More on Frobenius and Hopf

Paolo Saracco - Université Libre de Bruxelles

Introduction

Hopf algebras are the backbone of the algebraic approach to many questions in geometry, topology, representation theory, mathematical physics, and they are nowadays recognized as the algebraic counterpart of groups.

Frobenius algebras play a prominent role in representation theory, geometry, quantum group theory and, recently, in connection with TQFTs.

It is known that they are intimately related and that this relationship is preserved for some of the many existing extensions of these.

Is there a common wider framework ruling this Hopf-Frobenius connection?

The main characters

(♘) - A $\Bbbk$-bialgebra B is a one-sided Hopf algebra [1] if it admits a one-sided antipode S: $\forall\, b \in B$

$$\text{either} \quad S(b_1)b_2 = \varepsilon(b)1 \quad \text{or} \quad b_1 S(b_2) = \varepsilon(b)1.$$

If S is a two-sided antipode, B is a Hopf algebra.

(♖) - A $\Bbbk$-algebra A is a Frobenius algebra if there exists $\psi : A \to \Bbbk$ and $e \in A \otimes A$ such that

$$(\psi \otimes A)(e) = 1 = (A \otimes \psi)(e) \quad \text{and} \quad ae = ea$$

$\forall\, a \in A$. Equivalently, if $A \cong A^*$ as A-modules.

(♗) - A pair of functors

$$\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$$

is a Frobenius pair if $\mathcal{F} \dashv \mathcal{G} \dashv \mathcal{F}$. A functor is Frobenius if it is part of a Frobenius pair.

Previously on Frobenius-Hopf algebras

Morita (1965) - (♖) ⇔ (♗)

A $\Bbbk$-algebra is Frobenius if and only if

$$\mathsf{Forget} : \mathfrak{M}_A \rightleftarrows \mathfrak{M}_\Bbbk : - \otimes A$$

is a Frobenius pair.

Larson-Sweedler (1969) - (♘) ⇒ (♖)

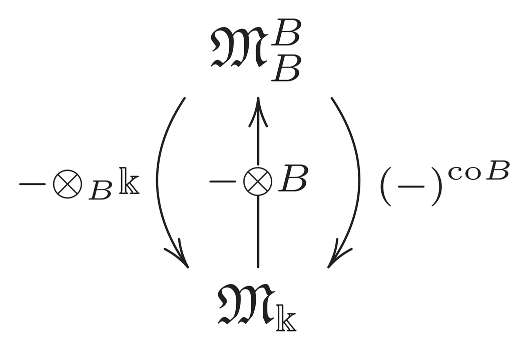
Any finitely generated and projective Hopf $\Bbbk$-algebra over a PID is Frobenius (via the Structure Theorem of Hopf Modules).

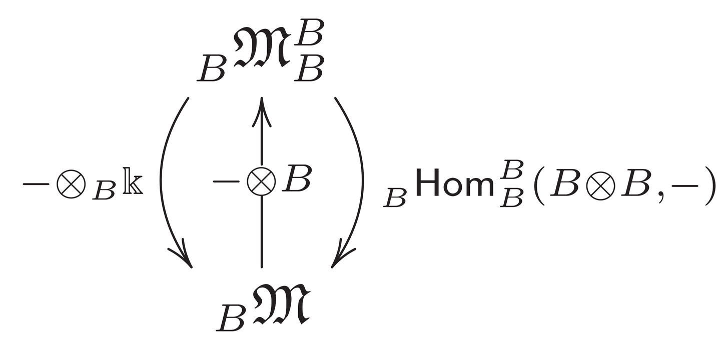
Pareigis (1971) - (♘) ⇔ (♖)

A $\Bbbk$-bialgebra B is a fgp Hopf $\Bbbk$-algebra with $\int B^* \cong \Bbbk$ if and only if it is a Frobenius $\Bbbk$-algebra and $\psi \in \int B^*$. Call them FH-algebras.

Reconsidering the Structure Theorem - (♘) ⇔ (♗)

For a bialgebra B we always have adjoint triples





and canonical natural transformations

$$\sigma_M : M^{\mathrm{co}B} \longrightarrow M \otimes_B \Bbbk, \qquad m \longmapsto m \otimes_B 1_\Bbbk.$$

$$\varsigma_M : {}_B\mathsf{Hom}_B^B(B \otimes B, M) \longrightarrow M \otimes_B \Bbbk, \qquad f \longmapsto f(1_B \otimes 1_B) \otimes_B 1_\Bbbk.$$

The following are equivalent:

(α) $- \otimes B$ is Frobenius;

(β) σ is a natural isomorphism;

(γ) $M \cong M^{\mathrm{co}B} \oplus MB^+$ for all $M \in \mathfrak{M}_B^B$;

(δ) B is a right Hopf algebra with anti-(co)multiplicative right antipode S.

(α') $- \otimes B$ is Frobenius;

(β') ς is a natural isomorphism;

(γ') $- \otimes B$ is a monoidal equivalence;

(δ') B is a Hopf algebra.

Some consequences

Pareigis' theorem for Frobenius functors [3]

The following are equivalent for a finitely generated and projective $\Bbbk$-bialgebra B

1. $- \otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ is Frobenius and $\int B^* \cong \Bbbk$.
2. B is a Hopf algebra with $\int B^* \cong \Bbbk$.
3. B is a FH-algebra.
4. $- \otimes B : \mathfrak{M}^B \to \mathfrak{M}_B^B$ is Frobenius and $\mathsf{Hom}^B(U_B(M), V^u) \cong \mathsf{Hom}(M^{\mathrm{co}B}, V)$.
5. $- \otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ is Frobenius and $\int B \cong \Bbbk$.
6. B^* is a Hopf algebra with $\int B^{**} \cong \Bbbk$.
7. B^* is a FH-algebra.
8. $- \otimes B : \mathfrak{M}_B \to \mathfrak{M}_B^B$ is Frobenius and $\mathsf{Hom}_B(V_\varepsilon, U^B(M)) \cong \mathsf{Hom}(V, M \otimes_B \Bbbk)$.

Hopf and Frobenius monads [2]

The following are equivalent for a $\Bbbk$-bialgebra B

1. B is a Hopf algebra;
2. $- \otimes_B \Bbbk \otimes B$ is a Frobenius monad on ${}_B\mathfrak{M}_B^B$;
3. $- \otimes_B \Bbbk \otimes B$ is a Hopf monad on ${}_B\mathfrak{M}_B^B$.

Frobenius functors and unimodularity [2]

The following are equivalent for a $\Bbbk$-bialgebra B

1. B is a fgp unimodular Hopf algebra and $\int B \cong \Bbbk$;
2. $- \otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$ is Frobenius, B is fgp and unimodular and $\int B \cong \Bbbk$;
3. $- \otimes B : {}_B\mathfrak{M}_B \to {}_B\mathfrak{M}_B^B$ is Frobenius and ${}_B\mathsf{Hom}_B(V_\varepsilon, U(M)) \cong {}_B\mathsf{Hom}(V, M \otimes_B \Bbbk)$.

References

[1] J. A. Green, W. D. Nichols, and E. J. Taft. Left Hopf algebras. *J. Algebra*, 1980.

[2] P. Saracco. Antipodes, preantipodes and Frobenius functors. *arXiv:1906.03435*, 2019.

[3] P. Saracco. Hopf modules, Frobenius functors and (one-sided) Hopf algebras. *arXiv:1904.13065*, 2019.

Further Questions

Could it be that Hopf monad and Frobenius monads are related as Hopf algebras and Frobenius algebras are?

Contact Information

Web sites.google.com/view/paolo-saracco

Email paolo.saracco@ulb.ac.be

Address Département de Mathématique, ULB, Bd du Triomphe, B-1050 Brussels, Belgium.