ULB UNIVERSITÉ LIBRE DE BRUXELLES

# A gentle introduction of the Connes-Moscovici's bialgebroid and its universal properties


Paolo Saracco

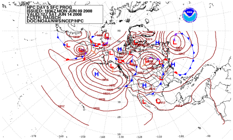
ULB - Université Libre de Bruxelles

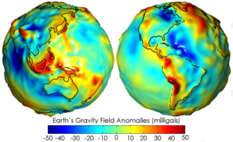
Fonds de la Recherche Scientifique - FNRS


Welcome Home 2020

21/12/2020

| **Geometry** | **Algebra** |
|---|---|
| Smooth manifolds $M$ | Smooth functions on the manifold $\mathcal{C}(M) = \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$ |



| Geometry | Algebra |
|---|---|
| Tangent space to $M$ at a point $p$: $T_pM$ | $p$-derivations on $\mathcal{C}(M)$ |

$X_p : \mathcal{C}(M) \to \mathbb{R}$ such that
$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$
for all $f, g \in \mathcal{C}(M)$

# From differential geometry to commutative algebra





| **Geometry** | | **Algebra** |
|---|---|---|
| $TM = \bigcup_{p \in M} T_p M$ | Tangent bundle | Derivations of $\mathcal{C}(M)$ $\mathrm{Der}(\mathcal{C}(M))$ |
| $TM \xrightarrow{\pi} M$, $X: M \dashrightarrow TM$ | Vector fields $\mathfrak{X}(M)$ | $X : \mathcal{C}(M) \to \mathcal{C}(M)$ such that $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$ for all $f, g \in \mathcal{C}(M)$ |
|  | | $[X, Y] := X \circ Y - Y \circ X$ is in $\mathrm{Der}(\mathcal{C}(M))$ |

$$[X, X] = 0 \quad \text{and} \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

# Lie algebras





## Definition

A Lie algebra is a vector space $L$ together with a bilinear map $[\text{-},\text{-}] : L \times L \to L$ such that $[X, X] = 0$ and

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad (\textit{Jacobi identity})$$

for all $X, Y, Z \in L$.

## Facts

- $\mathfrak{X}(M)$ is a Lie algebra with

$$[X, Y]_p(f) = X_p(Y(F)) - Y_p(X(f)).$$

- For any algebra $A$, $\mathrm{Der}(A)$ is a Lie algebra with

$$[X, Y] = X \circ Y - Y \circ X.$$

- Any algebra $A$ with $[a, b] = ab - ba$ is a Lie algebra.

# Universal enveloping algebras



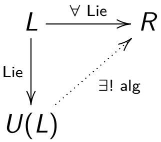
**Definition**

A universal enveloping algebra for $L$ is

- an associative algebra $U(L)$ together with
- a morphism of Lie algebras $L \xrightarrow{\iota} U(L)$

which is universal with respect to this property.

For every $\Bbbk$-algebra $R$ and every Lie algebra morphism $L \to R$,




**Remark**

$(U(L), \iota)$ is a universal arrow from $L$ to the functor $\mathcal{L} : \mathsf{Alg}_{\Bbbk} \to \mathsf{Lie}_{\Bbbk}$.

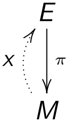
## Facts

- There is a one-to-one correspondence (isomorphism of categories) between representations of $L$ and modules over $U(L)$.
- $$U(L) \xrightarrow{\Delta} U(L) \otimes U(L) \qquad U(L) \xrightarrow{\varepsilon} \Bbbk$$
  $$X \longmapsto X \otimes 1 + 1 \otimes X \qquad X \longmapsto 0$$
  make of $U(L)$ a bialgebra.
- For $B$ a bialgebra,
  $$P(B) := \{b \in B \mid \Delta(b) = b \otimes 1 + 1 \otimes b\}$$
  is a Lie algebra. $(U(L), \iota)$ is also a universal arrow from $L$ to the functor $P : \mathrm{Bialg}_\Bbbk \to \mathrm{Lie}_\Bbbk$.

# From differential geometry to commutative algebra



| **Geometry** | **Algebra** |
|---|---|
| Vector bundles and global sections | Finitely generated and projective $\mathcal{C}(M)$-modules |




$E$ smooth

$\pi$ smooth

$\pi^{-1}(\{p\}) \cong V$

$\pi^{-1}(U) \cong U \times V$

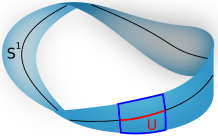
($V$ fixed f.d. vector space)

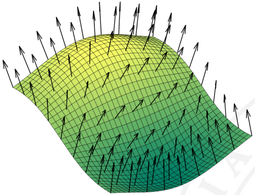
$\Gamma(E)$ = global sections

$f \in \mathcal{C}(M)$, $X \in \Gamma(E)$

$(f \cdot X)(p) := f(p)X_p$

for all $p \in M$



| Geometry | Algebra |
|---|---|
| Lie algebroids | Lie-Rinehart algebras |
| vector bundle $A \xrightarrow{\varpi} M$ | $\mathcal{C}(M)$-module $L$ |
| vector bundle morphism $A \xrightarrow{a} TM$ | morphism of $\mathcal{C}(M)$-modules $L \xrightarrow{\omega} \mathrm{Der}\big(\mathcal{C}(M)\big)$ |
| $[\text{-},\text{-}] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ | $[\text{-},\text{-}] : L \times L \to L$ |
| $\Gamma(A)$ Lie algebra<br>$a([X,Y]) = \big[a(X), a(Y)\big]$<br>$\big[X, fY\big] = f\big[X,Y\big] + a(X)(f)Y$ | $L$ Lie algebra<br>$\omega$ of Lie algebras<br>$\big[X, fY\big] = f\big[X,Y\big] + X(f)Y$ |

# Universal enveloping A-rings





## Definition

A universal enveloping $A$-ring for a Lie-Rinehart algebra $L$ is

- an associative $A$-ring $U$ with unit $A \xrightarrow{u} U$ together with
- a morphism of Lie algebras $L \xrightarrow{\iota} U$ satisfying

$$u(a)\iota(X) = \iota(aX) \quad \text{and} \quad \iota(X)u(a) - u(a)\iota(X) = u(X(a)) \quad \forall\, a \in A, X \in L$$

and which is universal with respect to these properties.

# Universal enveloping A-rings





## Facts

- **Rinehart**: a UE $A$-ring for $L$ exists - $\mathcal{U}(L)$.
- There is a one-to-one correspondence (isomorphism of categories) between representations of $L$ and modules over $\mathcal{U}(L)$.
- $$\begin{array}{ccc} \mathcal{U}(L) \xrightarrow{\Delta} \mathcal{U}(L) \otimes_A \mathcal{U}(L) & \qquad & \mathcal{U}(L) \xrightarrow{\varepsilon} A \\ X \longmapsto X \otimes_A 1 + 1 \otimes_A X & \qquad & X \longmapsto 0 \end{array}$$
  make of $\mathcal{U}(L)$ a cocommutative $A$-bialgebroid.
- For $\mathcal{B}$ a cocommutative $A$-bialgebroid,
  $$\mathcal{P}(\mathcal{B}) := \{b \in \mathcal{B} \mid \Delta(b) = b \otimes_A 1 + 1 \otimes_A b\}.$$
  is a Lie-Rinehart algebra. $(\mathcal{U}(L), \iota)$ is a universal arrow from $L$ to the functor $\mathcal{P} : \mathsf{CCBialgd}_A \to \mathsf{LieRin}_A$.

# Anchored Lie algebras





Fix: $\Bbbk$ field of $\mathrm{char}(\Bbbk) = 0$ $A$ a non-comm $\Bbbk$-algebra

## Definition

An $A$-anchored Lie algebra is a Lie algebra $L$ over $\Bbbk$ together with a morphism of Lie algebras $L \xrightarrow{\omega} \mathrm{Der}(A)$.

## Definition

A universal enveloping $A^{\mathrm{e}}$-ring for an $A$-anchored Lie algebra is

- an associative $A^{\mathrm{e}}$-ring $U$ with unit $\eta : A^{\mathrm{e}} \to U$ together with
- a morphism of Lie algebras $L \xrightarrow{\jmath} U$ satisfying

$$\left[\jmath(X), \eta(a \otimes b)\right] = \eta\big(X \cdot (a \otimes b)\big) \qquad \text{for all } a, b \in A, X \in L$$

and which is universal with respect to these properties.

## Facts

- $A$ is a representation of $L$ $\rightsquigarrow$ $A$ is a $U(L)$-module algebra.
- $$\Big(A\otimes U(L)\otimes A\Big)\otimes\Big(A\otimes U(L)\otimes A\Big) \to \Big(A\otimes U(L)\otimes A\Big)$$
$$(a\otimes u\otimes b)\otimes(a'\otimes u'\otimes b') \mapsto \sum a(u_1\cdot a')\otimes u_2u'\otimes(u_3\cdot b')b$$

$$\Delta:\Big(A\otimes U(L)\otimes A\Big) \to \Big(A\otimes U(L)\otimes A\Big)\otimes_A\Big(A\otimes U(L)\otimes A\Big)$$
$$(a\otimes u\otimes b) \mapsto \sum(a\otimes u_1\otimes 1)\otimes_A(1\otimes u_2\otimes b)$$

$$\varepsilon:\Big(A\otimes U(L)\otimes A\Big)\to A,\qquad (a\otimes u\otimes b)\mapsto a\varepsilon(u)b$$

makes of $A\otimes U(L)\otimes A$ an $A$-bialgebroid $A\odot U(L)\odot A$.

**Theorem (S. '20)**

The Connes-Moscovici's bialgebroid $A \odot U(L) \odot A$ is the universal enveloping $A^{e}$-ring of the $A$-anchored Lie algebra $L$.

**Remark**

$\mathcal{P}(\mathcal{B}) := \{b \in \mathcal{B} \mid \Delta(b) = b \otimes_A 1 + 1 \otimes_A b\}$ is an $A$-anchored Lie algebra.

**Theorem (S. '20)**

$(A \odot U(L) \odot A, \jmath)$ is a universal arrow from $L$ to the functor $\mathcal{P} : \mathrm{Bialgd}_A \to \mathrm{AnchLie}_A$.

ULB

UNIVERSITÉ LIBRE DE BRUXELLES

*Many thanks*

ULB

UNIVERSITÉ LIBRE DE BRUXELLES

fnrs LA LIBERTÉ DE CHERCHER

**Earth** By NASA/Apollo 17 crew; taken by either Harrison Schmitt or Ron Evans - https://web.archive.org/web/20160112123725/http://grin.hq.nasa.gov/ABSTRACTS/GPN-2000-001138.html; see also https://www.nasa.gov/multimedia/imagegallery/image_feature_329.html, Public Domain, https://commons.wikimedia.org/w/index.php?curid=43894484

**Preassure** By Hydrometeorological Prediction Center - http://www.wpc.ncep.noaa.gov/medr/day5nav.html, Public Domain, https://commons.wikimedia.org/w/index.php?curid=37193836

**Gravity** Public Domain, https://commons.wikimedia.org/w/index.php?curid=479365

**Winds** Greg Turk and David Banks. 1996. Image-guided streamline placement. In Proceedings of the 23rd annual conference on Computer graphics and interactive techniques (SIGGRAPH '96). Association for Computing Machinery, New York, NY, USA, 453–460.

**Moebius** By IkamusumeFan - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=41691537

**Normal bundle** By Nicoguaro - Own work, CC BY 4.0, https://commons.wikimedia.org/w/index.php?curid=46926592