phi_{(A,C)} : \mathsf{NAlg}_{\Bbbk}(A, C^*) \longrightarrow \mathsf{NCoalg}_{\Bbbk}(C, A^\bullet). \tag{12}$$

Therefore the functor $(-)^\bullet : \mathsf{NAlg}_{\Bbbk} \longrightarrow (\mathsf{NCoalg}_{\Bbbk})^{\mathrm{op}}$ is left adjoint to $(-)^* : (\mathsf{NCoalg}_{\Bbbk})^{\mathrm{op}} \longrightarrow \mathsf{NAlg}_{\Bbbk}$.

Proof. Denote by $\chi_V : V \longrightarrow V^{**}$ the canonical injection, defined for every vector space V by $\chi_V(v)(f) = f(v)$ for all $v \in V$ and $f \in V^*$. Recall that $\chi : (-) \longrightarrow (-)^{**}$ is a natural transformation and let us check that if C is a coalgebra then $\chi_C(C) \subseteq C^{*\bullet}$ (compare with [Mi1, Note on page 15]). This follows once proved that $\chi_C(C)$ is a good subspace of C^{**}. Given $c \in C$ and $\phi, \psi \in C^*$ we have that:

$$\begin{aligned}(m_{C^*})^*(\chi_C(c))(\phi \otimes \psi) &= \chi_C(c)(m_{C^*}(\phi \otimes \psi)) = m_{C^*}(\phi \otimes \psi)(c)\\ &= \sum \phi(c_1)\psi(c_2) = \sum \chi_C(c_1)(\phi)\chi_C(c_2)(\psi)\\ &= \varphi_{C^*,C^*}\Big(\sum \chi_C(c_1) \otimes \chi_C(c_2)\Big)(\phi \otimes \psi)\end{aligned}$$

so that $(m_{C^*})^*(\chi_C(c)) = \varphi_{C^*,C^*}(\sum \chi_C(c_1) \otimes \chi_C(c_2))$ for all $c \in C$ and

$$(m_{C^*})^*(\chi_C(C)) \subseteq \varphi_{C^*,C^*}(\chi_C(C) \otimes \chi_C(C))$$

so that $\chi_C(C)$ is good by definition. Note also that we have just proved that

$$(m_{C^*})^* \circ \chi_C = \varphi_{C^*,C^*} \circ (\chi_C \otimes \chi_C) \circ \Delta. \tag{13}$$

If we denote by $j_A : A^\bullet \longrightarrow A^*$ the inclusion of the finite dual of a non-associative algebra A into its ordinary dual, then we have just shown that for any non-coassociative coalgebra C, χ_C induces a $\Bbbk$-linear map $\epsilon_C : C \longrightarrow C^{*\bullet}$ that is still natural in C and it satisfies

$$j_{C^*} \circ \epsilon_C = \chi_C. \tag{14}$$

Let us check that ϵ_C actually is a comultiplicative and counital map. Denote by $\Delta_{C^{*\bullet}}$ the comultiplication of $C^{*\bullet}$. This is the only map that satisfies $\varphi_{C^*,C^*} \circ (j_{C^*} \otimes j_{C^*}) \circ \Delta_{C^{*\bullet}} = (m_{C^*})^* \circ j_{C^*}$.

As a consequence:

$$\begin{aligned}\varphi_{C^*,C^*} \circ (j_{C^*} \otimes j_{C^*}) \circ \Delta_{C^{*\bullet}} \circ \epsilon_C &= (m_{C^*})^* \circ j_{C^*} \circ \epsilon_C \overset{(14)}{=} (m_{C^*})^* \circ \chi_C\\ &\overset{(13)}{=} \varphi_{C^*,C^*} \circ (\chi_C \otimes \chi_C) \circ \Delta\\ &\overset{(14)}{=} \varphi_{C^*,C^*} \circ (j_{C^*} \otimes j_{C^*}) \circ (\epsilon_C \otimes \epsilon_C) \circ \Delta\end{aligned}$$

and, by injectivity of φ_{C^*,C^*} and of j_{C^*}, we have that $\Delta_{C^{*\bullet}} \circ \epsilon_C = (\epsilon_C \otimes \epsilon_C) \circ \Delta$. Moreover,

$$\varepsilon_{C^{*\bullet}}(\epsilon_C(c)) \overset{(\dagger)}{=} (u^\bullet_{C^*}(\epsilon_C(c)))(1_{\Bbbk}) = (\epsilon_C(c))(u_{C^*}(1_{\Bbbk})) = (\epsilon_C(c))(\varepsilon_C) = \varepsilon_C(c)$$

for any $c \in C$, where in (†) we identified $\Bbbk^*$ with $\Bbbk$. Hence, ϵ_C is comultiplicative and counital.

On the other hand, the injection $j_A : A^\bullet \hookrightarrow A^*$ induces a map $\eta_A : A \to A^{\bullet *}$ given by

$$\eta_A \coloneqq j_A^* \circ \chi_A. \tag{15}$$

We claim that this is an algebra morphism. Set $\eta = \eta_A$ for shortness. Then, for all $a, b \in A$ and for any $f \in A^\bullet$:

$$\begin{aligned}m_{A^{\bullet *}}(\eta(a) \otimes \eta(b))(f) &= \varphi_{A^\bullet,A^\bullet}(\eta(a) \otimes \eta(b))(\Delta_{A^\bullet}(f)) = \sum \eta(a)(f_1)\,\eta(b)(f_2)\\ &= \sum f_1(a)f_2(b) \overset{(6)}{=} f(ab) = \eta(ab)(f).\end{aligned}$$

Moreover, $\eta(u(1))(f) = f(1_A) = \varepsilon_{A^\bullet}(f) = (\varepsilon_{A^\bullet})^*(1)(f) = u_{A^{\bullet *}}(1)(f)$, for all $f \in A^\bullet$.

Let us check finally that η and ϵ satisfy the conditions to be the unit and the counit of the adjunction, respectively. By a direct calculation:

$$j_A \circ \eta_A^\bullet \circ \epsilon_{A^\bullet} \overset{(5)}{=} \eta_A^* \circ j_{A^{\bullet *}} \circ \epsilon_{A^\bullet} \overset{(14)}{=} \eta_A^* \circ \chi_{A^\bullet} \overset{(15)}{=} \chi_A^* \circ j_A^{**} \circ \chi_{A^\bullet} \overset{(*)}{=} \chi_A^* \circ \chi_{A^*} \circ j_A = j_A$$

where in $(*)$ we used the naturality of j and the last equality follows from the fact the $(-)^*$ is adjoint to itself at the level of vector spaces. Therefore, by injectivity of j_A, we have that $\eta_A^\bullet \circ \epsilon_{A^\bullet} = \mathrm{id}_{A^\bullet}$. For the other composition, let us compute

$$\epsilon_C^* \circ \eta_{C^*} \overset{(15)}{=} \epsilon_C^* \circ j_{C^*}^* \circ \chi_{C^*} \overset{(14)}{=} \chi_C^* \circ \chi_{C^*} = \mathrm{id}_{C^*}$$

and this concludes the proof. □

3.3. **The functor $(-)^\bullet$ and the tensor product.** Next we study how the functor $(-)^\bullet$ behaves with respect to the tensor product of two algebras.

Proposition 3.3. *Let A and B be in $\mathsf{NAlg}_{\Bbbk}$. Then the canonical injection $\varphi_{A,B} : A^* \otimes B^* \longrightarrow (A \otimes B)^*$ of equation* (3) *induces the natural isomorphism in $\mathsf{NCoalg}_{\Bbbk}$*

$$\varphi'_{A,B} \coloneqq (\varphi_{A^\bullet,B^\bullet} \circ (\eta_A \otimes \eta_B))^\bullet \circ \epsilon_{(A^\bullet \otimes B^\bullet)} : A^\bullet \otimes B^\bullet \overset{\cong}{\longrightarrow} (A \otimes B)^\bullet. \tag{16}$$

Proof. Observe that for any C, D in $\mathsf{NCoalg}_{\Bbbk}$, $\varphi_{C,D} : C^* \otimes D^* \to (C \otimes D)^*$ is a morphism in $\mathsf{NAlg}_{\Bbbk}$. Thus the morphism defined in equation (16) is well-defined. For all $f \in A^\bullet, g \in B^\bullet, a \in A, b \in B$ we have that

$$\begin{aligned}
((\varphi_{A^\bullet,B^\bullet} \circ (\eta_A \otimes \eta_B))^\bullet \circ \epsilon_{(A^\bullet \otimes B^\bullet)})(f \otimes g)(a \otimes b) &= (\varphi_{A^\bullet,B^\bullet})^\bullet(\epsilon_{(A^\bullet \otimes B^\bullet)}(f \otimes g))(\eta_A(a) \otimes \eta_B(b)) \\
&= \epsilon_{(A^\bullet \otimes B^\bullet)}(f \otimes g)(\varphi_{A^\bullet,B^\bullet}(\eta_A(a) \otimes \eta_B(b))) \\
&= \varphi_{A^\bullet,B^\bullet}(\eta_A(a) \otimes \eta_B(b))(f \otimes g) \\
&= \eta_A(a)(f)\eta_B(b)(g) \overset{(15)}{=} f(a)g(b) \\
&= \big(\varphi_{A,B}(f \otimes g)\big)(a \otimes b).
\end{aligned}$$

Therefore $j_{A\otimes B} \circ \varphi'_{A,B} = \varphi_{A,B} \circ (j_A \otimes j_B)$ and in particular $\varphi'_{A,B}$ is injective. It remains to find an inverse for $\varphi'_{A,B}$. To this aim consider the algebra morphisms

$$i_A : A \longrightarrow A \otimes B : a \longmapsto a \otimes 1 \qquad \text{and} \qquad i_B : B \longrightarrow A \otimes B : b \longmapsto 1 \otimes b.$$

Then we can consider the map

$$\psi_{A,B} \coloneqq (A \otimes B)^\bullet \xrightarrow{\Delta_{(A\otimes B)^\bullet}} (A \otimes B)^\bullet \otimes (A \otimes B)^\bullet \xrightarrow{(i_A)^\bullet \otimes (i_B)^\bullet} A^\bullet \otimes B^\bullet.$$

Note that $\psi_{A,B}$ satisfies, for all $f \in (A \otimes B)^\bullet$, $a \in A$, $b \in B$:

$$\begin{aligned}
\varphi'_{A,B}(\psi_{A,B}(f))(a \otimes b) &\coloneqq \varphi'_{A,B}(((i_A)^\bullet \otimes (i_B)^\bullet)(\Delta_{(A\otimes B)^\bullet}(f)))(a \otimes b) = \varphi'_{A,B}\Big(\sum f_1 \otimes f_2\Big)(i_A(a) \otimes i_B(b)) \\
&= \sum f_1(a \otimes 1) f_2(1 \otimes b) \overset{(6)}{=} f((a \otimes 1) \cdot (1 \otimes b)) = f(a \otimes b).
\end{aligned}$$

This means that $\varphi'_{A,B}$ is also surjective and hence, *a fortiori*, it is an isomorphism with inverse $\psi_{A,B}$. □

Remark 3.4. Let A be in $\mathsf{NAlg}_{\Bbbk}$. We can consider A° as defined in (7). It should be observed that in general A° is strictly contained in $A^\bullet$. To show this take $A = C^*$ for a non-coassociative coalgebra C that is not locally finite, then $A^\bullet$ cannot be locally finite (since $\epsilon_C : C \longrightarrow C^{*\bullet}$ is injective). At the same time, in view of Remark 2.2, $A^\circ = \mathrm{Loc}(A^\bullet)$ so that it is locally finite and hence it cannot coincide with $A^\bullet$. We now provide an example of a coalgebra which is not locally finite. Explicitly, consider $C = \Bbbk[X]$ the polynomial ring in the indeterminate X endowed with the comultiplication given by

$$\Delta(1) = 1 \otimes 1, \quad \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(X^n) = X^n \otimes 1 + 1 \otimes X^n + X^{n+1} \otimes X + X \otimes X^{n+1}, \quad n \geq 2$$

and the counit given by $\varepsilon(X^n) = \delta_{n,0}$ for all $n \geq 0$. It is easy to check that (C, Δ, ε) belongs to $\mathsf{NCoalg}_{\Bbbk}$. Note that factoring out by the coideal $\Bbbk 1$ and denoting, for $n \geq 0$, by x_n the class of X^{n+1} in the quotient yields the Lie coalgebra E considered in [Mi1, page 9]. As for E, one easily proves that X^2 does not lie in any finite-dimensional subcoalgebra of C. Thus C is not locally finite.

4. Contravariant adjunction between the categories $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$ and $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$.

In this section we recall the definitions of the category of coalgebras with multiplication and unit and of that of algebras with comultiplication and counit, denoted by $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$ and $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ respectively. For an object $C \in \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$, we investigate some basic properties of $C^\bullet$. The key result will be that, in this context, $C^\bullet$ is an algebra with comultiplication and counit. This will allow us to extend the contravariant adjunction of the previous section to a contravariant adjunction between the categories $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$ and $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$.

Let us start be recalling from [Ma, Preliminaries], [Ka, Definition XV.1.1 and Proposition XV.1.2], the following definitions.

Definition 4.1. A *coalgebra with multiplication and unit* is a datum $(C, \Delta, \varepsilon, m, u)$ where

(1) the triple (C, Δ, ε) is in $\mathsf{Coalg}_{\Bbbk}$;
(2) the maps $m\colon C \otimes C \to C$ and $u\colon \Bbbk \to C$ are morphisms in $\mathsf{Coalg}_{\Bbbk}$, called *multiplication* and *unit* respectively, such that m is unital with unit u.

In other words this is a *not necessarily associative monoid* (or algebra) inside the monoidal category of coassociative and counital coalgebras. A morphism of coalgebras with multiplication and unit is a linear map which is compatible with both structures, that is, simultaneously a morphism of coalgebras and of algebras. The category so obtained will be denoted by $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$.

Dualizing Definition 4.1 leads to the construction of the category of algebras with comultiplication and counit denoted by $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$, whose objects are denoted by $(A, m, u, \Delta, \varepsilon)$. Thus, an object in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ is a not necessarily coassociative comonoid inside the monoidal category of associative and unital algebras.

Proposition 4.2. *Let $(C, \Delta, \varepsilon, m, u)$ be a coalgebra with multiplication and unit. Then $(C^\bullet, \Delta^\bullet, \varepsilon^\bullet, m^\bullet, u^\bullet)$ is an algebra with comultiplication and counit. Moreover, this establishes a well-defined contravariant functor*

$$(-)^\bullet : \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk}) \longrightarrow \mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk}).$$

Proof. For the reader's sake, in this proof we will write explicitly the isomorphism $\varphi'_{C,D} : C^\bullet \otimes D^\bullet \to (C \otimes D)^\bullet$ of equation (16), although in the statement we identified the domain and the codomain of this map. For this reason the multiplication of $C^\bullet$ is more precisely $m_{C^\bullet} := \Delta^\bullet \circ \varphi'_{C,C}$ while its comultiplication is $\Delta_{C^\bullet} := \left(\varphi'_{C,C}\right)^{-1} \circ m^\bullet$. We compute

$$j_C \circ m_{C^\bullet} = j_C \circ \Delta^\bullet \circ \varphi'_{C,C} \overset{(5)}{=} \Delta^* \circ j_{C\otimes C} \circ \varphi'_{C,C} \overset{(16)}{=} \Delta^* \circ \varphi_{C,C} \circ (j_C \otimes j_C) = m_{C^*} \circ (j_C \otimes j_C).$$

Furthermore, since $\mathrm{id}_{\Bbbk} \in \Bbbk^* = \Bbbk^\bullet$, we can compute $\varepsilon_C^\bullet(\mathrm{id}_{\Bbbk})$ obtaining $\varepsilon_C = \varepsilon_C^\bullet(\mathrm{id}_{\Bbbk}) \in C^\bullet$. Thus we can set $1_{C^\bullet} := \varepsilon_C$ and we have that $j_C(1_{C^\bullet}) = 1_{C^*}$. Since j_C is injective, we deduce that $C^\bullet$ is an algebra. Explicitly, for every $f, g \in C^\bullet$ we have that $m_{C^\bullet}(f \otimes g)$ equals the convolution product $f * g$. Moreover $u^* : C^* \to \Bbbk^* \cong \Bbbk : f \mapsto f(1)$ restricts to a map $\varepsilon_{C^\bullet} := u^* : C^\bullet \to \Bbbk$.

For all $x, y \in C$ and $f, g \in C^\bullet$, we have

$$\begin{aligned}\sum (f*g)_1(x) \cdot (f*g)_2(y) &= (f*g)(xy) = \sum f((xy)_1) \cdot g((xy)_2) \\ &= \sum f(x_1y_1) \cdot g(x_2y_2) = \sum f_1(x_1) \cdot f_2(y_1) \cdot g_1(x_2) \cdot g_2(y_2) \\ &= \sum (f_1 * g_1)(x) \cdot (f_2 * g_2)(y),\end{aligned}$$

and $\sum \varepsilon_1(x) \cdot \varepsilon_2(y) = \varepsilon(xy) = \varepsilon(x) \cdot \varepsilon(y)$. This implies that

$$\sum (f*g)_1 \otimes (f*g)_2 = \sum (f_1 * g_1) \otimes (f_2 * g_2) \qquad \text{and} \qquad \sum \varepsilon_1 \otimes \varepsilon_2 = \varepsilon \otimes \varepsilon.$$

Thus $\Delta_{C^\bullet}$ is multiplicative and unital. Moreover

$$\varepsilon_{C^\bullet}(f*g) = (f*g)(1) = f(1) \cdot g(1) = \varepsilon_{C^\bullet}(f) \cdot \varepsilon_{C^\bullet}(g)$$

and $\varepsilon_{C^\bullet}(\varepsilon) = \varepsilon(1) = 1$ so that $\varepsilon_{C^\bullet}$ is multiplicative and unital as well.

Take a morphism $f : C \to D$ in $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$. By Lemma 3.1, we know that $f^\bullet : D^\bullet \to C^\bullet$ is a coalgebra map. It remains to check that it is multiplicative and unital. For every $x \in C$,

$$f^\bullet(1_{D^\bullet})(x) = 1_{D^\bullet}(f(x)) = \varepsilon_D(f(x)) = \varepsilon_C(x) = 1_{C^\bullet}(x),$$

where in the third equality we used the fact that f is a counital map. Furthermore, for every α and β in $D^\bullet$

$$f^\bullet(\alpha * \beta)(x) = (\alpha * \beta)(f(x)) = \sum \alpha(f(x)_1)\beta(f(x)_2) = \sum \alpha(f(x_1))\beta(f(x_2)) = (f^\bullet(\alpha) * f^\bullet(\beta))(x),$$

where we have used the fact that f is comultiplicative. This establishes the stated functor which is clearly a contravariant one. □

As we have mentioned in Remark 2.2, see also the references quoted therein, there is a functor

$$(-)^\circ : \mathsf{Alg}_{\Bbbk} \longrightarrow \mathsf{Coalg}_{\Bbbk}, \tag{17}$$

which is the restriction of the functor $(-)^\bullet$ defined in subsection 3.1. Even thought we could use the same notation for these two functors without ambiguity, we preferred to keep different notations, in order to make clear the distinction between the associative and the non-associative case.

Lemma 4.3. *The functor $(-)^\circ$ is lifted to a functor $(-)^\circ : \mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk}) \to \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$. That is, we have a commutative diagram*

$$\begin{array}{ccc} \mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk}) & \xrightarrow{(-)^\circ} & \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk}) \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{\Bbbk} & \xrightarrow{(-)^\circ} & \mathsf{Coalg}_{\Bbbk} \end{array}$$

where the vertical functors are the forgetful ones.

Proof. Take an object $(A, m, u, \Delta, \varepsilon)$ in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$. Analogously to subsection 3.3, we can consider the datum $(A^\circ, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ)$, where $(A^\circ, m^\circ, u^\circ)$, Δ° and ε° are the images, up to the natural isomorphism of Proposition 3.3, through the functor (17) of (A, m, u), Δ and ε, respectively. We know that $(A^\circ, m^\circ, u^\circ)$ is coassociative counital coalgebra, and also that Δ° and ε° are coalgebra maps. The unital property of Δ° with respect to ε° is automatically derived from the counitality of Δ with respect to ε. Summing up, we have that $(A^\circ, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ)$ is an object in the category $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$. To check that the stated functor is well-defined on morphisms one mimics the last part of the proof of Proposition 4.2. □

The following is our first main result.

Theorem 4.4. *There is a natural isomorphism:*

$$\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})\Big(A, C^\bullet\Big) \;\cong\; \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})\Big(C, A^\circ\Big)$$

for every pair of objects A in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ and C in $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$. That is, there is a contravariant adjunction

$$\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk}) \underset{(-)^\circ}{\overset{(-)^\bullet}{\rightleftarrows}} \mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$$

where the contravariant functor $(-)^\circ$ is defined by Lemma 4.3 and $(-)^\bullet$ by Proposition 4.2.

Proof. In order to prove the theorem let us consider the unit $\eta_A : A \to A^{\circ *}$ and the counit $\epsilon_C : C \to C^{*\circ}$ defined in (15) and (14) respectively, where A is in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ and C is in $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$.

We already know that η_A is multiplicative and unital. We claim that it lands into $A^{\circ\bullet}$, indeed let us show that $\mathrm{Im}(\eta_A)$ is a good subspace of $A^{\circ *}$. For all $a \in A$ and $f, g \in A^\circ$ we have that

$$\begin{aligned} m^{\circ *}(\eta_A(a))(f \otimes g) = \eta_A(a)(f * g) = (f * g)(a) &= \sum f(a_1)g(a_2) \\ &= \sum \eta_A(a_1)(f)\eta_A(a_2)(g) = \Big(\sum \eta_A(a_1) \otimes \eta_A(a_2)\Big)(f \otimes g) \end{aligned}$$

i.e. for all $a \in A$, $m^{\circ *}(\eta_A(a)) \in \varphi_{A^\circ, A^\circ}(\mathrm{Im}(\eta_A) \otimes \mathrm{Im}(\eta_A))$ where $\varphi_{-,-}$ is the canonical inclusion of equation (3). Therefore we denote by $\xi_A : A \to A^{\circ\bullet}$ the corestriction of η_A. Observe further that, in particular, this means that η_A (and hence ξ_A) is comultiplicative. Moreover, ξ_A is also counital since

$$u^\circ(\xi_A(a)) = \xi_A(a)(1_{A^\circ}) = \xi_A(a)(\varepsilon) = \varepsilon(a)$$

for all $a \in A$. By the foregoing, ξ_A is a morphism in the category $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$. Now we can check the naturality in A of ξ_A: pick a morphism $f\colon A \to B$ in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ and consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\xi_A} & A^{\circ\bullet} & \xrightarrow{j_{A^\circ}} & A^{\circ *} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\circ\bullet}} & & \downarrow{\scriptstyle f^{\circ *}} \\ B & \xrightarrow[\xi_B]{} & B^{\circ\bullet} & \xrightarrow[j_{B^\circ}]{} & B^{\circ *} \end{array}$$

The commutativity of the outer diagram encodes the naturality of η, while the right-hand side diagram follows by (5). Hence the left-hand side diagram commutes too whence the naturality of ξ is settled. To construct the counit one proceeds in a very similar way. Explicitly, for an object $(C, \Delta, \varepsilon, m, u)$ in $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$, the map ϵ_C induces the counit which is given by

$$\vartheta_C\colon C \longrightarrow C^{\bullet\circ},\ x \longmapsto \Big[C^\bullet \to \Bbbk,\ g \mapsto g(x)\Big]. \tag{18}$$

It remains to check the commutativity of the following two diagrams

$$\begin{array}{ccc} C^{\bullet\circ\bullet} & \xrightarrow{(\vartheta_C)^\bullet} & C^\bullet \\ {\scriptstyle \eta_{C^\bullet}}\uparrow & \nearrow_{=} & \\ C^\bullet & & \end{array} \qquad \begin{array}{ccc} A^{\circ\bullet\circ} & \xleftarrow{\vartheta_{A^\circ}} & A^\circ \\ {\scriptstyle (\eta_A)^\circ}\downarrow & \swarrow_{=} & \\ A^\circ & & \end{array}$$

As for the first one, for every $g \in C^\bullet$ and for every $c \in C$ a direct calculation shows that

$$(\vartheta_C)^\bullet(\eta_{C^\bullet}(g))(c) = \eta_{C^\bullet}(g)(\vartheta_C(c)) = \vartheta_C(c)(g) = g(c)$$

while for the second one, for every $f \in A^\circ$ and for every $a \in A$,

$$(\xi_A)^\circ(\vartheta_{A^\circ}(f))(a) = \vartheta_{A^\circ}(f)(\xi_A(a)) = \xi_A(a)(f) = f(a).$$

This finishes the proof. □

5. Contravariant adjunction between quasi-bialgebras and split dual quasi-bialgebras.

This section contains our main result. We show that the contravariant adjunction of Theorem 4.4 can be restricted to an adjunction between the category of quasi-bialgebras and that of split dual quasi-bialgebras. The latter is a full subcategory of the category of dual quasi-bialgebras characterized by the fact that the 3-cocycle (i.e. the reassociator) splits as a finite sum of tensor products of linear maps.

5.1. **Quasi-bialgebras and dual quasi-bialgebras: definitions and examples.**

We start by recalling the definition of the main objects of this section. The definitions presented here are quoted form [Dr, Ma, Ka].

Definition 5.1 ([Dr]). A *quasi-bialgebra* is an object $(H, m, u, \Delta, \varepsilon)$ in the category $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ (see after Definition 4.1), endowed with a counital 3-cocycle Φ, i.e. an invertible element in the algebra $H \otimes H \otimes H$ that satisfies

$$(H \otimes H \otimes \Delta)(\Phi) \cdot (\Delta \otimes H \otimes H)(\Phi) = (1 \otimes \Phi) \cdot (H \otimes \Delta \otimes H)(\Phi) \cdot (\Phi \otimes 1), \tag{19}$$

$$(\varepsilon \otimes H \otimes H)(\Phi) = (H \otimes \varepsilon \otimes H)(\Phi) = (H \otimes H \otimes \varepsilon)(\Phi) = 1 \otimes 1, \tag{20}$$

$$\Phi \cdot (\Delta \otimes H)(\Delta(h)) = (H \otimes \Delta)(\Delta(h)) \cdot \Phi. \tag{21}$$

The element Φ is called the *reassociator*[(1)] of the quasi-bialgebra. Obviously, if $\Phi = 1 \otimes 1 \otimes 1$, then $(H, m, u, \Delta, \varepsilon)$ is a usual bialgebra.

A linear map $\mathfrak{f} : (H, m, u, \Delta, \varepsilon, \Phi) \to (H', m', u', \Delta', \varepsilon', \Phi')$ is a *morphism of quasi-bialgebras* if it is a morphism in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ such that

$$(\mathfrak{f} \otimes \mathfrak{f} \otimes \mathfrak{f})(\Phi) = \Phi'. \tag{22}$$

The category of quasi-bialgebras and their morphisms will be denoted by $\mathsf{QBialg}_{\Bbbk}$.

(1) In [Ka, page 369] this is called the Drinfeld associator.

Example 5.2. Let C be in $\mathsf{NCoalg}_{\Bbbk}$ and consider the tensor algebra $T(C)$. By the universal property of the tensor algebra, the comultiplication and the counit of C induce a comultiplication and a counit on $T = T(C)$ respectively that make it into an object in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$. Suppose that T is in $\mathsf{QBialg}_{\Bbbk}$. Then it admits a reassociator $\Phi \in T^{\otimes 3}$ but, in view of Corollary B.4, $\Phi \in \Bbbk \cdot 1 \otimes 1 \otimes 1$. By (20), $\Phi = 1 \otimes 1 \otimes 1$ which means that T is in $\mathsf{Bialg}_{\Bbbk}$. This forces C to be in $\mathsf{Coalg}_{\Bbbk}$. Therefore, if we consider C in $\mathsf{NCoalg}_{\Bbbk}$ but not in $\mathsf{Coalg}_{\Bbbk}$, then $T(C)$ is in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$ but not in $\mathsf{QBialg}_{\Bbbk}$.

We now give an explicit example of such a C. Consider the alternative algebra A, see Example 2.3, constructed as follows. As a vector space, $A = \Bbbk e \oplus \Bbbk x \oplus \Bbbk y$, with multiplication table given as follows:

$\cdot$	e	x	y
e	e	x	y
x	x	y	x
y	y	x	x

This is a unital, commutative but not associative algebra. Let us take its ordinary linear dual $C = A^* = \Bbbk E \oplus \Bbbk X \oplus \Bbbk Y$ where $\{E, X, Y\}$ is the dual basis. It comes out to be a counital, cocommutative but not coassociative coalgebra. The induced comultiplication and counit are given by

$$\Delta(X) = X \otimes Y + Y \otimes X + Y \otimes Y + X \otimes E + E \otimes X, \qquad \varepsilon(X) = 0,$$
$$\Delta(Y) = X \otimes X + Y \otimes E + E \otimes Y, \qquad \varepsilon(Y) = 0,$$
$$\Delta(E) = E \otimes E, \qquad \varepsilon(E) = 1.$$

Observe further that cocommutativity ensures that $T(C)$ satisfies the condition to be a coalternative coalgebra, as given explicitly in Example 2.3 equation (8).

Besides algebras with comultiplication and counit that are not quasi-bialgebras, we also have quasi-bialgebras that are not bialgebras.

Example 5.3. Let us retrieve a couple of examples. First we exhibit a way to produce non-trivial quasi-bialgebras from ordinary bialgebras. Following [Dr], we consider a quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \Phi)$ and we recall that a *twist* on H (also referred to as *gauge transformation*, cf. [Ka, Definition XV.3.1]) is an invertible element F of $H \otimes H$ such that

$$(H \otimes \varepsilon)(F) = 1 = (\varepsilon \otimes H)(F).$$

Given a twist, we can construct the twisted quasi-bialgebra $H_F := (H, m, u, \Delta_F, \varepsilon, \Phi_F)$ where

$$\Delta_F(a) := F \cdot \Delta(a) \cdot F^{-1} \qquad \text{and} \qquad \Phi_F := (1 \otimes F) \cdot (A \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes A)(F^{-1}) \cdot (F^{-1} \otimes 1).$$

This is still a quasi-bialgebra (cf. [Ka, Proposition XV.3.2], [Dr, page 1422]). Now, assume we have $(G, m, u, \Delta, \varepsilon)$ an ordinary bialgebra. We can endow it with a trivial structure of quasi-bialgebra by considering $\Phi = 1 \otimes 1 \otimes 1$. If we take a non-trivial twist F on G, then G_F is a quasi-bialgebra, but it is not necessarily an ordinary bialgebra. Indeed:

$$\Phi_F = (1 \otimes F) \cdot (G \otimes \Delta)(F) \cdot (\Delta \otimes G)(F^{-1}) \cdot (F^{-1} \otimes 1)$$

does not equal $1 \otimes 1 \otimes 1$ in general, and Δ_F is not coassociative. In such cases, G_F is a non-trivial example of quasi-bialgebra.

Next, let us show a case in which an ordinary bialgebra can be endowed with a non-trivial structure of quasi-bialgebra without changing its underlying structure. This example comes from [EG, Preliminaries 2.3] (see also [BCT, Example 2.5]). Let $C_2 = \langle g \rangle$ be the cyclic group of order 2 with generator g and let $\Bbbk$ be a field of characteristic different from 2. Consider the group algebra $H(2) := \Bbbk C_2$ with its ordinary bialgebra structure, i.e., $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Observe that $H(2)$ is a two-dimensional commutative algebra. Now, set $p := \frac{1}{2}(1 - g)$ and consider the non-trivial reassociator

$$\Psi := (1 \otimes 1 \otimes 1) - 2(p \otimes p \otimes p).$$

It can be easily verified that $(H(2), m, u, \Delta, \varepsilon, \Psi)$ satisfies the conditions to be a quasi-bialgebra.

Actually this example can tell us more: $H(2)$ with this non-trivial quasi-bialgebra structure turns out to be not twist equivalent to any ordinary bialgebra (by *twist equivalent* to a bialgebra G' we mean that there exists a twist F on $H(2)$ and an isomorphism of quasi-bialgebras $G' \cong H(2)_F$; cf. [Dr, page 1422]).

Hence this is a genuine example of a quasi-bialgebra. To see this note that $H(2)$ can be endowed with a *quasi-antipode*, i.e. a triple (s, α, β) composed by an algebra anti-homomorphism $s : H(2) \to H(2)$ and two distinguished elements α and β such that

$$\sum s(a_1)\alpha a_2 = \varepsilon(a)\,\alpha, \quad \sum a_1\beta s(a_2) = \varepsilon(a)\,\beta, \quad \sum \Phi^1\beta s(\Phi^2)\alpha\Phi^3 = 1, \quad \sum s(\phi^1)\alpha\phi^2\beta s(\phi^3) = 1.$$

where $\sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 := \Phi$ and $\sum \phi^1 \otimes \phi^2 \otimes \phi^3 := \Phi^{-1}$ (a quasi-bialgebra with quasi-antipode is usually called a *quasi-Hopf algebra*; cf. [Dr, page 1424]). In particular, $H(2)$ can be endowed with the quasi-antipode $(\mathrm{id}_{H(2)}, g, 1)$. By [Dr, page 1425], it is possible to twist a quasi-Hopf algebra too, and get again a quasi-Hopf algebra. As a consequence, if there were a twist F on $H(2)$ such that $H(2)_F \cong G'$ where G' is an ordinary bialgebra, then $H(2)_F$ would turn out to be an ordinary Hopf algebra. In particular $\beta_F \cdot \alpha_F = 1$. However, writing $F := \sum F^1 \otimes F^2$, $F^{-1} := \sum f^1 \otimes f^2$ and recalling that $H(2)$ is commutative we get

$$\beta_F := \sum F^1 \cdot \beta \cdot s(F^2) = \sum F^1 \cdot F^2, \qquad \alpha_F := \sum s(f^1) \cdot \alpha \cdot f^2 = \sum f^1 \cdot f^2 \cdot g,$$

and

$$\beta_F \cdot \alpha_F = \Big(\sum F^1 \cdot F^2\Big) \cdot \Big(\sum f^1 \cdot f^2 \cdot g\Big) = \sum F^1 \cdot f^1 \cdot F^2 \cdot f^2 \cdot g = g \neq 1.$$

Therefore, $H(2)$ cannot be twist equivalent to any ordinary bialgebra.

For any quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \Phi)$, we denote by

$$\Phi := \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 \;=\; \sum \Psi^1 \otimes \Psi^2 \otimes \Psi^3 \;=\; \sum \Theta^1 \otimes \Theta^2 \otimes \Theta^3,$$

the reassociator Φ of Definition 5.1, whose inverse is

$$\phi := \sum \phi^1 \otimes \phi^2 \otimes \phi^3 \;=\; \sum \psi^1 \otimes \psi^2 \otimes \psi^3 \;=\; \sum \theta^1 \otimes \theta^2 \otimes \theta^3.$$

This notations will be soon understood. Explicitly, equations (19), (20) and (21) can be rewritten as

$$\begin{aligned}
\sum \Phi^1 \cdot \Psi^1_{\ 1} \otimes \Phi^2 \cdot \Psi^1_{\ 2} \otimes \Phi^3_{\ 1} \cdot \Psi^2 \otimes \Phi^3_{\ 2} \cdot \Psi^3 &= \sum \Psi^1 \cdot \Theta^1 \otimes \Phi^1 \cdot \Psi^2_{\ 1} \cdot \Theta^2 \otimes \Phi^2 \cdot \Psi^2_{\ 2} \cdot \Theta^3 \otimes \Phi^3 \cdot \Psi^3, \\
\sum \Phi^1 \varepsilon(\Phi^2) \otimes \Phi^3 &= 1 \otimes 1, \\
\sum \Phi^1 h_{1,1} \otimes \Phi^2 h_{1,2} \otimes \Phi^3 h_2 &= \sum h_1 \Phi^1 \otimes h_{2,1} \Phi^2 \otimes h_{2,2} \Phi^3.
\end{aligned}$$

The dual version of Definition 5.1, led in [Ma] to the notion of *dual quasi-bialgebra*, which, for sake of reader convenience, we recall here with details.

Definition 5.4. A *dual quasi-bialgebra* is an object $(U, \Delta, \varepsilon, m, u)$ in the category $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$, endowed with a unital 3-cocycle ω, i.e. a convolution invertible element $\omega\colon U \otimes U \otimes U \to \Bbbk$ that satisfies

$$\Big(\omega \circ (U \otimes U \otimes m)\Big) * \Big(\omega \circ (m \otimes U \otimes U)\Big) = (\varepsilon \otimes \omega) * \Big(\omega \circ (U \otimes m \otimes U)\Big) * (\omega \otimes \varepsilon) \tag{23}$$

$$\omega(h \otimes k \otimes l) = \varepsilon(h)\,\varepsilon(k)\,\varepsilon(l), \quad \text{whenever } 1_U \in \{h, k, l\} \subset U \tag{24}$$

$$\Big(u \circ \omega\Big) * \Big(m \circ (m \otimes U)\Big) = \Big(m \circ (U \otimes m)\Big) * \Big(u \circ \omega\Big), \tag{25}$$

where the star $*$ in equation (23) stands for the convolution product of the algebra $(U^{\otimes 4})^*$, while in equation (25) it is the convolution product of the non-associative algebra $\mathsf{Vect}_{\Bbbk}(U^{\otimes 3}, U)$. The map ω is also called the *reassociator* of the dual quasi-bialgebra (this is an invertible element in the convolution algebra $(U^{\otimes 3})^*$).

A linear map $\mathfrak{g}\colon (U, m, u, \Delta, \varepsilon, \omega) \to (U', m', u', \Delta', \varepsilon', \omega')$ is a morphism of dual quasi-bialgebras if it is a morphism in the category $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$ satisfying:

$$\omega' \circ (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) = \omega. \tag{26}$$

The category of dual quasi-bialgebras and their morphisms will be denoted by $\mathsf{DQBialg}_{\Bbbk}$.

On elements, the equations (23), (24) and (25) are written, for all $x, y, z, t \in U$, as

$$\begin{gathered}
\sum \omega(x_1 \otimes y_1 \otimes z_1 t_1)\omega(x_2 y_2 \otimes z_2 \otimes t_2) = \sum \omega(y_1 \otimes z_1 \otimes t_1)\omega(x_1 \otimes y_2 z_2 \otimes t_2)\omega(x_2 \otimes y_3 \otimes z_3), \\
\omega(x \otimes y \otimes 1) = \omega(x \otimes 1 \otimes y) = \omega(1 \otimes x \otimes y) = \varepsilon(x)\varepsilon(y), \\
\sum \omega(x_1 \otimes y_1 \otimes z_1)(x_2 y_2)z_2 = \sum x_1(y_1 z_1)\omega(x_2 \otimes y_2 \otimes z_2).
\end{gathered}$$

5.2. **The contravariant adjunction.** We first check that the functor of Lemma 4.3, leads to a functor from the category $\mathsf{QBialg}_{\Bbbk}$ to $\mathsf{DQBialg}_{\Bbbk}$. Consider a quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \Phi)$. Since the underlying object $(H, m, u, \Delta, \varepsilon)$ is in $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$, we can consider its image $(H^\circ, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ)$ in $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$ by the functor of Lemma 4.3. Set $U = H^\circ$ and consider the natural transformation of (15) at $H^{\otimes 3}$

$$H^{\otimes 3} \xrightarrow{\eta_{H^{\otimes 3}}} \Big((H^{\otimes 3})^\circ\Big)^* \overset{(16)}{\cong} \Big((H^\circ)^{\otimes 3}\Big)^* = \Big(U^{\otimes 3}\Big)^*,$$

which by construction is an algebra map. Therefore, the following $\Bbbk$-linear map

$$\omega := \eta_{H^{\otimes 3}}(\Phi) : U^{\otimes 3} \longrightarrow \Bbbk, \quad \Big(f \otimes g \otimes h \longmapsto \sum f(\Phi^1) g(\Phi^2) h(\Phi^3)\Big), \tag{27}$$

is an invertible element in the convolution algebra $(U^{\otimes 3})^*$, since Φ is so in the algebra $H^{\otimes 3}$.

We claim that $(U, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ, \omega)$ is now a dual quasi-bialgebra. Taken an element $x \in H$, we can compute

$$\begin{aligned}
\Big(\sum \omega(f_1 \otimes g_1 \otimes h_1)(f_2 g_2) h_2\Big)(x) &= \sum f_1(\Phi^1) g_1(\Phi^2) h_1(\Phi^3) f_2(x_{1,1}) g_2(x_{1,2}) h_2(x_2) \\
&\overset{(21)}{=} \sum f_1(\Phi^1) g_1(\Phi^2) h_1(\Phi^3) f_2(\phi^1 x_1 \Phi^1) g_2(\phi^2 x_{2,1} \Phi^2) h_2(\phi^3 x_{2,2} \Phi^3) \\
&\overset{(6)}{=} \sum f(\Phi^1 \phi^1 x_1 \Phi^1) g(\Phi^2 \phi^2 x_{2,1} \Phi^2) h(\Phi^3 \phi^3 x_{2,2} \Phi^3) \\
&= \sum f(x_1 \Phi^1) g(x_{2,1} \Phi^2) h(x_{2,2} \Phi^3) \\
&\overset{(6)}{=} \sum f_1(x_1) g_1(x_{2,1}) h_1(x_{2,2}) f_2(\Phi^1) g_2(\Phi^2) h_2(\Phi^3) \\
&= \sum f_1(x_1) g_1(x_{2,1}) h_1(x_{2,2}) \omega(f_2 \otimes g_2 \otimes h_2) \\
&= \Big(\sum f_1(g_1 h_1) \omega(f_2 \otimes g_2 \otimes h_2)\Big)(x).
\end{aligned}$$

This gives equation (25) for (U, ω). Equation (24) for (U, ω), follows by:

$$\omega(f \otimes 1 \otimes h) = f(\Phi^1) \varepsilon(\Phi^2) h(\Phi^3) \overset{(20)}{=} f(1) h(1) = \varepsilon(f) \varepsilon(h),$$

and similarly when 1 appears in the other entries.

Let us check equation (23) for (U, ω). Considered $f, g, h, e \in U$, we have

$$\begin{aligned}
&\sum \omega(f_1 \otimes g_1 \otimes h_1 e_1) \omega(f_2 g_2 \otimes h_2 \otimes e_2) \\
&\quad = \sum f_1(\Phi^1) g_1(\Phi^2) (h_1 e_1)(\Phi^3) \, (f_2 g_2)(\Psi^1) h_2(\Psi^2) e_2(\Psi^3) \\
&\quad = \sum f_1(\Phi^1) g_1(\Phi^2) h_1(\Phi^3{}_1) e_1(\Phi^3{}_2) \, f_2(\Psi^1{}_1) g_2(\Psi^1{}_2) h_2(\Psi^2) e_2(\Psi^3) \\
&\quad = \sum f(\Phi^1 \Psi^1{}_1) g(\Phi^2 \Psi^1{}_2) h(\Phi^3{}_1 \Psi^2) e(\Phi^3{}_2 \Psi^3) \\
&\quad \overset{(20)}{=} \sum f(\Psi^1 \Theta^1) g(\Phi^1 \Psi^2{}_1 \Theta^2) h(\Phi^2 \Psi^2{}_2 \Theta^3) e(\Phi^3 \Psi^3) \\
&\quad = \sum f_1(\Psi^1) f_2(\Theta^1) \, g_1(\Phi^1) g_2(\Psi^2{}_1) g_3(\Theta^2) \, h_1(\Phi^2) h_2(\Psi^2{}_2) h_3(\Theta^3) \, e_1(\Phi^3) e_2(\Psi^3) \\
&\quad = \sum \Big(g_1(\Phi^1) h_1(\Phi^2) e_1(\Phi^3)\Big)\Big(f_1(\Psi^1) g_2(\Psi^2{}_1) h_2(\Psi^2{}_2) e_2(\Psi^3)\Big)\Big(f_2(\Theta^1) g_3(\Theta^2) h_3(\Theta^3)\Big) \\
&\quad = \sum \Big(g_1(\Phi^1) h_1(\Phi^2) e_1(\Phi^3)\Big)\Big(f_1(\Psi^1) (g_2 h_2)(\Psi^2) e_2(\Psi^3)\Big)\Big(f_2(\Theta^1) g_3(\Theta^2) h_3(\Theta^3)\Big) \\
&\quad = \sum \omega(g_1 \otimes h_1 \otimes e_1) \, \omega(f_1 \otimes g_2 h_2 \otimes e_2) \, \omega(f_2 \otimes g_3 \otimes h_3),
\end{aligned}$$

where we have used the convolution product and the formula (6). This completes the proof of the claim.

Furthermore, it is by definition that any morphism $\mathfrak{f} : (H, m, u, \Delta, \varepsilon, \Phi) \longrightarrow (H', m', u', \Delta', \varepsilon', \Phi')$ of quasi-bialgebras, is a morphism in the category $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$. Then, by applying the functor $(-)^\circ$ of Lemma 4.3, we get that $\mathfrak{f}^\circ : (H'^\circ, \Delta'^\circ, \varepsilon'^\circ, m'^\circ, u'^\circ) \longrightarrow (H^\circ, \Delta^\circ, \varepsilon^\circ, m^\circ, u^\circ)$ is a morphism in the category $\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})$. Therefore, we only need to check the compatibility condition with reassociators constructed in equation (27), which is derived as follows:

$$\begin{aligned}
\omega\Big((\mathfrak{f}^\circ \otimes \mathfrak{f}^\circ \otimes \mathfrak{f}^\circ)(f \otimes g \otimes h)\Big) &= \omega\Big((f \circ \mathfrak{f}) \otimes (g \circ \mathfrak{f}) \otimes (h \circ \mathfrak{f})\Big) = (f \otimes g \otimes h)\Big((\mathfrak{f} \otimes \mathfrak{f} \otimes \mathfrak{f})(\Phi)\Big) \\
&\overset{(22)}{=} (f \otimes g \otimes h)(\Phi') = \omega'(f \otimes g \otimes h).
\end{aligned}$$

Hence † satisfies (26) and it is a morphism of dual quasi-bialgebras. Then, we have established a contravariant functor

$$(-)^\circ : \mathsf{QBialg}_{\Bbbk} \longrightarrow \mathsf{DQBialg}_{\Bbbk}, \tag{28}$$

which obviously converts the following diagram

$$\begin{array}{ccc} \mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk}) & \xrightarrow{\quad (-)^\circ \quad} & \mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk}) \\ \uparrow & & \uparrow \\ \mathsf{QBialg}_{\Bbbk} & \xrightarrow{\quad (-)^\circ \quad} & \mathsf{DQBialg}_{\Bbbk} \end{array}$$

commutative, where the vertical functors are the canonical forgetful functors.

In the other way around, take an object $(U, \Delta, \varepsilon, m, u, \omega)$ in the category $\mathsf{DQBialg}_{\Bbbk}$. Thus, we can take the image of its underlying object $(U, \Delta, \varepsilon, m, u)$ by the functor of Proposition 4.2, that is the object $(U^\bullet, m^\bullet, u^\bullet, \Delta^\bullet, \varepsilon^\bullet)$ of the category $\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})$. The problem now is to construct a reassociator Φ for $U^\bullet$, i.e. a unital 3-cocycle. It seems that *a priori* there is no obvious way to deduce this cocycle directly from the starting datum $(U, \Delta, \varepsilon, m, u, \omega)$. To this aim, an assumption should be postulated. First, we consider the following natural transformation:

$$\zeta : (U^\bullet)^{\otimes 3} \xrightarrow{(j_U)^{\otimes 3}} (U^*)^{\otimes 3} \xrightarrow{\varphi_{U,U}\otimes U^*} (U \otimes U)^* \otimes U^* \xrightarrow{\varphi_{U\otimes U,U}} (U^{\otimes 3})^*, \tag{29}$$

which, up to the isomorphism $(U^{\otimes 3})^\bullet \cong (U^\bullet)^{\otimes 3}$ of equation (16), coincides with the canonical injection of the total good subspace of $(U^{\otimes 3})^*$. Notice that ζ is an algebra map, as it is a composition of algebra maps. Moreover, it is easily seen that ζ is in fact a natural transformation at U.

Proposition 5.5. *Let $(U, \Delta, \varepsilon, m, u, \omega)$ be a dual quasi-bialgebra. Assume there exists an invertible element $\Phi \in (U^\bullet)^{\otimes 3}$ such that $\zeta(\Phi) = \omega$, then $(U^\bullet, m^\bullet, u^\bullet, \Delta^\bullet, \varepsilon^\bullet, \Phi)$ is a quasi-bialgebra.*

Proof. Write $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3$. Then $\omega(x \otimes y \otimes z) = \sum \Phi^1(x)\Phi^2(y)\Phi^3(z)$, for every $x, y, z \in U$. Using this equality, equations (23),(24)) and (25) are easily transferred to equations (19), (20) and (21), respectively. This concludes the proof. □

Corollary 5.6. *Let $(H, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. Then $(H^{\circ\bullet}, m^{\circ\bullet}, u^{\circ\bullet}, \Delta^{\circ\bullet}, \varepsilon^{\circ\bullet})$ is still a quasi-bialgebra with reassociator $\Psi := (\xi_H)^{\otimes 3}(\Phi)$, where ξ is the unit of the adjunction of Theorem 4.4.*

Proof. We already know from (28) that $(H^\circ, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ, \omega)$ is a dual quasi-bialgebra with reassociator given by $\omega = \zeta\left((\xi_H)^{\otimes 3}(\Phi)\right)$. Now apply Proposition 5.5 to conclude. □

Let us denote by $\mathsf{SDQBialg}_{\Bbbk}$ the full subcategory of the category $\mathsf{DQBialg}_{\Bbbk}$ whose objects are *split dual quasi-bialgebras*, i.e. dual quasi-bialgebras $(U, \Delta, \varepsilon, m, u, \omega)$ such that there exists an invertible element $\Phi \in (U^\bullet)^{\otimes 3}$ with $\zeta(\Phi) = \omega$. In this way the assignment described in Proposition 5.5 yields the functor

$$(-)^\bullet : \mathsf{SDQBialg}_{\Bbbk} \longrightarrow \mathsf{QBialg}_{\Bbbk}, \tag{30}$$

acting on morphisms as in Proposition 4.2 (the compatibility with reassociators follows by using the natural transformation of (29)). We are led to the following main result.

Theorem 5.7. *The contravariant adjunction of Theorem 4.4 induces the contravariant adjunction*

$$\mathsf{SDQBialg}_{\Bbbk} \underset{(-)^\circ}{\overset{(-)^\bullet}{\rightleftarrows}} \mathsf{QBialg}_{\Bbbk}$$

where the contravariant functor $(-)^\circ$ is the one of (28)*, and $(-)^\bullet$ is the one of* (30)*.*

Proof. The only thing we need to check is that the unit and the counit of the adjunction of Theorem 4.4 preserve the reassociator of a quasi-bialgebra and the one of a dual quasi-bialgebra respectively. For the unit, which is given by

$$\xi : \mathrm{id}_{\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})} \longrightarrow (-)^\bullet \circ (-)^\circ$$

as in the proof of Theorem 4.4, this follows directly from Corollary 5.6.

As for the counit

$$\vartheta : \mathrm{id}_{\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})} \longrightarrow (-)^\circ \circ (-)^\bullet$$

which is given by (18), consider a dual quasi-bialgebra $(U, \Delta, \varepsilon, m, u, \omega)$ in $\mathsf{SDQBialg}_{\Bbbk}$; this means that there exists an element $\Phi \in (U^\bullet)^{\otimes 3}$ such that $\zeta(\Phi) = \omega$ and that $(U^\bullet, \Delta^\bullet, \varepsilon^\bullet, m^\bullet, u^\bullet, \Phi)$ is a quasi-bialgebra, where ζ is the natural transformation of (29). From the definition of the functor in (28), we have that the reassociator for the dual quasi-bialgebra $(U^{\bullet\circ}, \Delta^{\bullet\circ}, \varepsilon^{\bullet\circ}, m^{\bullet\circ}, u^{\bullet\circ})$ is clearly given by $\zeta\left((\xi_{U^\bullet})^{\otimes 3}(\Phi)\right)$. Now observe that the following computation

$$\begin{aligned}\zeta\left((\xi_{U^\bullet})^{\otimes 3}(\Phi)\right) \circ (\vartheta_U)^{\otimes 3} &= \left((\vartheta_U)^{\otimes 3}\right)^* \left(\zeta\left((\xi_{U^\bullet})^{\otimes 3}(\Phi)\right)\right) \overset{\text{(nat. of } \zeta)}{=} \zeta\left(\left((\vartheta_U)^\bullet\right)^{\otimes 3}\left((\xi_{U^\bullet})^{\otimes 3}(\Phi)\right)\right) \\ &= \zeta\left(\left((\vartheta_U)^\bullet \circ \xi_{U^\bullet}\right)^{\otimes 3}(\Phi)\right) = \zeta(\Phi) = \omega\end{aligned}$$

shows that ϑ preserves reassociators as desired. Hence, the unit comes out to be a quasi-bialgebra map and the counit a dual quasi-bialgebra map, settling the adjunction. □

Remark 5.8. Recall that a subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is *closed under sources* whenever for any morphism $f\colon a \to b$ in $\mathcal{A}$, if b is in $\mathcal{B}$ then a is in $\mathcal{B}$. Let us check that $\mathsf{SDQBialg}_{\Bbbk}$ is closed under sources when regarded as a subcategory of $\mathsf{DQBialg}_{\Bbbk}$. Let $\mathfrak{g}\colon (U', \omega') \to (U, \omega)$ be a morphism in $\mathsf{DQBialg}_{\Bbbk}$ such that (U, ω) is an object in $\mathsf{SDQBialg}_{\Bbbk}$. By assumption, there exists $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 \in (U^\bullet)^{\otimes 3}$ such that $\omega = \zeta(\Phi)$. Since $\mathfrak{g}$ preserves the reassociator, we have that

$$\omega' = \omega \circ \left(\mathfrak{g}^{\otimes 3}\right) = \zeta(\Phi) \circ \left(\mathfrak{g}^{\otimes 3}\right) = \varphi'_{U'\otimes U', U'} \circ (\varphi'_{U', U'} \otimes U'^*)\left(\left(\Phi^1 \circ \mathfrak{g}\right) \otimes \left(\Phi^2 \circ \mathfrak{g}\right) \otimes \left(\Phi^3 \circ \mathfrak{g}\right)\right) = \zeta\left((\mathfrak{g}^\bullet)^{\otimes 3}(\Phi)\right),$$

where $\varphi'_{-,-}$ is the natural transformation of equation (16). This means that ω' itself comes out to be the image by ζ of $(\mathfrak{g}^\bullet)^{\otimes 3}(\Phi)$ that lies in $(U')^\bullet$. Therefore, (U', ω') belongs to $\mathsf{SDQBialg}_{\Bbbk}$.

Let us observe briefly that $\mathsf{SDQBialg}_{\Bbbk}$ is a proper subcategory of $\mathsf{DQBialg}_{\Bbbk}$: in fact the subsequent example exhibits a dual quasi-bialgebra whose reassociator does not split. This means moreover that this particular dual quasi-bialgebra cannot be the finite dual of a quasi-bialgebra (in view of the definition of the reassociator given in (27)).

Example 5.9. Let $\Bbbk$ be a field and consider $\Bbbk[X]$ the ring of polynomials in one indeterminate X with the monoid bialgebra structure, i.e. $\Delta(X) = X \otimes X$, $\varepsilon(X) = 1$. Let us consider a map $\varphi : \Bbbk[X] \longrightarrow \Bbbk$ not in $\Bbbk[X]^\circ$, the ordinary finite dual of $\Bbbk[X]$ (which, in this case, coincides with $\Bbbk[X]^\bullet$), and such that $\varphi(1) = 1$, $\varphi(X^n) \neq 0$ for all $n \geq 1$. Let us build a 3-cocycle ω that does not split by mean of φ. Recalling that a basis for $\Bbbk[X] \otimes \Bbbk[X] \otimes \Bbbk[X]$ is given by the elements $X^n \otimes X^k \otimes X^m$ for $m, k, n \geq 0$, let us define ω on this basis as follows, and then extend it by linearity. For all $m, n, k \geq 0$ let us set:

$$\omega(1 \otimes X^n \otimes X^m) = \omega(X^n \otimes 1 \otimes X^m) = \omega(X^n \otimes X^m \otimes 1) := 1;$$
$$\omega\left(X^n \otimes X^{k+1} \otimes X^m\right) := \varphi\left(X^k\right)^{-2} \varphi\left(X^{n+k}\right) \varphi\left(X^{m+k}\right).$$

Observe that the given comultiplication ensures that we have

$$\omega^{-1}\left(X^n \otimes X^k \otimes X^m\right) = \omega\left(X^n \otimes X^k \otimes X^m\right)^{-1} = \frac{1}{\omega(X^n \otimes X^k \otimes X^m)}$$

for all $m, k, n \geq 0$. Now, let us show that ω is actually a unital 3-cocycle. It is unital by definition. If $0 \in \{m, n, r, s\}$ then we trivially have

$$\omega(X^m \otimes X^r \otimes X^s)\,\omega(X^n \otimes X^{m+r} \otimes X^s)\,\omega(X^n \otimes X^m \otimes X^r) = \omega(X^n \otimes X^m \otimes X^{r+s})\,\omega(X^{n+m} \otimes X^r \otimes X^s).$$

For all $m, n, r, s \geq 1$ we have

$$\begin{aligned}&\omega(X^m \otimes X^r \otimes X^s)\,\omega(X^n \otimes X^{m+r} \otimes X^s)\,\omega(X^n \otimes X^m \otimes X^r) \\ &= \varphi\left(X^{r-1}\right)^{-2} \varphi\left(X^{m+r-1}\right) \varphi\left(X^{s+r-1}\right) \varphi\left(X^{m+r-1}\right)^{-2} \varphi\left(X^{n+m+r-1}\right) \varphi\left(X^{s+m+r-1}\right) \varphi\left(X^{m-1}\right)^{-2} \varphi\left(X^{n+m-1}\right) \varphi\left(X^{r+m-1}\right) \\ &= \varphi\left(X^{m-1}\right)^{-2} \varphi\left(X^{n+m-1}\right) \varphi\left(X^{s+m+r-1}\right) \varphi\left(X^{r-1}\right)^{-2} \varphi\left(X^{s+r-1}\right) \varphi\left(X^{n+m+r-1}\right) \\ &= \omega(X^n \otimes X^m \otimes X^{r+s})\,\omega(X^{n+m} \otimes X^r \otimes X^s).\end{aligned}$$

This proves that ω is a 3-cocycle. If $\omega \in \Bbbk[X]^\bullet \otimes \Bbbk[X]^\bullet \otimes \Bbbk[X]^\bullet$, then

$$\varphi = \omega(- \otimes X \otimes X) = (\Bbbk[X]^\bullet \otimes \eta(X) \otimes \eta(X))(\omega) \in \Bbbk[X]^\bullet$$

where $\eta = \eta_{\Bbbk[X]}$ is the map defined in equation (15), a contradiction. Since the comultiplication Δ is cocommutative, the datum $(\Bbbk[X], m, u, \Delta, \varepsilon, \omega)$ defines a dual quasi-bialgebra whose reassociator does not split, as desired. An example of a map φ as above is exhibited in Lemma B.6.

Remark 5.10. As we mentioned above, starting from a quasi-bialgebra $(U, \Delta, \varepsilon, m, u, \omega)$, the construction of a reassociator for $U^\bullet$ is not at all clear and perhaps an impossible task. This in fact is connected to a certain problem of localization in non-commutative algebras as follows. Precisely, we are asking for the construction of an invertible element Φ in a certain algebra R (in our case $R = (U^\bullet)^{\otimes 3}$), by only knowing the existence of an invertible element ω in an algebra extension T of R (in our case $T = (U^{\otimes 3})^*$ using the algebra map $\zeta : R \to T$ of equation (29)). In our opinion this construction is not at all realistic except perhaps in some very concrete situation. This is why we think that Theorem 5.7 was not established in a naive way and that it is the best result which can be extracted from this theory.

Appendix A. New characterization

In this section we give an alternative description of the finite dual in the non-associative case. Given a linear map, several useful criteria are shown in order to guarantee that this map belongs to the finite dual. Further characterizations can be found in [ACM].

Given a vector space V and $S \subseteq V^*$, we denote by

$$S^\perp := \{v \in V |\ s(v) = 0, \forall s \in S\}.$$

For every $a \in A$ in an algebra A and $f \in A^*$, we define in A^* the elements $a \rightharpoonup f$ and $f \leftharpoonup a$ by setting, for every $b \in A$

$$(a \rightharpoonup f)(b) := f(ba) \qquad \text{and} \qquad (f \leftharpoonup a)(b) := f(ab). \tag{31}$$

Furthermore, the vector subspace of A^* generated by the set $\{a \rightharpoonup f | a \in A\}$ will be simply denoted by $A \rightharpoonup f$. A similar notation will be adopted for the right action $\leftharpoonup$. The subsequent lemma is an analogue of [Sw, Proposition 6.0.3] or [Mo, Lemma 9.1.1] and can be proved by the same argument.

Lemma A.1. *Let $f \in A^*$. Then the following are equivalent.*

(1) $m^*(f) \in \mathrm{Im}(\varphi_{A,A})$.
(2) $\dim_\Bbbk(A \rightharpoonup f) < \infty$.
(3) $\dim_\Bbbk(f \leftharpoonup A) < \infty$.

One cannot expect, as in the case of associative algebras [Mo, Lemma 9.1.1], that the equivalent conditions (1)-(3) in Lemma A.1, imply either that $\dim_\Bbbk\Big(A \rightharpoonup (f \leftharpoonup A)\Big) < \infty$ or that $\dim_\Bbbk\Big((A \rightharpoonup f) \leftharpoonup A\Big) < \infty$. Nevertheless, the converse remains true.

A.1. **The tensor algebra and finite codimensional subspaces.** Let V and W be vector spaces endowed with a $\Bbbk$-linear map $\phi^1_{V,W} : V \to \mathrm{End}_\Bbbk(W)$. Then this map induces a unique algebra map $\phi_{V,W} : T(V) \to \mathrm{End}_\Bbbk(W)^{\mathrm{op}}$ such that $(\phi_{V,W})_{|V} = \phi^1_{V,W}$ and $(\phi_{V,W})_{|\Bbbk}$ is the unit $\Bbbk \to \mathrm{End}_\Bbbk(W)^{\mathrm{op}} : k \mapsto k\mathrm{id}_w$, where $T(-)$ stands for the tensor algebra functor.

Then W becomes a right $T(V)$-module via $\blacktriangleleft$ defined, for every $z \in T(V), w \in W$, by setting

$$w \blacktriangleleft z := \phi_{V,W}(z)(w).$$

Hence we can consider the left $T(V)$-module structure on W^* uniquely defined by setting

$$(z \blacktriangleright f)(w) := f(w \blacktriangleleft z), \text{ for every } z \in T(V), w \in W, f \in W^*.$$

Example A.2. Consider the so-called enveloping algebra $A^e := A \otimes A^{\mathrm{op}}$ as V and A as W. Then one can consider the map

$$\phi^1_{V,W} : A^e \to \mathrm{End}_\Bbbk(A) : l \otimes r \mapsto [a \mapsto r(al)].$$

For shortness, we set

$$\phi^1_A := \phi^1_{V,W} \qquad \text{and} \qquad \phi_A := \phi_{V,W}.$$

In particular, for every $l, r \in A$, we get

$$x \blacktriangleleft (l \otimes r) = \phi_A(l \otimes r)(x) = \phi^1_A(l \otimes r)(x) = r(xl) \tag{32}$$

and

$$((l \otimes r) \blacktriangleright f)(a) = f(a \blacktriangleleft (l \otimes r)) \overset{(32)}{=} f(r(al)) = (l \rightharpoonup (f \leftharpoonup r))(a)$$

so that

$$((l \otimes r) \blacktriangleright f) = (l \rightharpoonup (f \leftharpoonup r)). \tag{33}$$

For a subset $S \subseteq T(A^e)$ and an element $f \in A^*$, we denote by $S \blacktriangleright f$ the vector subspace of A^* spanned by the set of elements $\{s \blacktriangleright f | \, s \in S\}$.

Proposition A.3. *Let (A, m, u) be in $\mathsf{NAlg}_{\Bbbk}$. Then*

$$A^\bullet \overset{(4)}{=} \sum_{V \in \mathcal{G}} V \;=\; \Big\{ f \in A^* \mid \dim_{\Bbbk}\Big((A^e)^{\otimes n} \blacktriangleright f\Big) < \infty, \text{ for every } n \in \mathbb{N} \Big\}.$$

Proof. Set $T := T(A^e)$. We write a generator of $(A^e)^{\otimes i}$ in the form $(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)$ where $l_1, \ldots, l_i \in A$ and $r_1, \ldots, r_i \in A^{\mathrm{op}}$. Note that

$$\begin{aligned} [\phi_A(1 \otimes r) \circ \phi_A(l \otimes 1)](a) &= \left[\phi_A^1(1 \otimes r) \circ \phi_A^1(l \otimes 1)\right](a) \\ &= \phi_A^1(1 \otimes r)(al) = r(al) \\ &= \phi_A^1(l \otimes r)(a) = \phi_A(l \otimes r)(a) \end{aligned}$$

and hence

$$\phi_A(l \otimes r) = \phi_A(1 \otimes r) \circ \phi_A(l \otimes 1) = \phi_A(l \otimes 1) \circ^{\mathrm{op}} \phi_A(1 \otimes r),$$

where the notation $\circ^{\mathrm{op}}$ stands for the multiplication of $\mathrm{End}_{\Bbbk}(A)^{\mathrm{op}}$. Thus

$$\begin{aligned} \phi_A[(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] &= \phi_A[(l_1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes r_i)] \\ &= \phi_A(l_1 \otimes r_1) \circ^{\mathrm{op}} \cdots \circ^{\mathrm{op}} \phi_A(l_i \otimes r_i) \\ &= \phi_A(l_1 \otimes 1) \circ^{\mathrm{op}} \phi_A(1 \otimes r_1) \circ^{\mathrm{op}} \cdots \circ^{\mathrm{op}} \phi_A(l_i \otimes 1) \circ^{\mathrm{op}} \phi_A(1 \otimes r_i) \\ &= \phi_A[(l_1 \otimes 1) \cdot_T (1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes 1) \cdot_T (1 \otimes r_i)] \\ &= \phi_A[(l_1 \otimes 1) \otimes (1 \otimes r_1) \cdots \otimes (l_i \otimes 1) \otimes (1 \otimes r_i)] \end{aligned}$$

where the notation $.\,_T$ stands for the multiplication of T. Therefore

$$\begin{aligned} a \blacktriangleleft [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] &= \phi_A[(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)](a) \\ &= \phi_A[(l_1 \otimes 1) \otimes (1 \otimes r_1) \cdots \otimes (l_i \otimes 1) \otimes (1 \otimes r_i)](a) \\ &= a \blacktriangleleft [(l_1 \otimes 1) \cdot_T (1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes 1) \cdot_T (1 \otimes r_i)] \end{aligned}$$

Set $L := A \otimes 1$ and $R := 1 \otimes A^{\mathrm{op}}$. For shortness we write $l \in L$ for $l \otimes 1$ and $r \in R$ for $1 \otimes r$. We also omit the product over T. Using this notation, we obtain

$$a \blacktriangleleft [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] = a \blacktriangleleft (l_1 r_1 \cdots l_i r_i).$$

For every $n \geq 1$, $f \in A^*$, we set

$$W_n(f) := \mathrm{Span}_{\Bbbk}\Big\{ (a_1 a_2 \cdots a_{n-1} a_n) \blacktriangleright f \mid a_1, \ldots, a_n \in L \cup R \Big\}.$$

Set also $W_0(f) := \Bbbk f$. Since both A and A^{op} contain 1, it is clear that $W_i(f) \subseteq W_j(f)$ for $i \leq j$.

Note further that $W_i(f) \subseteq \Big((A^e)^{\otimes i} \blacktriangleright f\Big) \subseteq W_{2i}(f)$ for every $i \in \mathbb{N}$ so that $\dim_{\Bbbk}\Big((A^e)^{\otimes n} \blacktriangleright f\Big) < \infty$ if and only if $\dim_{\Bbbk}(W_n(f)) < \infty$ for every $n \in \mathbb{N}$. Set

$$B := \Big\{ f \in A^* \mid \dim_{\Bbbk}(W_n(f)) < \infty \text{ for every } n \in \mathbb{N} \Big\}.$$

It remains to prove that $A^\bullet = B$.

$\subseteq$) It suffices to prove that $V \subseteq B$ for every $V \in \mathcal{G}$. Let us prove that $W_n(f)$ is finite dimensional for every $f \in V$ by induction on $n \in \mathbb{N}$. For $n = 0$ there is nothing to prove.

Let $n > 0$ be such that $W_{n-1}(v)$ is finite-dimensional for every $v \in V$. Let $f \in V$. Write $\Delta_V(f) = \sum_{i=1}^t g_i \otimes h_i \in V \otimes V$. Let $a_1, \ldots, a_n \in L \cup R$ and $w := a_1 a_2 \cdots a_{n-1}$. Then

$$((w a_n) \blacktriangleright f)(x) = f(x \blacktriangleleft (w a_n)) = f((x \blacktriangleleft w) \blacktriangleleft a_n)$$

If $a_n = l \in L$, then

$$((w a_n) \blacktriangleright f)(x) = f((x \blacktriangleleft w) \blacktriangleleft l) \overset{(32)}{=} f((x \blacktriangleleft w) l) = \sum_{i=1}^t g_i (x \blacktriangleleft w) h_i(l) = \sum_{i=1}^t (w \blacktriangleright g_i)(x) h_i(l)$$

so that $(wa_n) \blacktriangleright f = \sum_{i=1}^{n} h_i(l) \cdot (w \blacktriangleright g_i) \in \sum_{i=1}^{n} W_{n-1}(g_i)$. If $a_n = r \in R$, then

$$((wa_n) \blacktriangleright f)(x) = f((x \blacktriangleleft w) \blacktriangleleft r) \overset{(32)}{=} f(r(x \blacktriangleleft w)) = \sum_{i=1}^{t} g_i(r) h_i(x \blacktriangleleft w) = \sum_{i=1}^{t} g_i(r)(w \blacktriangleright h_i)(x)$$

so that

$$((wa_n) \blacktriangleright f) = \sum_{i=1}^{t} g_i(r) \cdot (w \blacktriangleright h_i) \in \sum_{i=1}^{t} W_{n-1}(h_i)$$

Thus

$$(a_1 a_2 a_3 a_4 \cdots a_{n-1} a_n) \blacktriangleright f \in \sum_{i=1}^{t} W_{n-1}(g_i) + \sum_{i=1}^{t} W_{n-1}(h_i)$$

for every $a_1, \ldots, a_n \in L \cup R$, which means that

$$W_n(f) \subseteq \sum_{i=1}^{t} W_{n-1}(g_i) + \sum_{i=1}^{t} W_{n-1}(h_i).$$

Since, by inductive hypothesis, the latter is finite-dimensional so is $W_n(f)$.

$\supseteq$) Let $f \in B$ and let us prove that $V := (T \blacktriangleright f)$ is good (this implies $f = (1 \blacktriangleright f) \in V \subseteq A^\bullet$). Consider an element $v \in V$. Then there is $z \in T$ such that $v = z \blacktriangleright f$. Write $z := \sum_{i=0}^{n} z_i$ with $z_i \in (A^e)^{\otimes i}$ so that

$$v = z \blacktriangleright f = \sum_{i=0}^{n} z_i \blacktriangleright f \in \sum_{i=0}^{n} \left((A^e)^{\otimes i} \blacktriangleright f\right) \subseteq \left((A^e)^{\otimes n} \blacktriangleright f\right).$$

Henceforth it is not restrictive to assume $z \in (A^e)^{\otimes n}$. We have then that

$$(A \rightharpoonup v) \overset{(33)}{\subseteq} (A^e) \blacktriangleright v \subseteq (A^e) \blacktriangleright (z \blacktriangleright f) \subseteq (A^e) \blacktriangleright \left((A^e)^{\otimes n} \blacktriangleright f\right) \subseteq (A^e)^{\otimes (n+1)} \blacktriangleright f$$

and the latter is finite-dimensional. Hence $(A \rightharpoonup v)$ is finite-dimensional and, by Lemma A.1, we have that $m^*(v) \in Im(\varphi_{A,A})$. Write $m^*(v) = \sum_{i=1}^{n} g_i \otimes h_i \in A^* \otimes A^*$. By the proof of the same lemma, we can choose $g_1, \ldots, g_n$ to form a basis of $(A \rightharpoonup v)$. Thus there exist $a_1, \ldots, a_n \in A$ such that $g_i(a_j) = \delta_{i,j}$. We compute

$$\left(\left(1 \otimes a_j\right) \blacktriangleright v\right)(x) \overset{(33)}{=} \left(v \leftharpoonup a_j\right)(x) = v\left(a_j x\right) = \sum_{i=1}^{n} g_i\left(a_j\right) h_i(x) = h_j(x)$$

so that $h_j = \left(1 \otimes a_j\right) \blacktriangleright v \in (A^e \blacktriangleright v) \subseteq V$. We have so proved that $m^*(v) = \sum_{i=1}^{n} g_i \otimes h_i \in A^* \otimes V$. A similar argument shows that $m^*(v) \in V \otimes A^*$ and hence $m^*(v) \in (A^* \otimes V) \cap (V \otimes A^*) = V \otimes V$. □

Remark A.4. Let $f \in A^*$ be such that $f(I) = 0$ for some finite codimensional ideal in A (an ideal in a non-associative algebra is just a $\Bbbk$-vector subspace such that $aI \subseteq I$ and $Ia \subseteq I$ for all $a \in A$). Let $l \otimes r \in A^e$ and let $x \in I$. We have that

$$((l \otimes r) \blacktriangleright f)(x) = f(r(xl)) \subseteq f(I) = 0.$$

Inductively, if $z \in (A^e)^{\otimes n}$, $z = (l_1 \otimes r_1) \otimes \cdots \otimes (l_{n-1} \otimes r_{n-1}) \otimes (l_n \otimes r_n) = w \otimes (l_n \otimes r_n)$, then

$$(z \blacktriangleright f)(x) = ((l_n \otimes r_n) \blacktriangleright f)(x \blacktriangleleft w) = f(r_n((x \blacktriangleleft w) l_n)) \subseteq f(r_n(I l_n)) \subseteq f(I) = 0.$$

Therefore $(A^e)^{\otimes n} \blacktriangleright f$ is contained in $I^\perp$, that injects into $\left(\frac{A}{I}\right)^*$, which has finite dimension for all $n \in \mathbb{N}$. Hence, if f vanishes on a finite codimensional ideal of A, then $f \in A^\bullet$. This is an alternative way to show that A° is contained in $A^\bullet$, see Remark 2.2.

Remark A.5. Another description of $A^\bullet$ by using the so-called *standard filtration* $(T_{(n)})_{n \in \mathbb{N}}$ of $T := T(A^e)$, is also possible. Precisely, this filtration is defined by setting $T_{(n)} := \bigoplus_{i=0}^{n} (A^e)^{\otimes i}$, where $(A^e)^{\otimes 0} := \Bbbk$. Then

$$A^\bullet = \left\{f \in A^* \mid \dim_{\Bbbk}(T_{(n)} \blacktriangleright f) < \infty \text{ for every } n \in \mathbb{N}\right\}.$$

In fact

$$(T_{(n)} \blacktriangleright f) \subseteq \left(\left(\bigoplus_{i=0}^{n} (A^e)^{\otimes i}\right) \blacktriangleright f\right) \subseteq \sum_{i=0}^{n} \left((A^e)^{\otimes i} \blacktriangleright f\right) \subseteq \left((A^e)^{\otimes n} \blacktriangleright f\right)$$

so that $(T_{(n)} \blacktriangleright f) = \left((A^e)^{\otimes n} \blacktriangleright f\right)$.

We now give a characterization of $A^\bullet$ in the spirit of [Mo, Definition 1.2.3].

Proposition A.6. *Let (A, m, u) be in $\mathsf{NAlg}_{\Bbbk}$ and let $f \in A^*$. Then the following are equivalent*

(i) $f \in A^\bullet$;

(ii) There is a family $(I_n)_{n\in\mathbb{N}}$ of subspaces of A of finite codimension such that, for each $n \geq 1$,

$$(I_n \blacktriangleleft A^{\mathrm{e}}) \subseteq I_{n-1}, \quad \text{and} \quad f(I_0) = 0.$$

Moreover, if one the these conditions holds true, then we can choose

$$I_0 = \mathrm{Ker}(f) \quad \text{and} \quad I_n = \left\{a \in A \mid a \blacktriangleleft A^{\mathrm{e}} \subseteq I_{n-1}\right\}, \quad \text{for every} \quad n > 0.$$

Proof. ($\Rightarrow$). Assume $f \in A^\bullet$ and set $I_n := \left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right)^\perp$. For every $n \geq 1$, $u \in I_n, z \in A^{\mathrm{e}}, w \in (A^{\mathrm{e}})^{\otimes(n-1)}$,

$$(w \blacktriangleright f)(u \blacktriangleleft z) = (z \blacktriangleright (w \blacktriangleright f))(u) = ((zw) \blacktriangleright f)(u) \in \left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right)(u) = 0$$

so that $u \blacktriangleleft z \in \left((A^{\mathrm{e}})^{\otimes(n-1)} \blacktriangleright f\right)^\perp = I_{n-1}$ and hence $(I_n \blacktriangleleft A^{\mathrm{e}}) \subseteq I_{n-1}$. Since $\left((A^{\mathrm{e}})^{\otimes 0} \blacktriangleright f\right) = (\Bbbk \blacktriangleright f) = \Bbbk f$ we get that $f(I_0) = 0$.

($\Leftarrow$). Inductively one proves that $\left(I_n \blacktriangleleft (A^{\mathrm{e}})^{\otimes n}\right) \subseteq I_0$ so that we have

$$\left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right)(I_n) \subseteq f\left(I_n \blacktriangleleft (A^{\mathrm{e}})^{\otimes n}\right) \subseteq f(I_0) = 0.$$

Therefore $\left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right) \subseteq I_n^\perp$ which is finite-dimensional as I_n has finite codimension, which by Proposition A.3 implies that $f \in A^\bullet$.

Let us check the last statement. For $n = 0$ we have that

$$I_0 = \left((A^{\mathrm{e}})^{\otimes 0} \blacktriangleright f\right)^\perp = (\Bbbk \blacktriangleright f)^\perp = (\Bbbk f)^\perp = \mathrm{Ker}(f),$$

and for $n > 0$ we have:

$$\begin{aligned} I_n &:= \left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right)^\perp = \left\{a \in A \mid \left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f\right)(a) = 0\right\} \\ &= \left\{a \in A \mid \left((A^{\mathrm{e}})^{\otimes(n-1)} \blacktriangleright f\right)(a \blacktriangleleft A^{\mathrm{e}}) = 0\right\} \\ &= \left\{a \in A \mid (a \blacktriangleleft A^{\mathrm{e}}) \subseteq \left((A^{\mathrm{e}})^{\otimes(n-1)} \blacktriangleright f\right)^\perp\right\} \\ &= \left\{a \in A \mid (a \blacktriangleleft A^{\mathrm{e}}) \subseteq I_{n-1}\right\}, \end{aligned}$$

and this finishes the proof. □

Let C be a coalgebra. Then the coalgebra structure of C, through the universal property of the tensor algebra, induces a bialgebra structure on $T(C)$ so that it makes sense to use the notation $\Delta_{T(C)}(z) := \sum z_1 \otimes z_2$ for any $z \in T(C)$, for the comultiplication of $T(C)$; see e.g. [Ra, Theorem 5.3.1].

Lemma A.7. *Let C and D be two coalgebras with a $\Bbbk$-linear map $\phi^1_{C,D} : C \to \mathrm{End}_{\Bbbk}(D)$ as in subsection A.1. Assume that $D \otimes C \to D : d \otimes c \mapsto d \blacktriangleleft c$ is a coalgebra map. Then D is a right $T(C)$-module coalgebra through $\blacktriangleleft$ and (D^*, m_{D^*}, u_{D^*}) is a left $T(C)$-module algebra through $\blacktriangleright$ where*

$$m_{D^*}(f \otimes g) = f * g \text{ (convolution product), and } u_{D^*}(k) = k\varepsilon_D$$

for every $f, g \in D^*, k \in \Bbbk$.

Proof. By hypothesis for every $c \in C, d \in D$, we have that

$$\begin{aligned} (d_1 \blacktriangleleft c_1) \otimes (d_2 \blacktriangleleft c_2) &= (d \blacktriangleleft c)_1 \otimes (d \blacktriangleleft c)_2, \\ \varepsilon_D(d)\varepsilon_C(c) &= \varepsilon_D(d \blacktriangleleft c). \end{aligned}$$

We need to prove that for every $z \in T(C), d \in D$, we have

$$\begin{aligned} (d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) &= (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2, \\ \varepsilon_D(d)\varepsilon_{T(C)}(z) &= \varepsilon_D(d \blacktriangleleft z). \end{aligned}$$

For $k \in \Bbbk$ we have

$$d \blacktriangleleft k = \phi_{C,D}(k)(d) = (k\mathrm{id}_D)(d) = kd.$$

Then, for every $z \in \Bbbk$ we have

$$\begin{aligned}(d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) &= (d_1 \blacktriangleleft z1) \otimes (d_2 \blacktriangleleft 1) = zd_1 \otimes d_2 \\ &= (zd)_1 \otimes (zd)_2 = (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2 ,\end{aligned}$$

and

$$\varepsilon_D (d)\, \varepsilon_{T(C)} (z) = \varepsilon_D (d)\, z = \varepsilon_D (dz) = \varepsilon_D (d \blacktriangleleft z) .$$

Let $c_1, \dots, c_n \in C$. Let us prove, by induction on $n \geq 1$, that

$$(d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) = (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2 \quad \text{and} \quad \varepsilon_D (d)\, \varepsilon_{T(C)} (z) = \varepsilon_D (d \blacktriangleleft z) ,$$

where $z \coloneqq c_1 \cdots c_n$ is the multiplication of the c_i's, each one viewed as an element in $T(C)$.

For $n = 1$ there is nothing to prove. Let $n > 1$ and assume the statement true for $n - 1$. If we set $z' \coloneqq c_1 \cdots c_{n-1}$, then we get from the one hand that

$$\begin{aligned}(d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) &= (d_1 \blacktriangleleft (z'c_n)_1) \otimes (d_2 \blacktriangleleft (z'c_n)_2) = (d_1 \blacktriangleleft z'_1 (c_n)_1) \otimes (d_2 \blacktriangleleft z'_2 (c_n)_2) \\ &= ((d_1 \blacktriangleleft z'_1) \blacktriangleleft (c_n)_1) \otimes ((d_2 \blacktriangleleft z'_2) \blacktriangleleft (c_n)_2) \\ &= ((d \blacktriangleleft z')_1 \blacktriangleleft (c_n)_1) \otimes ((d \blacktriangleleft z')_2 \blacktriangleleft (c_n)_2) \\ &= ((d \blacktriangleleft z') \blacktriangleleft c_n)_1 \otimes ((d \blacktriangleleft z') \blacktriangleleft c_n)_2 \\ &= (d \blacktriangleleft (z'c_n))_1 \otimes (d \blacktriangleleft (z'c_n))_2 = (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2\end{aligned}$$

and from the other hand that

$$\begin{aligned}\varepsilon_D (d)\, \varepsilon_{T(C)} (z) &= \varepsilon_D (d)\, \varepsilon_{T(C)} (z'c_n) = \varepsilon_D (d)\, \varepsilon_{T(C)} (z')\, \varepsilon_{T(C)} (c_n) \\ &= \varepsilon_D (d \blacktriangleleft z')\, \varepsilon_{T(C)} (c_n) = \varepsilon_D ((d \blacktriangleleft z') \blacktriangleleft c_n) \\ &= \varepsilon_D (d \blacktriangleleft (z'c_n)) = \varepsilon_D (d \blacktriangleleft z) .\end{aligned}$$

This shows the claimed formulae for every $z \in T(C)$. Therefore D is a right $T(C)$-module coalgebra through $\blacktriangleleft$. Since $(D, \Delta_D, \varepsilon_D)$ is a coassociative coalgebra, we know that (D^*, m_{D^*}, u_{D^*}) is an associative algebra. Let us check that it is a left $T(C)$-module algebra through $\blacktriangleright$. For all $f, g \in D^*, z \in T(C), d \in D$ we have

$$\begin{aligned}\sum \Big[(z_1 \blacktriangleright f) * (z_2 \blacktriangleright g) \Big] (d) &= \sum (z_1 \blacktriangleright f)(d_1)\, (z_2 \blacktriangleright g)(d_2) \\ &= \sum f (d_1 \blacktriangleleft z_1)\, g (d_2 \blacktriangleleft z_2) = \sum f\Big((d \blacktriangleleft z)_1 \Big) g\Big((d \blacktriangleleft z)_2 \Big) \\ &= (f * g)(d \blacktriangleleft z) = (z \blacktriangleright (f * g))(d)\end{aligned}$$

so that $\sum (z_1 \blacktriangleright f) * (z_2 \blacktriangleright g) = (z \blacktriangleright (f * g))$. Moreover

$$(z \blacktriangleright \varepsilon_D)(d) = \varepsilon_D (d \blacktriangleleft z) = \varepsilon_D (d)\, \varepsilon_{T(C)} (z)$$

so that $z \blacktriangleright \varepsilon_D = \varepsilon_{T(C)} (z)\, \varepsilon_D$. This proves that (D^*, m_{D^*}, u_{D^*}) is a left $T(C)$-module algebra through $\blacktriangleright$. □

Remark A.8. More generally, given a bialgebra B, the obvious contravariant functor $(-)^* : \mathfrak{M}_B \to {}_B\mathfrak{M}$, from the category of right to the category of left B-modules, is lax monoidal so that it induces the covariant functor $(-)^* : \mathfrak{M}_B \to ({}_B\mathfrak{M})^{\mathrm{op}}$ which is colax monoidal. Thus the latter functor induces a functor $\mathrm{Coalg}((-)^*) : \mathrm{Coalg}(\mathfrak{M}_B) \to \mathrm{Coalg}(({}_B\mathfrak{M})^{\mathrm{op}}) \equiv (\mathrm{Alg}({}_B\mathfrak{M}))^{\mathrm{op}}$ which means that $(-)^*$ maps right B-module coalgebras to left B-module algebras as in the particular case of Lemma A.7.

Appendix B. Complementary results

The following result is probably well-known but we were not able to find a reference.

Lemma B.1. *Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be an $\mathbb{N}$-graded ring. Suppose that the product of two non-zero homogeneous elements is non-zero. Then the invertible element of A are concentrated in A_0. Moreover, A is a domain.*

Proof. Let $x, y \in A$ be non-zero elements. Write $x = x_0 + x_1 + \cdots + x_s$, where $x_i \in A_i$, and $y = y_0 + y_1 + \cdots + y_t$, where $y_i \in A_i$, with $x_s \neq 0$ and $y_t \neq 0$. By assumption, $x_s y_t \neq 0$ and it is clearly the homogeneous element with greatest degree of xy.

If $xy = 1$, then the only possibility is $s + t = 0$ whence $s = 0$ which means $x \in A_0$.

If $xy = 0$, then we must have $x_s y_t = 0$, which is a contradiction. □

Corollary B.2. *Given a vector space V, the group of units of the tensor algebra $T(V)$ is $\Bbbk \setminus \{0\}$ and $T(V)$ is a domain.*

Proof. We have that $T = T(V)$ is graded with respect to $T_n := V^{\otimes n}$. Given $x \in T_s$ and $y \in T_t$ non-zero elements, we have that $x \cdot y = x \otimes y$ which is non-zero. □

Lemma B.3. *Let R be a $\Bbbk$-algebra that is also a domain. Then $T(V) \otimes R$ is a domain.*

Proof. Set $T = T(V)$. By the Axiom of Choice we can choose a totally ordered basis $\mathcal{B}_V := \{v_i \mid i \in I\}$ for V. Mimicking [Gr, Example 2 and 3] we can construct an admissible graded lexicographic order on the basis $\mathcal{B}_T := \{v_{i_1} v_{i_2} \cdots v_{i_n} \mid n \geq 1 \text{ and } i_1, \ldots, i_n \in I\} \cup \{1_\Bbbk\}$ of $T(V)$ as follows

$$v_{i_1} v_{i_2} \cdots v_{i_n} < v_{j_1} v_{j_2} \cdots v_{j_m}$$

if $n < m$ or $n = m$, $v_{i_s} = v_{j_s}$ for $0 \leq s \leq (t-1) < n$ and $v_{i_t} < v_{j_t}$ with respect to the total order on $\mathcal{B}$. Let $x, y \in T \otimes R$ with $x \neq 0$ and $y \neq 0$. We can write $x = b_{i_1} \otimes x_1 + \cdots + b_{i_s} \otimes x_s$ where $x_1, \ldots, x_s \in R$ with $x_s \neq 0$, $b_{i_1}, \cdots, b_{i_s} \in \mathcal{B}_T$ and $b_{i_1} < \cdots < b_{i_s}$. Analogously write $y = b_{j_1} \otimes y_1 + \cdots + b_{j_t} \otimes y_t$ where $y_1, \ldots, y_t \in R$ with $y_t \neq 0$, $b_{j_1}, \cdots, b_{j_t} \in \mathcal{B}_T$ and $b_{j_1} < \cdots < b_{j_t}$. Since R is a domain, $x_s y_t \neq 0$. Moreover, $b_{i_s} b_{j_t} \in \mathcal{B}_T$ whence $(b_{i_s} \otimes x_s)\left(b_{j_t} \otimes y_t\right) = b_{i_s} b_{j_t} \otimes x_s y_t \neq 0$. Note that $xy = b_{i_1} b_{j_1} \otimes x_1 y_1 + \cdots + b_{i_s} b_{j_t} \otimes x_s y_t$ where $b_{i_s} b_{j_t}$ is the greatest of all the first entries of the summands involved. Thus $xy \neq 0$. □

Corollary B.4. *Given a vector space V and $n \in \mathbb{N}$, the group of units of the tensor algebra $T(V)^{\otimes n}$ is $\Bbbk \setminus \{0\}$ and $T(V)^{\otimes n}$ is a domain.*

Proof. By induction on n, in view of Lemma B.3, $T(V)^{\otimes n}$ is a domain. Moreover, since $T(V)$ is a graded algebra, $T(V)^{\otimes n}$ is graded too. By Lemma B.1, the group of units of $T(V)^{\otimes n}$ is concentrated in degree zero. □

Remark B.5. Obviously Corollary B.2 follows also by Corollary B.4.

Lemma B.6. *Let $\Bbbk$ be a field and consider $\Bbbk[X]$ the (bi)algebra of polynomials in one indeterminate X. The map $\varphi : \Bbbk[X] \longrightarrow \Bbbk$ given by $\varphi(X^n) := n!$ and extended by linearity does not belong to $\Bbbk[X]^\circ$, the ordinary finite dual of $\Bbbk[X]$.*

Proof. Assume, by contradiction, that $\varphi \in \Bbbk[X]^\circ$. Then there exists a (finite-codimensional) ideal $I := \langle p(X) \rangle$ in $\Bbbk[X]$ with $\varphi(I) = 0$. Consider the system of the equations $\varphi\left(X^i p(X)\right) = 0$ for $i = 0, \ldots, n$. If we write $p(X) = \sum_{j=0}^n p_j X^j$, these equations become $\sum_{j=0}^n p_j \varphi\left(X^{i+j}\right) = 0$ for $i = 0, \ldots, n$. The matrix associated to this system is

$$T := \begin{pmatrix} 0! & 1! & \cdots & (n-1)! & n! \\ 1! & 2! & \cdots & n! & (n+1)! \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1)! & n! & \cdots & (2n-2)! & (2n-1)! \\ n! & (n+1)! & \cdots & (2n-1)! & (2n)! \end{pmatrix}$$

Thus $T = \left((i+j)!\right)$ for i, j that run from 0 to n. We claim that $\det(T) \neq 0$, or equivalently that T is invertible, which is impossible since $p(X) \neq 0$, as I is finite-codimensional. To show this, let us consider the n-th Pascal matrix $Q_n = \left(q_{ij}\right)$, i.e. the matrix whose entries are given by the relation $q_{ij} := \binom{i+j}{i}$. Then:

$$\det(T) = \det\left((i+j)!\right) = \det\left(i!\, j! q_{ij}\right) = \prod_{i=0}^n i! \prod_{j=0}^n j! \det(Q_n) = \left(0!1! \cdots (n-1)! n!\right)^2 \det(Q_n).$$

In view of [BP, Discussion preceding Theorem 4], we know that $\det(Q_n) = 1$, whence $\det(T) \neq 0$ and the claim is proved. □

Remark B.7. The fact that the map $\varphi(X^n) = n!$ is not in $\Bbbk[X]^\circ$ seems to be well-known, see [FMT, Section 2]. This depends on the correspondence between elements in $\Bbbk[X]^\circ$ and linearly recursive sequences, see e.g. [LT]. Since we could not find an explicit proof that $n!$ defines a non-linearly recursive sequence, we included the previous lemma.

Acknowledgements. L. El Kaoutit would like to thank all the members of the Department of Mathematics "Giuseppe Peano" for the warm hospitality and for the unsurpassable atmosphere during his visit.

References

[Ab] E. Abe, *Hopf algebras*. Cambridge University Press, Cambridge, 1980.

[ACM] J. A. Anquela, T. Cortés, F. Montaner, *Nonassociative coalgebras*. Comm. Algebra **22** (1994), no. 12, 4693-4716.

[AGW] J. Y. Abuhlail, J. Gómez-Torrecillas and R. Wisbauer, *Dual coalgebras of algebras over commutative rings*. J. Pure Appl. Alg. **153**, 107–120, (2000).

[BCT] D. Bulacu and S. Caenepeel, B. Torrecillas, *Involutory Quasi-Hopf Algebras*. Algebr. Represent. Theory **12** (2009), no. 2-5, 257-285.

[Bo] N. Bourbaki, *Algebra I. Chapters 1–3*. Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[BP] R. Brawer, M. Pirovino, *The linear algebra of the Pascal matrix*. (English summary) Linear Algebra Appl. **174** (1992), 13–23.

[Dr] V. G. Drinfel'd, *Quasi-Hopf algebras*. (Russian) Algebra i Analiz **1** (1989), no. 6, 114-148; translation in Leningrad Math. J. **1** (1990), no. 6, 1419-1457.

[EG] P. Etingof, S. Gelaki, *Finite dimensional quasi-Hopf algebras with radical of codimension 2*. Math. Res. Lett.**11** (2004) no. 5-6, 685-696.

[FMT] C. A. Futia, E. F. Müller, E. J. Taft, *Bialgebras of recursive sequences and combinatorial identities*. Adv. in Appl. Math. **28** (2002), no. 2, 203-230.

[Gr] E. L. Green, *An introduction to noncommutative Groebner bases*. Computational algebra (Fairfax, VA, 1993), 167-190, Lecture Notes in Pure and Appl. Math., 151, Dekker, New York, 1994.

[Ka] C. Kassel, *Quantum Groups*. Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[LT] R. G. Larson, E. J. Taft, *The algebraic structure of linearly recursive sequences under Hadamard product*. Hopf algebras. Israel J. Math. **72** (1990), no. 1-2, 118–132.

[Ma] S. Majid, *Tannaka–Krein theorem for quasi-Hopf algebras*. In Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990) (1992), M. Gerstenhaber and J. Stasheff, Eds., vol. **134** of Contemp. Math., AMS, pp. 219–232.

[Mi1] W. Michaelis, *Lie coalgebras*. Adv. in Math. **38** (1980), no. 1, 1–54.

[Mi2] W. Michaelis, *The primitives of the continuous linear dual of a Hopf algebra as the dual Lie algebra of a Lie coalgebra*. (1990), vol. **110** of Contemp. Math., AMS, pp. 125–176.

[MM] J. W. Milnor, J. C. Moore, *On the structure of Hopf algebras*. Ann. of Math. (2) **81** 1965 211-264.

[Mo] S. Montgomery, *Hopf algebras and their actions on rings*. CBMS Regional Conference Series in Mathematics, **82**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[Ra] D. E. Radford, *Hopf algebras*. Series on Knots and Everything, 49. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Sw] M. E. Sweedler, *Hopf Algebras*. W. A. Benjamin, Inc., New York, 1969.

University of Turin, Department of Mathematics "Giuseppe Peano", via Carlo Alberto 10, I-10123 Torino, Italy
Email address: alessandro.ardizzoni@unito.it
URL: sites.google.com/site/aleardizzonihome

Universidad de Granada, Departamento de Álgebra. Facultad de Educación, Econonía y Tecnología de Ceuta. Cortadura del Valle, s/n. E-51001 Ceuta, Spain
Email address: kaoutit@ugr.es
URL: http://www.ugr.es/~kaoutit

University of Turin, Department of Mathematics "Giuseppe Peano", via Carlo Alberto 10, I-10123 Torino, Italy
Email address: p.saracco@unito.it
URL: sites.google.com/site/paolosaracco