On Frobenius and Hopf

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Introduction

Hopf algebras are the backbone of the algebraic approach to many questions in geometry, topology, representation theory, mathematical physics, and they are nowadays recognized as the algebraic counterpart of groups.

Frobenius algebras play a prominent role in representation theory, geometry, quantum group theory and, recently, in connection with TQFTs.

It is known that they are intimately related and that this relationship is preserved for some of the many existing extensions of these.

Is there a common wider framework ruling this Hopf-Frobenius connection?

The main characters

(♘) - A $\Bbbk$-bialgebra B is a one-sided Hopf algebra [1] if it admits a one-sided antipode S: for all $b \in B$

$$\text{either} \quad S(b_1)b_2 = \varepsilon(b)1 \quad \text{or} \quad b_1S(b_2) = \varepsilon(b)1.$$

If S is a two-sided antipode, B is a Hopf algebra.

(♖) - A $\Bbbk$-algebra A is a Frobenius algebra if there exists $\psi : A \to \Bbbk$ and $e \in A \otimes A$ such that

$$(\psi \otimes A)(e) = 1 = (A \otimes \psi)(e) \quad \text{and} \quad ae = ea$$

for all $a \in A$. Equivalently, if $A \cong A^*$ as A-modules.

(♗) - A pair of functors

$$\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$$

is a Frobenius pair if $\mathcal{F} \dashv \mathcal{G} \dashv \mathcal{F}$. A functor is Frobenius if it is part of a Frobenius pair.

Previously on Frobenius-Hopf algebras

Morita (1965) - (♖) ⇔ (♗)

A $\Bbbk$-algebra is Frobenius if and only if

$$\mathsf{Forget} : \mathfrak{M}_A \rightleftarrows \mathfrak{M}_\Bbbk : - \otimes A$$

is a Frobenius pair.

Larson-Sweedler (1969) - (♘) ⇒ (♖)

Any finitely generated and projective Hopf $\Bbbk$-algebra over a PID is Frobenius.

Pareigis (1971) - (♘) ⇔ (♖)

A $\Bbbk$-bialgebra B is a fgp Hopf $\Bbbk$-algebra with $\int B^* \cong \Bbbk$ if and only if it is a Frobenius $\Bbbk$-algebra and $\psi \in \int B^*$. Call them FH-algebras.

The original argument

For a $\Bbbk$-bialgebra B we always have an adjunction

$$-\otimes B : \mathfrak{M}_\Bbbk \rightleftarrows \mathfrak{M}_B^B : (-)^{\mathrm{co}B}$$

$$\gamma_V : V \xrightarrow{\cong} (V \otimes B)^{\mathrm{co}B}$$

$$\vartheta_M : M^{\mathrm{co}B} \otimes B \longrightarrow M$$

where $M^{\mathrm{co}B} = \{m \in M \mid \delta(m) = m \otimes 1\}$.

Structure theorem for Hopf modules

B is a Hopf algebra iff ϑ is a natural isomorphism.

When B is finitely generated and projective, we have $B^* \in \mathfrak{M}_B^B$ and

$$\vartheta_{B^*} : \int B^* \otimes B \cong B^* \quad \Rightarrow \quad B_B \cong B_B^*.$$

Recall: $\lambda \in \int B^* \;\Leftrightarrow\; \lambda * f = \lambda f(1) \quad (\forall\, f \in B^*)$
$\Leftrightarrow\; \lambda(a_1)a_2 = \lambda(a)1 \quad (\forall a \in B)$.

A truth revealed

In fact, we always have an adjoint triple

$$-\otimes_B \Bbbk \dashv -\otimes B \dashv (-)^{\mathrm{co}B}, \qquad \mathfrak{M}_B^B \rightleftarrows \mathfrak{M}_\Bbbk$$

and a canonical natural map

$$\sigma_M : M^{\mathrm{co}B} \to M \otimes_B \Bbbk, \quad m \mapsto m \otimes_B 1_\Bbbk.$$

When is $- \otimes B$ Frobenius?

The following are equivalent for a bialgebra B

(α) $- \otimes B$ is Frobenius;

(β) σ is a natural isomorphism;

(γ) $M \cong M^{\mathrm{co}B} \oplus MB^+$ for all $M \in \mathfrak{M}_B^B$.

Example

$B = \Bbbk\left\langle e_{ij}^{(k)} \mid 1 \le i,j \le n, k \ge 0\right\rangle / I$ where I is

$$\left\langle \sum_h e_{hi}^{(k+1)} e_{hj}^{(k)} - \delta_{ij}, \sum_h e_{ih}^{(l)} e_{jh}^{(l+1)} - \delta_{ij} \;\middle|\; \begin{matrix} k \ge 1 \\ l \ge 0 \end{matrix} \right\rangle$$

and with $s\left(e_{ij}^{(k)}\right) = e_{ji}^{(k+1)}$ is a right Hopf algebra.

The main results [2]

A new structure theorem - (♘) ⇔ (♗)

The following are equivalent for a bialgebra B

1. B is a right Hopf algebra with anti-(co)multiplicative right antipode S;
2. σ is a natural isomorphism;
3. $\sigma_{B \otimes B}$ is invertible.

In such a case, for all $a \in B$

$$S(a) := (B \otimes \varepsilon)\left(\sigma_{B \otimes B}^{-1}\left((1 \otimes a) \otimes_B 1_\Bbbk\right)\right).$$

How far from Hopf?

B is a Hopf $\Bbbk$-algebra if and only if $\sigma_{B \otimes B}$ is invertible and $\vartheta_{B \otimes B}$ is surjective.

Larson-Sweedler (1969)

If B is finitely generated and projective, then B is Hopf if and only if it is one-sided Hopf.

Pareigis' theorem for Frobenius functors

The following are equivalent for a finitely generated and projective $\Bbbk$-bialgebra B

1. $- \otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ is Frobenius and $\int B^* \cong \Bbbk$.
2. B is a Hopf algebra with $\int B^* \cong \Bbbk$.
3. B is a FH-algebra.
4. $- \otimes B : \mathfrak{M}^B \to \mathfrak{M}_B^B$ is Frobenius and $\mathsf{Hom}^B(U_B(M), V^u) \cong \mathsf{Hom}(M^{\mathrm{co}B}, V)$.
5. $- \otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ is Frobenius and $\int B \cong \Bbbk$.
6. B^* is a Hopf algebra with $\int B^{**} \cong \Bbbk$.
7. B^* is a FH-algebra.
8. $- \otimes B : \mathfrak{M}_B \to \mathfrak{M}_B^B$ is Frobenius and $\mathsf{Hom}_B(V_\varepsilon, U^B(M)) \cong \mathsf{Hom}(V, M \otimes_B \Bbbk)$.

Further questions

The functor $- \otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$ is part of an adjoint triple as well with right adjoint ${}_B\mathsf{Hom}_B^B(B \otimes B, -)$. When is $- \otimes B$ Frobenius?

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References

[1] J. A. Green, W. D. Nichols, and E. J. Taft. Left Hopf algebras. *J. Algebra*, 1980.

[2] P. Saracco. Hopf modules, Frobenius functors and (one-sided) Hopf algebras. *arXiv:1904.13065*, 2019.