A duality result for (dual) quasi-bialgebras

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The classical algebra/coalgebra duality

Fix $\Bbbk$ a field. We assume to work in the category $\mathfrak{M} := \mathsf{Vect}_\Bbbk$ of $\Bbbk$-vector spaces.

An associative algebra is a triple $(A, m : A \otimes A \to A, u : \Bbbk \to A)$ s.t.

$$m \circ (m \otimes A) = m \circ (A \otimes m), \quad m \circ (u \otimes A) = \mathrm{id}_A = m \circ (A \otimes u).$$

A coassociative coalgebra is a triple $(C, \Delta : C \to C \otimes C, \varepsilon : \Bbbk \to C)$ s.t.

$$(\Delta \otimes C) \circ \Delta = (C \otimes \Delta) \circ \Delta, \quad (\varepsilon \otimes C) \circ \Delta = \mathrm{id}_C = (C \otimes \varepsilon) \circ \Delta.$$

If C is a coassociative coalgebra, then $(C^*, \Delta^*, \varepsilon^*)$ is an associative algebra.

The finite (or Sweedler, or restricted) dual

$$A^\circ = \{f \in A^* \mid \mathrm{Ker}(f) \text{ contains a finite-codimensional ideal of } A\}$$

of A is the largest subspace of A^* for which $\Delta_{A^\circ} = m^* : A^* \to (A \otimes A)^*$ defines a comultiplication. The pair of functors

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Motivation I

Definition

A bialgebra is a datum $(H, m, u, \Delta, \varepsilon)$ where

- (H, m, u) is an associative algebra;
- Δ and ε are algebra maps s.t. (H, Δ, ε) is a coassociative coalgebra;

If further we have $S : H \to H$ s.t. $m \circ (S \otimes H) \circ \Delta = u \circ \varepsilon = m \circ (H \otimes S) \circ \Delta$ then S is called the antipode and $(H, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra.

Theorem (Sweedler, 1960)

If H is a Hopf algebra then H° is a Hopf algebra.

Remark

- The set $\mathcal{G}(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$ of group-like elements is a group with product induced by H, unit 1_H and inverse $h^{-1} := S(h)$.
- The space $\mathcal{P}(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ of primitive elements is a Lie algebra with bracket $[f, g] := fg - gf$.

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Motivation II

Example (Hopf a. of representative functions on a topological group)

Let (G, μ, ι, e) be a topological group and $\mathcal{C}_{\mathbb{R}}(G)$ be the algebra of real-valued continuous functions on G. A $f \in \mathcal{C}_{\mathbb{R}}(G)$ is a representative function iff there exists V fin. dim. representation of G, $v \in V$ and $\varphi \in V^*$ s.t. $f(x) = \varphi(x \cdot v)$ for all $x \in G$. Let $\mathcal{R}_{\mathbb{R}}(G)$ be the algebra of representative functions. The maps

- $\Delta : \mathcal{R}_{\mathbb{R}}(G) \to \mathcal{R}_{\mathbb{R}}(G) \otimes \mathcal{R}_{\mathbb{R}}(G)$, $f \mapsto \sum_i g_i \otimes h_i$
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Conversely, if H is Hopf then $\mathcal{G}(H^\circ) = \mathrm{Alg}_{\mathbb{R}}(H, \mathbb{R})$ is a topological group. The pair of functors

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where $\sum_i g_i \otimes h_i$ is defined uniquely by the relation $\sum_i g_i(x)h_i(y) = f(xy)$, endow $\mathcal{R}_{\mathbb{R}}(G)$ with an Hopf algebra structure.

Conversely, if H is Hopf then $\mathcal{G}(H^\circ) = \mathrm{Alg}_{\mathbb{R}}(H, \mathbb{R})$ is a topological group. The pair of functors

$$\mathsf{TopGrp} \underset{\mathcal{G}(-^\circ)}{\overset{\mathcal{R}_{\mathbb{R}}(-)}{\rightleftarrows}} \mathsf{CHopf}_{\mathbb{R}}$$

defines a duality that restricts to an anti-equivalence between compact Lie groups and finitely generated commutative $\mathbb{R}$-Hopf algebras (plus other properties).

Motivation III

Example (Hopf a. of regular functions on an affine algebraic group)

Let us denote by (G, μ, ι, e) an (affine) algebraic group over an algebraically closed field $\Bbbk$. The algebra $\Bbbk[G]$ of global regular functions on G has the same Hopf algebra structure of the previous example, i.e.

- $\Delta : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$ induced by μ,
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Conversely, if H is a finitely generated reduced commutative Hopf algebra over an algebraically closed field $\Bbbk$ then $\mathcal{G}(H^\circ)$ is an affine algebraic group. The pair of functors

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defines an anti-equivalence of categories. $\mathcal{P}(H^\circ)$ is the Lie algebra of $\mathcal{G}(H^\circ)$.

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The main construction I

A (non-associative) algebra is simply a (unital) algebra (A, m, u).
Dually, a (non-coassociative) coalgebra is a (counital) coalgebra (C, Δ, ε).
If C is a coalgebra, then $(C^*, \Delta^*, \varepsilon^*)$ is an algebra.

Obstruction

In general we have:

$$\begin{array}{ccc} A^* & \xrightarrow{\;m^*\;} & (A \otimes A)^* \\ & \searrow^{\Delta(?)} & \uparrow \varphi_{A,A} \\ & & A^* \otimes A^* \end{array}$$

where for every $V, W \in \mathfrak{M}$

$$\varphi_{V,W} : V^* \otimes W^* \to (V \otimes W)^*\,,\ f \otimes g \mapsto m_{\Bbbk} \circ (f \otimes g).$$

Definition (cf. [Mi1, page 13])

A subspace $V \subseteq A^*$ is good if $m^*(V) \subseteq \varphi_{A,A}(V \otimes V)$.

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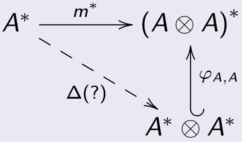

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On a good subspace $V \subseteq A^*$ we can define $\Delta_V(f) = \sum_{i=1}^{m} g_i \otimes h_i$, uniquely determined by $f(ab) = \sum_{i=1}^{m} g_i(a)h_i(b)$.

Remark

The maximal good subspace of A^*,

$$A^\bullet := \sum_{V \text{ good}} V,$$

turns out to be a (non-coassociative and counital) coalgebra.

Definition (cf. [ACM, section 2])

The coalgebra $A^\bullet$ is the finite dual coalgebra of A.

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A couple of examples

Example

Let A be an algebra and $A^\circ = \left\{ f \in A^* \mid \mathrm{Ker}(f) \supseteq I \text{ s.t. } \dim_{\Bbbk}\left(\frac{A}{I}\right) < \infty \right\}$ its Sweedler dual. Then $A^\circ \subseteq A^\bullet$. If moreover A is associative, then $A^\circ = A^\bullet$.

C coalgebra is locally finite if every $x \in C$ lies in a finite-dimensional subcoalgebra.

Example

Let C be a non-locally finite coalgebra and $A := C^*$. Since $C \hookrightarrow C^{*\bullet}$, $A^\bullet$ is non-locally finite. On the other hand, $A^\circ = \mathrm{Loc}(A^\bullet)$, the biggest locally finite subcoalgebra. Hence $A^\circ \subsetneq A^\bullet$.
An example of such C is given by $\Bbbk[X]$ with

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(X) = X \otimes 1 + 1 \otimes X,$$
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X^2 does not lie in any finite-dimensional subcoalgebra.

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The first adjunction

Set $\mathrm{NAlg}_\Bbbk$ and $\mathrm{NCoalg}_\Bbbk$ for the categories of algebras and coalgebras respectively. $(-)^* : \mathrm{Coalg}_\Bbbk \to \mathrm{Alg}_\Bbbk$ extends to a contravariant functor $(-)^* : \mathrm{NCoalg}_\Bbbk \to \mathrm{NAlg}_\Bbbk$ and the finite dual induces $(-)^\bullet : \mathrm{NAlg}_\Bbbk \to \mathrm{NCoalg}_\Bbbk$.

Theorem (cf. [ACM, section 2])

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An intermediate step between (co)algebras and (dual) quasi-bialgebras is given by:

Definition

A associative algebra with comultiplication and counit is a datum $(A, m, u, \Delta, \varepsilon)$ s.t

- $(A, m, u) \in \mathrm{Alg}_{\Bbbk}$;
- $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are algebra maps s.t. $(A, \Delta, \varepsilon) \in \mathrm{NCoalg}_{\Bbbk}$.

Dually

Definition

A coassociative coalgebra with multiplication and unit is a datum $(C, \Delta, \varepsilon, m, u)$ s.t

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The finite dual functor $(-)^\bullet$ restricts to a contravariant functor

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(Dual) quasi-bialgebras

Definition (Drinfel'd, 1989)

A quasi-bialgebra is an object $(H, m, u, \Delta, \varepsilon)$ in the category $\mathrm{NCoalg}\,(\mathrm{Alg}_{\Bbbk})$, endowed with a counital 3-cocycle $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3$ called the reassociator, i.e. an invertible element in the algebra $H \otimes H \otimes H$ that satisfies

$$(H \otimes H \otimes \Delta)(\Phi) \cdot (\Delta \otimes H \otimes H)(\Phi) = (1 \otimes \Phi) \cdot (H \otimes \Delta \otimes H)(\Phi) \cdot (\Phi \otimes 1),$$
$$(\varepsilon \otimes H \otimes H)(\Phi) = (H \otimes \varepsilon \otimes H)(\Phi) = (H \otimes H \otimes \varepsilon)(\Phi) = 1 \otimes 1,$$
$$\Phi \cdot (\Delta \otimes H)(\Delta(h)) = (H \otimes \Delta)(\Delta(h)) \cdot \Phi.$$

Definition (Majid, 1990)

A dual quasi-bialgebra is an object $(U, \Delta, \varepsilon, m, u)$ in the category $\mathrm{NAlg}\,(\mathrm{Coalg}_{\Bbbk})$, endowed with a unital 3-cocycle ω called the reassociator, i.e. a convolution invertible element $\omega \in (U \otimes U \otimes U)^*$ that satisfies

$$(\omega \circ (U \otimes U \otimes m)) * (\omega \circ (m \otimes U \otimes U)) = (\varepsilon \otimes \omega) * (\omega \circ (U \otimes m \otimes U)) * (\omega \otimes \varepsilon)$$
$$\omega\,(h \otimes k \otimes l) = \varepsilon\,(h)\,\varepsilon\,(k)\,\varepsilon\,(l)\,, \qquad \text{whenever } 1_U \in \{h, k, l\}$$
$$(u \circ \omega) * (m \circ (m \otimes U)) = (m \circ (U \otimes m)) * (u \circ \omega).$$

(Dual) quasi-bialgebras

Definition (Drinfel'd, 1989)

A quasi-bialgebra is an object $(H, m, u, \Delta, \varepsilon)$ in the category $\mathrm{NCoalg}\,(\mathrm{Alg}_{\Bbbk})$, endowed with a counital 3-cocycle $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3$ called the reassociator, i.e. an invertible element in the algebra $H \otimes H \otimes H$ that satisfies

$$(H \otimes H \otimes \Delta)(\Phi) \cdot (\Delta \otimes H \otimes H)(\Phi) = (1 \otimes \Phi) \cdot (H \otimes \Delta \otimes H)(\Phi) \cdot (\Phi \otimes 1),$$
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The last adjunction I

Lemma

$(-)^\circ : \mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk) \to \mathsf{NAlg}\,(\mathsf{Coalg}_\Bbbk)$ *restricts further to a contravariant functor*

$$(-)^\circ : \mathsf{QBialg}_\Bbbk \to \mathsf{DQBialg}_\Bbbk .$$

If $(H, \Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3)$ is a quasi-bialgebra and η denotes the unit of the adjunction $\mathsf{Alg}_\Bbbk \underset{(-)^*}{\overset{(-)^\circ}{\rightleftarrows}} (\mathsf{Coalg}_\Bbbk)^{\mathrm{op}}$ then we have an algebra map

$$\eta_{H^{\otimes 3}} : H^{\otimes 3} \longrightarrow \left(\left(H^{\otimes 3}\right)^\circ\right)^* \cong \left(\left(H^\circ\right)^{\otimes 3}\right)^*, \ \Phi \longmapsto \omega,$$

whence $\omega(f \otimes g \otimes h) = \sum f\left(\Phi^1\right) g\left(\Phi^2\right) h\left(\Phi^3\right)$ defines a reassociator for H°.

Remark

$$\omega = \sum \mathrm{ev}_{\Phi^1} \otimes \mathrm{ev}_{\Phi^2} \otimes \mathrm{ev}_{\Phi^3} \in \left(H^\circ\right)^* \otimes \left(H^\circ\right)^* \otimes \left(H^\circ\right)^* .$$

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The last adjunction II

Proposition

Let $(U, \Delta, \varepsilon, m, u, \omega) \in \mathrm{DQBialg}_{\Bbbk}$ and assume that the following holds:

$$(*) \quad \exists\, \Phi \in (U^{\bullet})^{\otimes 3} \text{ invertible s.t. } \omega \text{ is the image of } \Phi \text{ via } \zeta_U : (U^{\bullet})^{\otimes 3} \hookrightarrow (U^{\otimes 3})^{*}.$$

Then $(U^{\bullet}, m^{\bullet}, u^{\bullet}, \Delta^{\bullet}, \varepsilon^{\bullet}, \Phi)$ is a quasi-bialgebra.

Definition

A dual quasi-bialgebra that satisfies $(*)$ is called a split dual quasi-bialgebra.

Split dual quasi-bialgebras form a full subcategory $\mathrm{SDQBialg}_{\Bbbk}$ of $\mathrm{DQBialg}_{\Bbbk}$ and $(-)^{\bullet}$ yields a contravariant functor $(-)^{\bullet} : \mathrm{SDQBialg}_{\Bbbk} \to \mathrm{QBialg}_{\Bbbk}$.

Theorem

The duality between $\mathrm{NAlg}\left(\mathrm{Coalg}_{\Bbbk}\right)$ *and* $\mathrm{NCoalg}\left(\mathrm{Alg}_{\Bbbk}\right)$ *induces the adjunction*

$$\mathrm{SDQBialg}_{\Bbbk} \underset{(-)^{\circ}}{\overset{(-)^{\bullet}}{\rightleftarrows}} \left(\mathrm{QBialg}_{\Bbbk}\right)^{\mathrm{op}}.$$

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To split or not to split

Remark

$\mathsf{SDQBialg}_{\Bbbk}$ is closed under sources in $\mathsf{DQBialg}_{\Bbbk}$, i.e. if $f : (U', \omega') \to (U, \omega)$ is a morphism in $\mathsf{DQBialg}_{\Bbbk}$ and $(U, \omega) \in \mathsf{SDQBialg}_{\Bbbk}$, then $(U', \omega') \in \mathsf{SDQBialg}_{\Bbbk}$.

Example ($\mathsf{SDQBialg}_{\Bbbk}$ is a proper subcategory)

Let $\Bbbk[X]$ be the polynomial algebra in one indeterminate X with the monoid bialgebra structure, i.e. $\Delta(X) = X \otimes X$ and $\varepsilon(X) = 1$. Let $\varphi : \Bbbk[X] \to \Bbbk$ not in $\Bbbk[X]^\circ$ $(= \Bbbk[X]^\bullet)$. E.g. $\varphi(X^n) = n!$. For all $m, n, k \geq 0$ we can define inductively

$$\omega(1 \otimes X^n \otimes X^m) = \omega(X^n \otimes 1 \otimes X^m) = \omega(X^n \otimes X^m \otimes 1) := 1,$$
$$\omega(X^n \otimes X^{k+1} \otimes X^m) := \varphi(X^k)^{-2} \varphi(X^{n+k}) \varphi(X^{m+k}).$$

The constructed ω is a reassociator. If $\omega \in \Bbbk[X]^\bullet \otimes \Bbbk[X]^\bullet \otimes \Bbbk[X]^\bullet$, then

$$\varphi = \omega(- \otimes X \otimes X) = \left(\Bbbk[X]^\bullet \otimes \mathrm{ev}_X \otimes \mathrm{ev}_X\right)(\omega) \in \Bbbk[X]^\bullet,$$

which is a contradiction. $(\Bbbk[X], \Delta, \varepsilon, \omega)$ is a dual quasi-bialgebra that is not split.

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Alternative descriptions

Lemma

Let A be associative and set $(a \rightharpoonup f)(b) := f(ba)$ and $(f \leftharpoonup a)(b) := f(ab)$. The following are equivalent for $f \in A^*$:

- $f \in A^\circ$;
- $\dim(A \rightharpoonup f \leftharpoonup A) < \infty$;
- $\ker(f) \supseteq I$ *s.t.* $\dim\left(\frac{A}{I}\right) < \infty$.

Let A be any algebra. $A^e := A \otimes A^{\mathrm{op}}$. Consider the left action of $T(A^e)$ on A^* and the right one on A respectively induced by

$$(l \otimes r) \blacktriangleright f := (l \rightharpoonup (f \leftharpoonup r)) \qquad \text{and} \qquad a \blacktriangleleft (l \otimes r) := r(al).$$

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