On Lie-Rinehart algebras and complete duals of cocommutative Hopf algebroids

Paolo Saracco

Brussels Hopf Algebra Workshop 2017

L. El Kaoutit, P. Saracco, *Topological Tensor Product of Bimodules, Complete Hopf Algebroids and Convolution Algebras*. Preprint, arXiv:1705.06698, (2017).

- The integration problem

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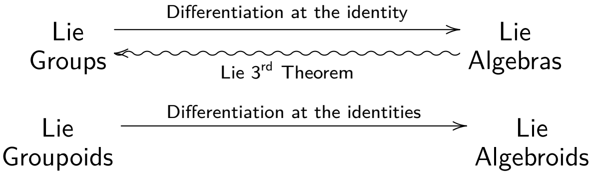

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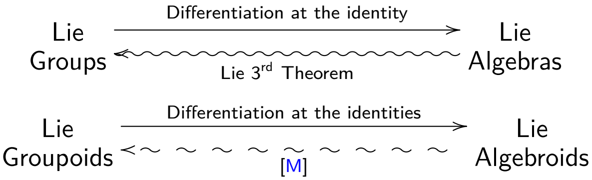

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Fix a field $\mathbb{k}$ of characteristic 0. Assume that A is a commutative $\mathbb{k}$-algebra and denote by $\mathrm{Der}(A)$ the Lie algebra of $\mathbb{k}$-derivations of A.

A Lie-Rinehart algebra over A is a triple (A, L, ω) where L is a Lie algebra which is also an A-module and $\omega : L \to \mathrm{Der}(A)$ is a Lie algebra map (the anchor) such that for all $a \in A$ and $X, Y \in L$

$$\omega(a \cdot X) = a \cdot X \qquad \text{and} \qquad [X, a \cdot Y] = \omega(X)(a) \cdot Y + a \cdot [X, Y].$$

Example: A Lie algebroid is a vector bundle $\mathcal{L} \to \mathcal{M}$ over a smooth manifold $\mathcal{M}$ with a structure of Lie algebra in the space $\Gamma(\mathcal{L})$ of sections of $\mathcal{L}$ and a morphism of vector bundles $\omega : \mathcal{L} \to T\mathcal{M}$ such that $\Gamma(\omega) : \Gamma(\mathcal{L}) \to \Gamma(T\mathcal{M})$ is a Lie algebra map and

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for all $f \in \mathcal{C}^\infty(\mathcal{M})$ and $X, Y \in \Gamma(\mathcal{L})$.

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Universal enveloping algebra and filtered Hopf algebroids

The universal enveloping algebra of a Lie-Rinehart algebra (A, L, ω) is a triple $\left(\mathcal{V}_A(L), \iota_A, \iota_L\right)$ composed by a $\Bbbk$-algebra $\mathcal{V}_A(L)$, an algebra map $\iota_A : A \to \mathcal{V}_A(L)$ and a Lie algebra map $\iota_L : L \to \mathcal{V}_A(L)$ satisfying

$$\iota_L(a \cdot X) = \iota_L(X)\iota_A(a) \quad \text{and} \quad \left[\iota_L(X), \iota_A(a)\right] = \iota_A(\omega(X)(a)), \qquad (\dagger)$$

that enjoys the following universal property:

For any triple (U, ϕ_A, ϕ_L) as above satisfying $(\dagger)$ there exists a unique algebra map $\Phi : \mathcal{V}_A(L) \to U$ such that $\Phi \circ \iota_A = \phi_A$ and $\Phi \circ \iota_L = \phi_L$.

Explicitly, $$\mathcal{V}_A(L) := \frac{T_A(A \otimes L)}{\left\langle [\eta(X), \eta(Y)] - \eta([X, Y]),\ [\eta(X), a] - \omega(X)(a) \right\rangle}$$

with $\iota_A : A \to \mathcal{V}_A(L);\ a \mapsto a$ and $\iota_L : L \to \mathcal{V}_A(L);\ X \mapsto \eta(X) := 1 \otimes X$.

Example: If $A = \mathcal{C}^\infty(\mathcal{M})$, then $\mathcal{V}_A(\mathrm{Der}(A))$ is the algebra of differential operators on $\mathcal{M}$.

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The $\Bbbk$-algebra $\mathcal{V}_A(L)$ comes endowed with

(HA1) an (injective) $\Bbbk$-algebra map $\iota_A : A \to \mathcal{V}_A(L)$;

(HA2) a cocommutative A-coring structure $\left(\mathcal{V}_A(L)_{\iota_A}, \Delta, \varepsilon\right)$ given by

$$\varepsilon(\iota_A(a)) = a, \qquad \Delta(\iota_A(a)) = \iota_A(a) \otimes_A 1 = 1 \otimes_A \iota_A(a),$$
$$\varepsilon(\iota_L(X)) = 0, \qquad \Delta(\iota_L(X)) = \iota_L(X) \otimes_A 1 + 1 \otimes_A \iota_L(X),$$

such that $\varepsilon(uv) = \varepsilon(\varepsilon(u)v)$ for all $u, v \in \mathcal{V}_A(L)$ and Δ factors through an A-ring map $\Delta : \mathcal{V}_A(L) \to \mathcal{V}_A(L) \times_A \mathcal{V}_A(L)$;

(HA3) an inverse for the map $\mathfrak{can} : \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L) \to \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L)$, $\mathfrak{can}(u \otimes_A v) = uv_1 \otimes_A v_2$, which is uniquely determined by

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A pair of $\Bbbk$-algebras $(A, \mathcal{U})$ satisfying (HA1) - (HA2) is called a cocommutative (right) bialgebroid. If it satisfies (HA3) as well, then it is a cocommutative (right) Hopf algebroid (Schauenburg [S]).

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Even more:

(FHA1) The algebra $\mathcal{V}_A(L)$ carries an exhaustive ascending filtration

$$0 \subset F^0(\mathcal{V}_A(L)) \subset F^1(\mathcal{V}_A(L)) \subset F^2(\mathcal{V}_A(L)) \subset \cdots$$

where $F^0(\mathcal{V}_A(L)) = A$ and $F^p(\mathcal{V}_A(L))$ is the right A-submodule of $\mathcal{V}_A(L)$ generated by products of at most p elements of $\iota_L(L)$. If we assume A to be filtered with the discrete filtration $F^nA = 0$ for all $n < 0$ and $F^nA = A$ for all $n \geq 0$, then the structure maps of $\mathcal{V}_A(L)$ turn out to be filtered. In particular, it does so the translation map

$$\delta : \mathcal{V}_A(L) \to \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L),\ u \mapsto \mathfrak{can}^{-1}(1 \otimes_A u) := u_- \otimes_A u_+.$$

(FHA2) If L is a finitely generated and projective A-module, then the quotient modules $F^n(\mathcal{V}_A(L))/F^{n-1}(\mathcal{V}_A(L))$ are finitely generated and projective right A-modules as well (e.g. $L = \Gamma(\mathcal{L})$, $\mathcal{L}$ a Lie algebroid).

A cocommutative Hopf algebroid $(A, \mathcal{U})$ satisfying (FHA1) is said to be filtered. If it satisfies (FHA2) as well, then it is said to have an admissible filtration.

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A cocommutative Hopf algebroid $(A, \mathcal{U})$ satisfying (FHA1) is said to be filtered. If it satisfies (FHA2) as well, then it is said to have an admissible filtration.

Even more:

(FHA1) The algebra $\mathcal{V}_A(L)$ carries an exhaustive ascending filtration

$$0 \subset F^0(\mathcal{V}_A(L)) \subset F^1(\mathcal{V}_A(L)) \subset F^2(\mathcal{V}_A(L)) \subset \cdots$$

where $F^0(\mathcal{V}_A(L)) = A$ and $F^p(\mathcal{V}_A(L))$ is the right A-submodule of $\mathcal{V}_A(L)$ generated by products of at most p elements of $\iota_L(L)$. If we assume A to be filtered with the discrete filtration $F^nA = 0$ for all $n < 0$ and $F^nA = A$ for all $n \geq 0$, then the structure maps of $\mathcal{V}_A(L)$ turn out to be filtered. In particular, it does so the translation map

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The completion 2-functor

- For a decreasingly filtered bimodule $(M, F_n M)$ over filtered algebras S and R, $\widehat{M} := \varprojlim (M/F_n M)$ is a complete $\big(\widehat{S}, \widehat{R}\big)$-bimodule.
- The bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{flt}}$ has filtered $\Bbbk$-algebras as 0-cells and filtered bimodules over filtered algebras as $\{1, 2\}$-cells. The horizontal composition is given by the usual tensor product, filtered by

$$F_n(M \otimes_R N) = \sum_{p+q=n} \operatorname{im}(F_p M \otimes_R F_q N).$$

- The bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{c}}$ has complete $\Bbbk$-algebras as 0-cells and complete bimodules over complete algebras as $\{1, 2\}$-cells. The horizontal composition is given by the completed tensor products

$$M \mathbin{\widehat{\otimes}_R} N = \widehat{M \otimes_R N}.$$

- The completion procedure induces a bifunctor $\widehat{(-)} : \mathscr{B}im_{\Bbbk}^{\mathsf{flt}} \to \mathscr{B}im_{\Bbbk}^{\mathsf{c}}$,

$$\text{0-cells}: \ R \longmapsto \widehat{R} \qquad \text{1-cells}: \ {}_S M_R \longmapsto {}_{\widehat{S}}\widehat{M}_{\widehat{R}}$$

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The full linear dual and complete Hopf algds

Let $(A,\mathcal{U})$ be a cocommutative Hopf algd with an admissible filtration. Consider its full (right) linear dual $\mathcal{U}^* = \mathrm{Hom}_{-,A}(\mathcal{U}, A) \cong \varprojlim \left(F^n(\mathcal{U})^*\right)$.

Kapranov [Ka]: $\mathcal{U}^*$ inherits a natural decreasing filtration

$$G_0\left(\mathcal{U}^*\right) = \mathcal{U}^* \quad \text{and} \quad G_{n+1}\left(\mathcal{U}^*\right) = \mathfrak{Ann}\left(F^n(\mathcal{U})\right),\ n \geq 0.$$

such that $\mathcal{U}^*$ is a complete commutative $\Bbbk$-algebra w.r.t. the convolution product. The counit induces

$$\eta = s \otimes t : A \otimes A \to \mathcal{U}^*, \quad (a \otimes b \mapsto [u \mapsto \varepsilon(bu)a]).$$

The unit and the multiplication of $\mathcal{U}$ induce a counit $\varepsilon_* : \mathcal{U}^* \to A$ and a comultiplication $\Delta_* : \mathcal{U}^* \to \mathcal{U}^* \widehat{\otimes}_A \mathcal{U}^*$ which make of $\mathcal{U}^*$ a coalgebra in the monoidal category $\left({}_A\mathrm{Bim}^c_A, \widehat{\otimes}_A, A\right)$ of complete A-bimodules.

Even more, the translation map δ induces a complete $\Bbbk$-algebra map

$$\mathcal{S} : \mathcal{U}^* \to \mathcal{U}^*, \quad f \mapsto [u \mapsto \varepsilon(f(u_-)u_+)].$$

[Ka] M. Kapranov, *Free Lie algebroids and the space of paths*. Sel. Math., New ser. **13** (2007).

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Summing up, $\mathcal{U}^*$ is a complete commutative algebra with a diagram

$$A \underset{t}{\overset{s}{\rightrightarrows}} \overset{\varepsilon_*}{\leftarrow} \mathcal{U}^* \circlearrowleft_{\mathcal{S}} \xrightarrow{\Delta_*} \mathcal{U}^* \widehat{\otimes}_A \mathcal{U}^* \qquad (\dagger)$$

of complete algebra maps such that

(CHA1) $(\mathcal{U}^*, \Delta_*, \varepsilon_*)$ is a coalgebra in ${}_A\mathrm{Bim}^c_A$;

(CHA2) $\mathcal{S} \circ s = t$, $\mathcal{S} \circ t = s$ and $\mathcal{S}^2 = \mathrm{Id}_{\mathcal{U}^*}$;

(CHA3) $\sum \mathcal{S}(f_1)f_2 = (t \circ \varepsilon_*)(f)$ and $\sum f_1\mathcal{S}(f_2) = (s \circ \varepsilon_*)(f)$.

A complete Hopf algebroid consists of a pair of complete commutative algebras $(A, \mathcal{H})$ together with a diagram of algebra maps $(\dagger)$ that satisfy (CHA1) - (CHA3). The loop map is called the antipode.

Equivalently, a complete Hopf algebroid is a cogroupoid object in the category of complete commutative algebras (see e.g. [De]).

[De] E. S. Devinatz, *Morava's change of rings theorem*. The Čech centennial (Boston, MA, 1993), pp. 83–118, Contemp. Math., **181**, Amer. Math. Soc., Providence, RI, 1995.

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The finite dual

Let $(A,\mathcal{U})$ be a cocommutative Hopf algebroid.

El Kaoutit, Gómez-Torrecillas [EG]: The category $\mathcal{A}_{\mathcal{U}}$ of those right $\mathcal{U}$-modules whose underlying A-module is finitely generated and projective is a symmetric rigid monoidal $\Bbbk$-linear category and the forgetful functor $\omega : \mathcal{A}_{\mathcal{U}} \to \mathfrak{proj}(A)$ is a strict monoidal additive faithful functor. As a consequence, the Tannaka reconstruction process provides us for a commutative Hopf algebroid $(A,\mathcal{U}^{\circ})$ (the finite dual) and a strict monoidal functor $\chi : \mathcal{A}_{\mathcal{U}} \to \mathcal{A}^{\mathcal{U}^{\circ}}$.

$$\text{Namely, } \mathcal{U}^{\circ} := \frac{\bigoplus_{M\in\mathcal{A}_{\mathcal{U}}} M^* \otimes_{T_M} M}{\langle \varphi \otimes_{T_N} f(m) - \varphi \circ f \otimes_{T_M} m \mid \varphi \in N^*, m \in M, f \in T_{M,N} \rangle}$$

where $T_{M,N} = \mathrm{Hom}_{\mathcal{A}_{\mathcal{U}}}(M,N)$ and $T_M = T_{M,M}$. Furthermore, there is a canonical $A \otimes A$-algebra map

$$\zeta : \mathcal{U}^{\circ} \to \mathcal{U}^*,\ \overline{\varphi \otimes_{T_M} m} \mapsto [u \mapsto \varphi(m \cdot u)]$$

whose injectivity implies that χ is an isomorphism.

[EG] L. El Kaoutit, J. Gómez-Torrecillas, *On the finite dual of a cocommutative Hopf algebroid. Application to linear differential matrix equations*. Preprint, arXiv:1607.07633v2 (2016).

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The main morphism of complete Hopf algds

Assume that $(A, \mathcal{U})$ is endowed with an admissible filtration $\{F^n\mathcal{U}\}_{n\geq 0}$.

The commutative Hopf algebroid $(A, \mathcal{U}^\circ)$ can be filtered with the augmentation filtration $G_0(\mathcal{U}^\circ) = \mathcal{U}^\circ$ and $G_n(\mathcal{U}^\circ) = \ker(\varepsilon_\circ)^n$ and its completion $\left(A, \widehat{\mathcal{U}^\circ}\right)$ is a complete Hopf algebroid (A discretely filtered).

Theorem

The canonical map $\zeta : \mathcal{U}^\circ \to \mathcal{U}^*$ is filtered and hence it can be lifted to a morphism $\widehat{\zeta} : \widehat{\mathcal{U}^\circ} \to \mathcal{U}^*$ of complete Hopf algebroids such that

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$$\begin{array}{ccc} \mathcal{U}^\circ & \xrightarrow{\quad\zeta\quad} & \mathcal{U}^* \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle\widehat{\zeta}} & \\ & \widehat{\mathcal{U}^\circ} & \end{array}$$

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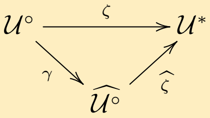

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Idea

If $\mathcal{V}_A(L)^\circ$ is separated and $\widehat{\zeta}$ is an isomorphism, ζ is injective. It follows then that $\widehat{\mathcal{V}_A(L)^\circ}$ can be seen as a formal groupoid which integrates L and that is "canonically" associated with a groupoid whose category of representations is equivalent to the category of modules of L.

Proposition

Let $(A,\mathcal{U})$ be a cocommutative Hopf algebroid with an admissible filtration and assume that $\zeta : \mathcal{U}^\circ \to \mathcal{U}^*$ is injective. TFAE

(a) $\widehat{\zeta} : \widehat{\mathcal{U}^\circ} \to \mathcal{U}^*$ is a filtered isomorphism,

(b) $\widehat{\zeta}$ is surjective and the augmentation filtration on $\mathcal{U}^\circ$ coincides with the induced one,

Moreover, the following assertions are equivalent as well

(c) $\widehat{\zeta} : \widehat{\mathcal{U}^\circ} \to \mathcal{U}^*$ is an homeomorphism,

(d) $\widehat{\zeta} : \widehat{\mathcal{U}^\circ} \to \mathcal{U}^*$ is open and injective and $\mathcal{U}^\circ$ is dense in $\mathcal{U}^*$,

(e) the augmentation topology on $\mathcal{U}^\circ$ is equivalent to the induced one and $\mathcal{U}^\circ$ is dense in $\mathcal{U}^*$.

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If we endow $A \otimes A$ with the $\mathcal{K}$-adic filtration induced by $\mathcal{K} := \ker(m : A \otimes A \to A)$, then the algebra extension $\eta : A \otimes A \to \mathcal{U}^*$ is filtered and we may consider $\widehat{\eta} : \widehat{A \otimes A} \to \mathcal{U}^*$.

If the morphism $\widehat{\eta}$ is a filtered isomorphism, then all the assertions from (a) to (e) are equivalent.

Example: Let A be a commutative $\mathbb{k}$-algebra such that the modules ${}_A\mathcal{J}^k(A) := (A \otimes A)/\mathcal{K}^{k+1}$ of k-jets over A are finitely generated and projective. By Krasil'shchik [K], $\mathrm{Diff}_k(A, A) \cong {}^*\mathcal{J}^k(A)$. Then $\widehat{\eta}$ becomes a filtered isomorphism between $\widehat{A \otimes A} = \mathcal{J}(A)$, the algebra of infinite jets of A, and the dual of $\mathcal{V}_A(\mathrm{Der}(A)) = \mathrm{Diff}(A)$, the algebra of differential operators on A (e.g. $A = \mathbb{C}[X]$).

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Ref. Nestruev [N, §§11.59-11.62].
Let $\mathcal{M}$ be a smooth manifold and $A = \mathcal{C}^\infty(\mathcal{M})$. Let P be a projective A-module, which is the same as the module of sections of a vector bundle $\pi_P : \mathcal{E} \to \mathcal{M}$. Let $\mu_z \subseteq A$ be the maximal ideal of A associated with the point $z \in \mathcal{M}$.
Set $J^l_z P := P/\mu_z^{l+1} P$ with projection $P \twoheadrightarrow J^l_z P : p \mapsto [p]^l_z := p + \mu_z^{l+1} P$. $J^l P := \bigcup_{z\in\mathcal{M}} J^l_z P$ is a vector bundle and it is called the bundle of l-jets of the bundle π_P. Its projection is

$$\pi_{J^l P} : J^l P \to \mathcal{M}, J^l_z P \to z \in \mathcal{M}$$

The module of smooth sections of the vector bundle $\pi_{J^l P}$ is called the module of l-jets of the bundle π_P and it is denoted by $\mathcal{J}^l(P)$. The elements of this module are the l-jets of P.

[N] J. Nestruev, *Smooth manifolds and observables*. Graduate Texts in Mathematics, **220**. Springer-Verlag, New York, 2003.

Take as P the algebra A itself. [N, §9.64]: $J^l_z\mathcal{M} := \mathcal{C}^\infty(\mathcal{M})/\mu_z^{l+1}$ is the vector space of l-jets of smooth functions on $\mathcal{M}$ at $z \in \mathcal{M}$.
$J^l\mathcal{M} := \bigcup_{z\in\mathcal{M}} J^l_z\mathcal{M}$ is the manifold of l-jets and $\pi_{J^l\mathcal{M}} : J^l\mathcal{M} \to \mathcal{M}$ is the bundle of l-jets of smooth functions on $\mathcal{M}$ ([N, 10.11(IX)]).
[N, 9.67]: P, Q two A-modules and set $\mathrm{Diff}_l(P, Q)$ is the A-module of differential operators of order $\leq l$ acting from P to Q.
[N, 11.46]: $\mathcal{J}^l(\mathcal{M})$ is the module of sections of the vector bundle $\pi_{J^l\mathcal{M}}$. This is the module of l-jets of smooth functions on $\mathcal{M}$. With respect to differential operators, they play a role similar to the role of the Kahler module with respect to derivation.
[N, 11.64]: $\mathrm{Diff}_l(P, Q) \cong \mathrm{Hom}_A(\mathcal{J}^l(P), Q)$.

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Not an example

Even when ζ is injective and $A = \Bbbk$, $\widehat{\zeta}$ may not be an isomorphism.

Example (from [ES])

Let $L = \mathbb{C}X$ be the one dimensional (abelian) complex Lie algebra.

- It is trivially a Lie-Rinehart algebra over $\mathbb{C}$
- Its universal enveloping algebra is the Hopf algebra $\mathbb{C}[X]$
- The finite dual of $\mathbb{C}[X]$ coincides with the usual Sweedler dual $\mathbb{C}[X]^\circ$
- The morphism ζ is the inclusion $\mathbb{C}[X]^\circ \subseteq \mathbb{C}[X]^*$
- Let $\xi \in \mathbb{C}[X]^\circ$ be given by $\xi(X^n) = \delta_{n,1}$ (Kronecker delta). Either the augmentation filtration on $\mathbb{C}[X]^\circ$ and the filtration on $\mathbb{C}[X]^*$ are the $\langle \xi \rangle$-adic ones

In this case, it turns out that $\widehat{\zeta}$ is surjective but the $\langle \xi \rangle$-adic filtration on $\mathbb{C}[X]^\circ$ is strictly finer then the one induced by $\mathbb{C}[X]^*$, whence $\widehat{\zeta}$ cannot be a filtered isomorphism (in fact, not even an homeomorphism).

[ES] L. El Kaoutit, P. Saracco, *Comparing Topologies on Linearly Recursive Sequences*. Preprint, arXiv:1705.03433, (2017).

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Thank you