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Globalization for geometric partial comodules

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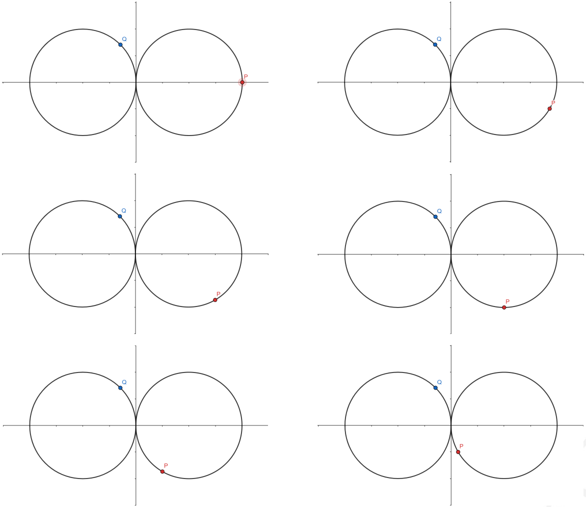
Outline

1. Partial actions of groups
2. Geometric partial comodules
3. Globalization of geometric partial comodules
4. Applications

Partial actions of groups



Partial actions of groups

Definition [Exel, 1998]

A partial action of a group G on a set X is a collection $\{X_g, \alpha_g \mid g \in G\}$ of subsets X_g of X and bijections $\alpha_g : X_{g^{-1}} \to X_g$ such that

- $X_e = X$ and $\alpha_e = \mathrm{id}_X$
- $\alpha_g^{-1}(X_g \cap X_{h^{-1}}) = X_{g^{-1}} \cap X_{(hg)^{-1}}$ for all $g, h \in G$
- $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $X_{g^{-1}} \cap X_{(hg)^{-1}}$ for all $g, h \in G$

Example

For $G = S^1$ and $X = \{\text{pair of tangent circumferences}\}$ take as $X_g = \{\text{right-hand side circumference}\}$ and as α_g the rotation by g clockwise around its center, for $g \neq e$.

Restrictions of global actions

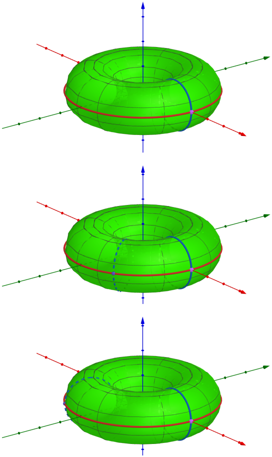
Definition

Let $\beta : G \times Y \to Y$ be a global action of G on Y and let $X \subseteq Y$. Set

- $X_{g^{-1}} := \beta_{g^{-1}}(X) \cap X$ and
- α_g given by restriction of β_g for all $g \in G$.

Fact: The collection $\{X_g, \alpha_g\}$ is a partial action of G on X, called the induced partial action.

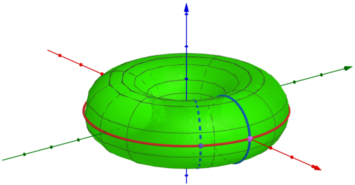
Restrictions of global actions

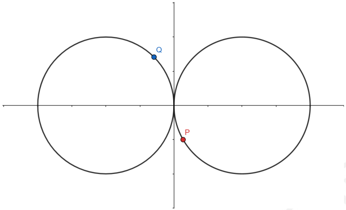


Restrictions of global actions

Example

Take $X = \{$blue circumference$\} \cup \{$red circumference$\}$ on the torus and the induced action of $G = S^1$ by clockwise rotation around the z-axis.





Globalization of partial actions of groups

Definition

A globalization for $\{X_g, \alpha_g \mid g \in G\}$ is a G-set Y with an injection $\epsilon : X \rightarrowtail Y$ st the partial action on X is induced by the global one on Y and Y is universal (initial) among the G-sets satisfying this property.

Theorem [Abadie, 2003]

Every partial action of a group G on a set X admits a globalization (unique up to iso) which can be realized as

$$G \times X / \sim$$

where $(g, x) \sim (h, y)$ iff $x \in X_{h^{-1}g}$ and $y = \alpha_{g^{-1}h}(x)$.

Example

The torus is the globalization of the partial action of S^1 on the tangent circumferences.

Round up of partial (co)actions

- Partial actions of groups on sets
- Partial actions of (topological) groups on topological spaces
- Partial actions of (C^*-quantum) groups on C^*-algebras
- Partial representations of groups in algebras
- ...

- Partial modules over Hopf algebras
- Partial comodules over Hopf algebras
- Partial comodule algebras over Hopf algebras
- Partial representations of Hopf algebras in algebras
- Partial actions of Hopf algebras on $\Bbbk$-linear categories
- Partial actions of multiplier Hopf algebras
- Partial actions of groupoids on rings
- ...

Outline

Setting the stage

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal cat with pushouts and (H, Δ, ε) a coalgebra.

Monoidal category

A category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{I}$ such that

$$(X \otimes Y) \otimes Z \cong Z \otimes (Y \otimes Z), \qquad \mathbb{I} \otimes X \cong X \cong X \otimes \mathbb{I}$$

and the pentagon and triangle axioms hold.

Coalgebra

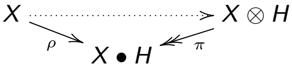
In a monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ a coalgebra is an object H together with

$$\Delta : H \to H \otimes H \qquad \text{and} \qquad \varepsilon : H \to \mathbb{I} \qquad \text{such that}$$

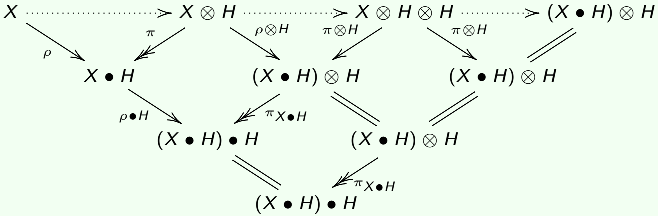
$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes H} \\ H \otimes H & \xrightarrow[H \otimes \Delta]{} & H \otimes H \otimes H \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} H & \xleftarrow{\varepsilon \otimes H} & H \otimes H & \xrightarrow{H \otimes \varepsilon} & H \\ & {\scriptstyle =}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle =} & \\ & & H & & \end{array}$$

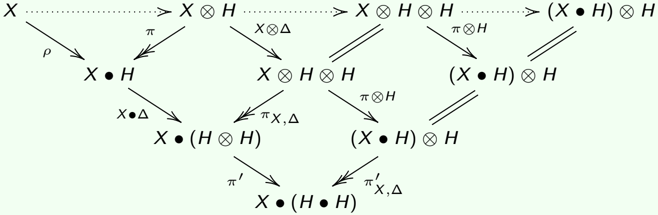
Partial comodule data

A partial comodule datum (pcd_H) is a cospan



Any partial comodule datum induces canonically the following pushouts





Geometric partial comodules

Definition [Hu-Vercruysse, 2018]

A geometric partial comodule (gpc_H) is a pcd_H $(X, X \bullet H, \pi, \rho)$ st

- there exists an isomorphism $\theta : X \bullet (H \bullet H) \to (X \bullet H) \bullet H$ such that $\theta \circ \pi'_{X,\Delta} = \pi_{X \bullet H}$ and the following diagram commutes

$$\begin{array}{ccccc} X & \xrightarrow{\rho} & X \bullet H & \xrightarrow{\rho \bullet H} & (X \bullet H) \bullet H \\ \downarrow{\scriptstyle \rho} & & & & \uparrow{\scriptstyle \theta} \\ X \bullet H & \xrightarrow[X \bullet \Delta]{} & X \bullet (H \otimes H) & \xrightarrow[\pi']{} & X \bullet (H \bullet H). \end{array}$$

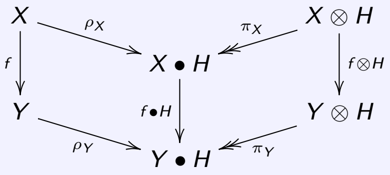
- there exists $X \bullet \varepsilon : X \bullet H \to X$ st the following diagram commutes

$$\begin{array}{ccccc} X & \xrightarrow{\rho} & X \bullet H & \xleftarrow{\pi} & X \otimes H \\ & {\scriptstyle \mathrm{Id}_X} \searrow & \downarrow{\scriptstyle X \bullet \varepsilon} & \swarrow {\scriptstyle X \otimes \varepsilon} & \\ & & X & & \end{array}$$

The category of geometric partial comodules

Definition

If $(X, X \bullet H, \pi_X, \rho_X)$ and $(Y, Y \bullet H, \pi_Y, \rho_Y)$ are gpcs$_H$, then a **morphism of geometric partial comodules** is a pair $(f, f \bullet H)$ of morphisms in $\mathcal{C}$ st



commutes. We denote by gPCom^H the category of geometric partial comodules over H and their morphisms.

Examples of geometric partial comodules

Example

Let G be a group and $\{X_g, \alpha_g\}$ a partial action of G on X. Set $G \bullet X := \{(g, x) \in G \times X \mid x \in X_{g^{-1}}\}$. Then

$$G \times X \longleftarrow G \bullet X \xrightarrow{\ \alpha\ } X$$

is a geometric partial G-comodule in $\mathsf{Set}^{\mathsf{op}}$.

- Partial actions of groups/monoids on sets ($\mathcal{C} = \mathsf{Set}^{\mathsf{op}}$)
- Partial actions of (topological) groups/monoids on topological spaces ($\mathcal{C} = \mathsf{Top}^{\mathsf{op}}$)
- Partial modules over Hopf algebras ($\mathcal{C} = \mathsf{Vect}_{\Bbbk}^{\mathsf{op}}$)
- Partial comodules over Hopf algebras ($\mathcal{C} = \mathsf{Vect}_{\Bbbk}$)
- Partial comodule algebras over Hopf algebras ($\mathcal{C} = \mathsf{Alg}_{\Bbbk}$)
- ...

Induced geometric partial comodules

Definition [Hu-Vercruysse, 2018]

Let (Y, δ) be an H-comodule and $p : Y \to X$ an epi in $\mathcal{C}$. The pushout

$$\begin{array}{ccccc} & & Y & & \\ & \overset{p}{\swarrow} & & \overset{(p\otimes H)\circ\delta}{\searrow} & \\ X & & & & X\otimes H \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\ & & X\bullet H & & \end{array}$$

is a gpc_H and p is a morphism of gpcs_H. We refer to this as the induced partial comodule structure from Y to X.

Example

If (Y, β) is a G-set and $j : X \subseteq Y$ is any subset, the pullback

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\beta\circ(G\times j)}{\nearrow} & & \overset{j}{\nwarrow} & \\ G\times X & & & & X \\ & \underset{\iota}{\nwarrow} & & \underset{\alpha}{\nearrow} & \\ & & G\bullet X & & \end{array}$$

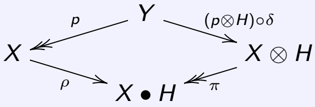
gives the induced partial action of G on X.

Globalization for geometric partial comodules

Definition

Given a gpc_H $(X, X \bullet H, \pi, \rho)$, a globalization for X is an H-comodule (Y, δ) with an epimorphism $p : Y \to X$ in $\mathcal{C}$ such that

- the diagram



  commutes and it is a pushout square in $\mathcal{C}$;
- Y is universal among all comodules admitting a morphism of gpcs_H to X: if (Z, δ') is global and $p' : Z \to X$ is of gpcs_H, then there exists a unique morphism of coms_H $\eta : Z \to Y$ such that $p \circ \eta = p'$.

X is globalizable if a globalization for X exists and we denote by gPCom^H_{gl} the full subcategory of gPCom^H of the globalizable partial comodules.

Are geometric partial comodules globalizable?

$\mathcal{I}(Y,\delta) := (Y, Y \otimes H, id, \delta)$ induces a functor $\mathcal{I} : \mathsf{Com}^H \to \mathsf{gPCom}^H$.

Let $(X, X \bullet H, \pi, \rho)$ be a gpc_H. We have a diagram of coms_H

$$(\dagger) \qquad X \otimes H \underset{(\pi\otimes H)\circ(X\otimes\Delta)}{\overset{\rho\otimes H}{\rightrightarrows}} X \bullet H \otimes H \, .$$

Lemma

For a gpc_H $(X, X \bullet H, \pi, \rho)$ and a com_H (Y,δ), there is a bijective correspondence

$$\mathsf{gPCom}^H(\mathcal{I}Y, X) \cong \{ f \in \mathsf{Com}^H(Y, X \otimes H) \mid f \text{ equalizes } (\dagger) \}$$
$$g \mapsto (g \otimes H) \circ \delta, \qquad (X \otimes \varepsilon) \circ f \leftarrow f.$$

Moreover, this correspondence is natural in both arguments Y and X.

Are geometric partial comodules globalizable?

Theorem [S.-Vercruysse]

Let H be a coalgebra in the monoidal category $\mathcal{C}$. Then a geometric partial comodule $(X, X \bullet H, \pi, \rho)$ is globalizable iff

- the equalizer

$$(Y_X, \delta) \xrightarrow{\kappa} (X \otimes H, X \otimes \Delta) \underset{(\pi\otimes H)\circ(X\otimes\Delta)}{\overset{\rho\otimes H}{\rightrightarrows}} (X \bullet H \otimes H, X \bullet H \otimes \Delta)$$

  exists in Com^H;
- the morphism $\epsilon = (X \otimes \varepsilon) \circ \kappa : Y_X \to X$ is an epimorphism in $\mathcal{C}$;
- the diagram

$$\begin{array}{ccccc} & & Y_X & & \\ & \overset{\epsilon}{\swarrow} & & \overset{\kappa = (\epsilon\otimes H)\circ\delta}{\searrow} & \\ X & & & & X \otimes H \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\ & & X \bullet H & & \end{array}$$

  is a pushout diagram in $\mathcal{C}$.

Under these equivalent conditions Y_X is the globalization of X.

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Outline

Partial actions of monoids and groups

Set $\mathcal{C} = \mathrm{Set}^{\mathrm{op}}$. A gpc_H is a partial action of a monoid H.

Corollary [Megrelishvili-Schröder, 2004]

If X is a partial action of H, then $Y_X = (X \times M)/R$ is the globalization of X, where $R \subseteq (X \times M) \times (X \times M)$ is the equivalence relation generated by

$$\left\{ \left((x \cdot m, n), (x, mn) \right) \mid m, n \in M, x \in X_m \right\}.$$

In particular, we have $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$ for every monoid H.

Y_X is the coequalizer of $X \bullet H \times H \underset{(\pi \times H) \circ (X \times \mu)}{\overset{\rho \times H}{\rightrightarrows}} X \times H$.

Corollary [Abadie, 2003]

For every group G, $\mathrm{gPCom}^G_{gl} = \mathrm{gPCom}^G$.

Partial actions of topological monoids and groups

Set $\mathcal{C} = \mathrm{Top}^{\mathrm{op}}$.

If we endow a group H acting on a set X with the indiscrete topology, we get a global action of a topological group $(H, \tau_H^{\mathfrak{i}})$ on a topological space $(X, \tau_X^{\mathfrak{i}})$. Set $X \bullet H := X \times H$ endowed with the discrete topology $\tau_{X \bullet H}^{\mathfrak{d}}$. Then $\big((X, \tau_X^{\mathfrak{i}}), (X \bullet H, \tau_{X\bullet H}^{\mathfrak{d}}), \mathrm{Id}, \delta\big)$ is a gpc_H in $\mathrm{Top}^{\mathrm{op}}$. However,

$$\begin{array}{ccccc} & & Y_X = (X, \tau_X^{\mathfrak{i}}) & & \\ & \overset{\delta}{\nearrow} & & \overset{\mathrm{Id}}{\nwarrow} & \\ (X \times H, \tau_X^{\mathfrak{i}} \times \tau_G^{\mathfrak{i}}) & & & & (X, \tau_X^{\mathfrak{i}}) \\ & \underset{\mathrm{Id}}{\nwarrow} & & \underset{\delta}{\nearrow} & \\ & & (X \bullet H, \tau_{X\bullet H}^{\mathfrak{d}}) & & \end{array}$$

cannot be a pullback square.

Therefore, in general, $\mathrm{gPCom}_{gl}^{H} \subsetneq \mathrm{gPCom}^{H}$ in $\mathrm{Top}^{\mathrm{op}}$. Nevertheless

Corollary [Abadie, 2003]

For a topological group (H, τ_H), $\mathrm{TopParAct}_H$ is a full subcat of gPCom_{gl}^{H}.

Geometric partial comodules in abelian categories

Let $\mathcal{C}$ be any abelian monoidal category.

Proposition [S.-Vercruysse]

Assume that $(X, X \bullet H, \pi, \rho)$ is a counital pcd_H. Consider the pullback

$$\begin{array}{ccccc} & & T & & \\ & \overset{\varpi}{\swarrow} & & \overset{\lambda}{\searrow} & \\ X & & & & X \otimes H \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\ & & X \bullet H & & \end{array}$$

in $\mathcal{C}$. Then $(X, X \bullet H, \pi, \rho)$ is a gpc_H iff

$$T \xrightarrow{\lambda} X \otimes H \underset{(\pi \otimes H)\circ(X \otimes \Delta)}{\overset{\rho \otimes H}{\rightrightarrows}} X \bullet H \otimes H$$

is an equalizer in $\mathcal{C}$.

Theorem [S.-Vercruysse]

If H is a coalgebra in $\mathcal{C}$ such that Com^H admits equalizers and $\mathrm{Com}^H \to \mathcal{C}$ preserves them, then $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$.

Geometric partial (co)modules in vector spaces

Let $\mathbb{k}$ be a field.

Corollary

- If H is a $\mathbb{k}$-algebra, then gpcs_H can be identified with H-modules together with a chosen generating subspace.
- If H is a $\mathbb{k}$-coalgebra, then gpcs_H can be identified with H-comodules together with a chosen co-generating quotient space.

- Partial modules over Hopf algebras are globalizable and the globalization coincides with their standard dilation [Alves-Batista-Vercruysse]
- Partial representations of finite groups are globalizable [D'Adderio-Hautekiet-S.-Vercruysse]
- Partial comodules over Hopf algebras are globalizable
- Partial comodule algebras over Hopf algebras are globalizable, but their globalization is not their enveloping coaction [Alves-Batista]

Partially graded representations

Let G be a group. $\mathcal{C} = \mathrm{Rep}_G$ is an abelian monoidal category and $\Bbbk G$ is a coalgebra therein. We call partially graded G-representation a partial comodule over $\Bbbk G$ in $\mathcal{C}$.

Partially graded G-representations are all and only of the following form. For a vector space V, consider $\Bbbk G \otimes V$ with regular action and coaction. Pick a $\Bbbk G$-submodule $N \subseteq \Bbbk G \otimes V$ and define $M := (\Bbbk G \otimes V)/N$. Then M with the induced structure

$$\begin{array}{ccccc} & & \Bbbk G\otimes V & & \\ & \overset{p}{\swarrow} & & \overset{(\Bbbk G\otimes p)\circ(\Delta\otimes V)}{\searrow} & \\ M & & & & \Bbbk G\otimes M \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\ & & \Bbbk G\bullet M & & \end{array}$$

is a partially graded G-representation.

In particular, this construction induces a bijective correspondence between structures of partially graded G-representation on the base field $\Bbbk$ (up to isomorphism) and linear characters of the group G.

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Many thanks

Selected bibliography

[1] Abadie, *Enveloping actions and Takai duality for partial actions*. J. Funct. Anal. 197 (2003), no. 1, 14-67.

[2] Alves, Batista, *Globalization theorems for partial Hopf (co)actions, and some of their applications*. Groups, algebras and applications, 13–30, Contemp. Math., 537, Amer. Math. Soc., Providence, RI, 2011.

[3] Alves, Batista, Vercruysse, *Partial representations of Hopf algebras*. J. Algebra 426 (2015), 137-187.

[4] D'Adderio, Hautekiet, Saracco, Vercruysse, *Partial and global representations of finite groups* (2020).

[5] Dokuchaev, *Recent developments around partial actions*. São Paulo J. Math. Sci. 13 (2019), no. 1, 195-247.

[6] Exel, *Partial actions of groups and actions of inverse semigroups*. Proc. Am. Math. Soc. 126, No. 12 (1998), 3481-3494.

[7] Hu, Vercruysse, *Geometrically Partial Actions*. Trans. Amer. Math. Soc. 373 (2020), 4085-4143.

[8] Megrelishvili, Schröder, *Globalization of confluent partial actions on topological and metric spaces*. Topology Appl. 145 (2004), no. 1-3, 119-145.

[9] Saracco, Vercruysse, *Globalization for geometric partial comodules, part I: General theory*. To appear.

[10] Saracco, Vercruysse, *Globalization for geometric partial comodules, part II: Applications*. In preparation.