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Tannaka-Kreĭn reconstruction and coquasi-bialgebras with preantipode

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SIC - Lille, 12 October 2018

Report on *Coquasi-bialgebras with Preantipode and Rigid Monoidal Categories* - arXiv:1611.06819

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Any locally compact abelian group G can be recovered from its (group of) one-dimensional unitary representations. Namely, there is a functorial isomorphism $G \cong \widehat{\widehat{G}}$ between G and its double dual.

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Any compact group G can be recovered from its (monoidal) category of finite-dimensional representations.

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Any affine group scheme can be recovered from its category of finite-dimensional representations. In other words, any commutative Hopf algebra can be recovered from its category of finite-dimensional comodules.

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Recall: monoidal categories and functors

Monoidal category: $\mathcal{C}$ endowed with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit $\mathbb{I} \in \mathcal{C}$ and natural isomorphisms

$$\mathfrak{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$
$$\mathfrak{l}_X : \mathbb{I} \otimes X \to X, \quad \mathfrak{r}_X : X \otimes \mathbb{I} \to X$$

that satisfy the *Pentagon* and the *Triangle Axioms*.

Monoidal functor: $\omega : \mathcal{C} \to \mathcal{C}'$ with an iso $\varphi_0 : \mathbb{I}' \to \omega(\mathbb{I})$ and a natural iso $\varphi_{X,Y} : \omega(X) \otimes' \omega(Y) \to \omega(X \otimes Y)$ in $\mathcal{C}'$ such that

$$\omega(\mathfrak{l}_X)\,\varphi_{\mathbb{I},X}(\varphi_0 \otimes' \omega(X)) = \mathfrak{l}'_{\omega(X)}, \quad \omega(\mathfrak{r}_X)\,\varphi_{X,\mathbb{I}}(\omega(X) \otimes' \varphi_0) = \mathfrak{r}'_{\omega(X)}$$
$$\omega(\mathfrak{a}_{X,Y,Z})\varphi_{X\otimes Y,Z}(\varphi_{X,Y} \otimes' \omega(Z)) = \varphi_{X,Y\otimes Z}(\omega(X) \otimes' \varphi_{Y,Z})\mathfrak{a}'_{\omega(X),\omega(Y),\omega(Z)}$$

Comonoids and comodules: In a monoidal category $\mathcal{C}$ a comonoid is $C \in \mathcal{C}$ with $\Delta : C \to C \otimes C$, $\varepsilon : C \to \mathbb{I}$ such that

$$(\Delta \otimes C) \circ \Delta = (C \otimes \Delta) \circ \Delta, \qquad (C \otimes \varepsilon) \circ \Delta = C = (\varepsilon \otimes C) \circ \Delta.$$

A (left) comodule over C is $N \in \mathcal{C}$ with $\delta : N \to C \otimes N$ such that

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$\Bbbk$ a commutative ring. $\mathfrak{M}$ the (monoidal) category of $\Bbbk$-modules.
A $\Bbbk$-(co)algebra is a (co)monoid in $\mathfrak{M}$.

Recall

A (co)algebra is a bialgebra if and only if its category of (co)modules is monoidal and the forgetful functor to $\Bbbk$-modules is a monoidal functor.

That is to say, it is a $\Bbbk$-module B together with $\Bbbk$-linear maps

$$\Bbbk \underset{u}{\overset{\varepsilon}{\rightleftarrows}} B \underset{m}{\overset{\Delta}{\rightleftarrows}} B \otimes B$$

such that (B, Δ, ε) is a coalgebra, (B, m, u) is an algebra and m, u are coalgebra morphisms (equiv m, u are algebra morphisms).

Larson, Sweedler: Structure Theorem for Hopf modules (1967)

A bialgebra B is a Hopf algebra if and only if the free Hopf module functor $-\otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ is an equivalence of categories.

I.e., there exists $S : B \to B$ $\Bbbk$-linear such that

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Reconstruction theorem for Hopf algebras

Ulbrich's Reconstruction Theorem [U] (1990)

Any (essentially small) rigid monoidal category $\mathcal{C}$ together with a monoidal functor $\omega : \mathcal{C} \to \mathfrak{M}_f$ to finitely-generated and projective $\Bbbk$-modules gives rise to a $\Bbbk$-Hopf algebra. In particular, every Hopf algebra over a field can be recovered from its category of finite-dimensional comodules.

Corollary

A coalgebra over a field is a Hopf algebra if and only if its category of finite-dimensional comodules is rigid monoidal with monoidal underlying functor.

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Definition (Majid (1991))

A coalgebra is a coquasi-bialgebra if and only if its category of comodules is monoidal and the forgetful functor to $\Bbbk$-modules is a (neutral) quasi-monoidal functor (i.e. it preserves tensor product, unit and unit constraints but it is not compatible with the associativity constraints).

In particular, it is a $\Bbbk$-module B with $\Bbbk$-linear maps

$$B \otimes B \otimes B \xrightarrow{\omega} \Bbbk \underset{u}{\overset{\varepsilon}{\leftrightarrows}} B \underset{m}{\overset{\Delta}{\rightleftarrows}} B \otimes B$$

such that (B, Δ, ε) is a coalgebra, m, u are coalgebra morphisms, ω is convolution invertible and

$$\omega\,(C \otimes C \otimes m) * \omega\,(m \otimes C \otimes C) = (\varepsilon \otimes \omega) * \omega\,(C \otimes m \otimes C) * (\omega \otimes \varepsilon)\,,$$

$$m\,(C \otimes m) * \omega = \omega * m\,(m \otimes C),$$

where $(f * g)(z) = \sum f(z_1)g(z_2)$.

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Majid's Reconstruction Theorem [M] (1991)

Any (essentially small) monoidal category $\mathcal{C}$ with a *quasi*-monoidal functor $\boldsymbol{\omega} : \mathcal{C} \to \mathfrak{M}_f$ to finitely-generated and projective $\Bbbk$-modules gives rise to a $\Bbbk$-coquasi-bialgebra.

- In the same paper, Majid claims that translating rigidity will provide a "good" candidate for the role of an antipode in the coquasi-case.
- A *coquasi-Hopf algebra* is a coquasi-bialgebra together with an anti-coalgebra endomorphism s and $\alpha, \beta \in H^*$ s.t.

$$\sum h_1 \beta(h_2) s(h_3) = \beta(h) 1, \quad \sum s(h_1) \alpha(h_2) h_3 = \alpha(h) 1,$$
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Majid's reconstruction, coquasi-Hopf algebras

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A preantipode for a coquasi-bialgebra B is a $\Bbbk$-linear endomorphism S s.t.

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Theorem (Structure Theorem for coquasi-Hopf bicomodules)

A coquasi-bialgebra B over a field admits a preantipode iff the free coquasi-Hopf bicomodule functor $- \otimes B : {}^B\mathfrak{M} \to {}^B\mathfrak{M}^B_B$ is an equivalence.

Theorem (Schauenburg [S] (2002))

For a coquasi-bialgebra B over a field, the Structure Theorem holds iff the category ${}^B\mathfrak{M}_f$ of finite-dimensional B-comodules is rigid.

[AP] Ardizzoni, Pavarin, *Preantipodes for Dual Quasi-Bialgebras*, Israel J. Math. **192** (2012).

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Naively, T-K reconstruction means to construct an object H in a suitable category $\mathcal{A}$ once given a functor $\omega : \mathcal{C} \to \mathcal{A}_0$ from a category $\mathcal{C}$ (with some properties) to the subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ of dualizable objects.

Let $\mathcal{C}$ be an essentially small category with a functor $\omega : \mathcal{C} \to \mathfrak{M}_f$.

The coalgebra structure

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Key example: If $\Bbbk$ is a field and $\mathcal{C} = {}^{C}\mathfrak{M}_f$, then $H \cong C$.

Tannaka-Kreĭn reconstruction

Naively, T-K reconstruction means to construct an object H in a suitable category $\mathcal{A}$ once given a functor $\boldsymbol{\omega} : \mathcal{C} \to \mathcal{A}_0$ from a category $\mathcal{C}$ (with some properties) to the subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ of dualizable objects.

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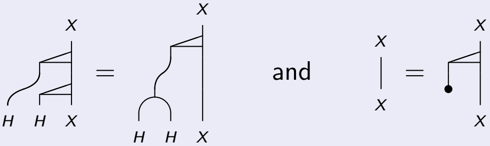

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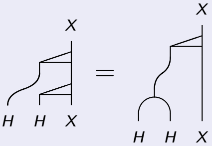

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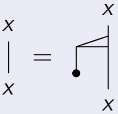

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Monoidal categories and coquasi-bialgebras

Assume that $(\mathcal{C}, \odot, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$ is monoidal and $\boldsymbol{\omega} : \mathcal{C} \to \mathfrak{M}_f$ is a (strict) quasi-monoidal functor, i.e. $X \odot Y = X \otimes Y$ and $\mathbb{I} = \Bbbk$.

The additional (coquasi-bialgebra) structure

The functors $\mathrm{Nat}(\boldsymbol{\omega}^n, - \otimes \boldsymbol{\omega}^n)$ are represented by $H^{\otimes n}$. H becomes a coquasi-bialgebra with multiplication $m =$, unit $u =$ and $\omega : H^3 \to \Bbbk$ uniquely determined by

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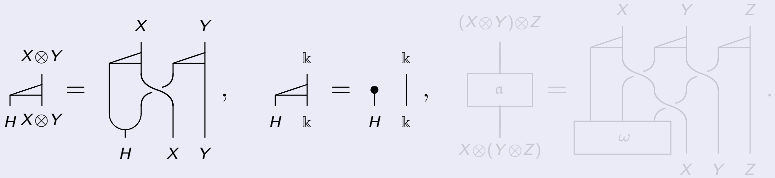

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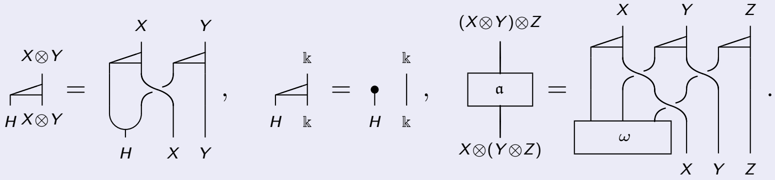

Theorem (Majid [M])

Let $\mathcal{C}$ be an essentially small monoidal category and $\omega : \mathcal{C} \to \mathfrak{M}_f$ a quasi-monoidal functor. There is a coquasi-bialgebra H s.t. ω factorizes through a monoidal functor $\chi : \mathcal{C} \to {}^H\mathfrak{M}$ followed by the forgetful functor

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\chi} & {}^H\mathfrak{M} \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\mathcal{F}} & \\ & \mathfrak{M}_f & \end{array}$$

If H' is another one and $\mathcal{G} : \mathcal{C} \to {}^{H'}\mathfrak{M}$ is a functor as above then there is a unique morphism of coquasi-bialgebras $\epsilon : H \to H'$ s.t.

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Assume that $\mathcal{C}$ is (right) rigid, i.e. for every X there is $(X^\star, \mathrm{ev}_X, \mathrm{db}_X)$ with

$$(\mathrm{ev}_X \otimes X) \circ (X \otimes \mathrm{db}_X) = \mathrm{Id}_X \qquad (X^\star \otimes \mathrm{ev}_X) \circ (\mathrm{db}_X \otimes X^\star) = \mathrm{Id}_{X^\star},$$

and that a choice $(-)^\star$ of dual objects has been performed. Denote by $\omega^\star : \mathcal{C}^{\mathrm{op}} \to \mathfrak{M}$ the functor sending X to $\omega(X^\star)$. Consider the maps

$$\mathrm{ev}_{\omega(X)} := \omega\left(\mathrm{ev}_X\right) =$$

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These induce a natural transformation

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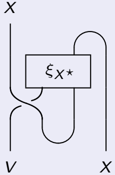
$$\nabla_V(\xi)_X =$$ 

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In light of Yoneda's Lemma

$$\mathrm{Nat}\Big(\mathrm{Nat}(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega}), \mathrm{Nat}(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega})\Big) \cong \mathrm{Nat}(\boldsymbol{\omega}, H \otimes \boldsymbol{\omega}) \cong \mathrm{End}_{\Bbbk}(H),$$

so that there exists a unique linear endomorphism S of H such that

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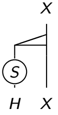

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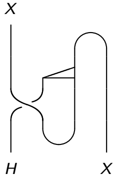

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Lemma

The endomorphism S above is a preantipode for H.

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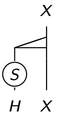

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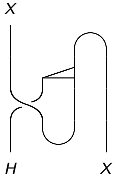

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Theorem (Reconstruction theorem for preantipodes)

Let $\mathcal{C}$ be an essentially small right rigid monoidal category together with a quasi-monoidal functor $\omega : \mathcal{C} \to \mathfrak{M}_f$. Then there exists a preantipode S for the universal coquasi-bialgebra H of $(\mathcal{C}, \omega)$.

For a coquasi-Hopf algebra H, the category ${}^H\mathfrak{M}_f$ is rigid monoidal with quasi-monoidal underlying functor. In fact, $N^\star = \mathrm{Hom}_{\Bbbk}(N, \Bbbk) = N^*$ with

$$\delta_{N^*}(f) = \sum_i s\big((e_i)_{-1}\big) f\big((e_i)_0\big) \otimes e^i,$$

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If we have $\omega(X^\star) \cong \omega(X)^*$ then we say that ω is *preserving duals*.

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Henceforth $\Bbbk$ is a field.

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Preantipodes are unique and coquasi-bialgebra morphisms preserve them.

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Remark

For B a coquasi-bialgebra with preantipode, the category ${}^B\mathfrak{M}_f$ of finite-dimensional B-comodules is a rigid monoidal category and the underlying functor to $\mathfrak{M}_f$ is quasi-monoidal. The dual of an object V is given by $(V^* \otimes B)^{\mathrm{co}B}$.

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A coalgebra C is a coquasi-bialgebra with preantipode if and only if ${}^C\mathfrak{M}_f$ is rigid monoidal and the forgetful functor $\mathcal{F}$ is quasi-monoidal. It is a coquasi-Hopf algebra if and only if in addition $\mathcal{F}$ preserves duals.

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Every coquasi-Hopf algebra H with antipode (s, α, β) admits a preantipode $S := \beta * s * \alpha$. The converse is not true [S].

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Every coquasi-Hopf algebra H with antipode (s, α, β) admits a preantipode $S := \beta * s * \alpha$. The converse is not true [S].

[S] P. Schauenburg, *Hopf algebra extensions and monoidal categories.* New directions in Hopf algebras (2002).

Remark

For B a coquasi-bialgebra with preantipode, the category ${}^B\mathfrak{M}_f$ of finite-dimensional B-comodules is a rigid monoidal category and the underlying functor to $\mathfrak{M}_f$ is quasi-monoidal. The dual of an object V is given by $(V^* \otimes B)^{\mathrm{co}B}$.

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The Sweedler dual of a quasi-bialgebra with preantipode

A quasi-bialgebra A is a bialgebra where Δ is coassociative only up to conjugation by an invertible element $\Phi \in A \otimes A \otimes A$. A preantipode for a quasi-bialgebra is a linear endomorphism S such that

$$\sum a_1 S(ba_2) = \varepsilon(a) S(b) = \sum S(a_1 b) a_2, \qquad \sum \Phi^1 S(\Phi^2) \Phi^3 = 1.$$

Consider $A^\circ = \{f \in A^* \mid \ker(f) \supseteq I \text{ s.t. } \dim_{\Bbbk}(A/I) < \infty\}$.

Proposition

The Sweedler dual A° of a quasi-bialgebra with preantipode A is a coquasi-bialgebra with preantipode.

proof: The category $\mathfrak{M}_A$ is monoidal with quasi-monoidal underlying functor and the full subcategory ${}_f\mathfrak{M}_A$ is rigid with duals given by

$$M^\star := \frac{A \otimes M^*}{A^+(A \otimes M^*)}.$$

${}_f\mathfrak{M}_A \cong {}^{A^\circ}\mathfrak{M}_f$ compatibly with the underlying functors.

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