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Globalization for geometric partial comodules

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Based on an ongoing project with J. Vercruysse (ULB) - arXiv:2001.07669

Outline

1. Partial actions of groups

2. Geometric partial comodules

3. Globalization of geometric partial comodules

4. Applications

Partial actions of groups

Definition [Exel, 1998]

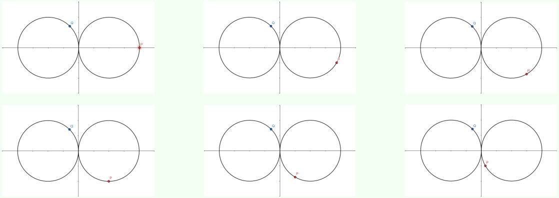
A partial action of a group G on a set X is a partial function

$$G \times X \dashrightarrow X \qquad G \times X \xleftarrow{\;\supseteq\;} G \bullet X \xrightarrow{\;\cdot\;} X$$

s.t.

- $\exists\ 1_G \cdot x$ for every x and $1_G \cdot x = x$;
- if $\exists\ g \cdot x$, then $\exists\ g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;
- if $\exists\ g \cdot (h \cdot x)$, then $\exists\ gh \cdot x$ and $g \cdot (h \cdot x) = gh \cdot x$.

Example



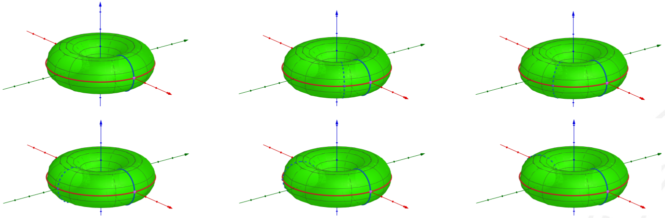
Restrictions of global actions

Definition

Let $\beta : G \times Y \to Y$ be a global action of G on Y and let $X \subseteq Y$.
Define $G \bullet X \dot{\to} X$ by declaring that $\exists\ g \cdot x$ if $\beta(g, x) \in X$.
This gives a partial action of G on X, called the induced partial action.

Example

Take $X = \{$blue circumference$\} \cup \{$red circumference$\}$ on the torus and the induced action of $G = S^1$ by clockwise rotation around the z-axis.



Globalization of partial actions of groups

Definition

A globalization for $G \bullet X \dot{\to} X$ is a G-set Y with an injection $\epsilon : X \rightarrowtail Y$ s.t. the partial action on X is induced by the global one on Y and Y is universal (initial) among the G-sets satisfying this property.

Theorem [Abadie, 2003]

Every partial action of a group G on a set X admits a globalization (unique up to iso) which can be realized as

$$G \times X / \sim$$

where $(g, x) \sim (h, y)$ iff $\exists\ g^{-1}h \cdot x \in X_{h^{-1}g}$ and $y = g^{-1}h \cdot x$.

Example

The torus is the globalization of the partial action of S^1 on the tangent circumferences.

Round up of partial (co)actions

- Partial actions of groups on sets
- Partial actions of (topological) groups on topological spaces
- Partial actions of (C^*-quantum) groups on C^*-algebras
- Partial representations of groups in algebras
- ...

- Partial modules over Hopf algebras
- Partial comodules over Hopf algebras
- Partial comodule algebras over Hopf algebras
- Partial representations of Hopf algebras in algebras
- Partial actions of Hopf algebras on $\mathbb{k}$-linear categories
- Partial actions of multiplier Hopf algebras
- Partial actions of groupoids on rings
- ...

Outline

Setting the stage

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal cat with pushouts and (H, Δ, ε) a coalgebra.

Monoidal category

A category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{I}$ such that

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \qquad \mathbb{I} \otimes X \cong X \cong X \otimes \mathbb{I}$$

and the pentagon and triangle axioms hold.

Coalgebra

An object H in $(\mathcal{C}, \otimes, \mathbb{I})$ together with

$$\Delta : H \to H \otimes H \qquad \text{and} \qquad \varepsilon : H \to \mathbb{I} \qquad \text{such that}$$

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes H} \\ H \otimes H & \xrightarrow[H \otimes \Delta]{} & H \otimes H \otimes H \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} H & \xleftarrow{\varepsilon \otimes H} & H \otimes H & \xrightarrow{H \otimes \varepsilon} & H \\ & {\scriptstyle =}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle =} & \\ & & H & & \end{array}$$

Geometric partial comodules

Definition [Hu-Vercruysse, 2018]

A geometric partial comodule (gpc_H) is a cospan

$$\xymatrix{X \ar@{.>}[rr] \ar[dr]_{\rho} & & X \otimes H \ar[dl]^{\pi} \\ & X \bullet H & }$$

such that

- the following pushouts

$$\xymatrix{X \ar@{.>}[rr] \ar[dr]_{\rho} & & X \otimes H \ar@{.>}[rr] \ar[dl]^{\pi} \ar[dr]_{X \otimes \varepsilon} & & X \\ & X \bullet H \ar[dr]_{X \bullet \varepsilon} & & X \ar@{=}[ur] \ar[dl]^{\pi_\varepsilon} & \\ & & X_\varepsilon & & }
\qquad \text{and} \qquad
\xymatrix{X \ar@{.>}[rr] \ar@{=}[dr] & & X \\ & X \ar@{=}[ur] & }$$

  coincide (up to isomorphism);
- ...

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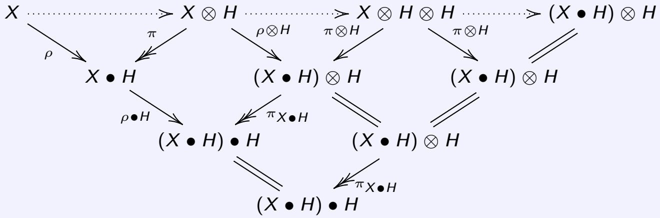
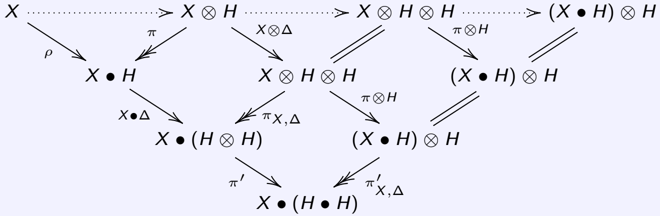
Geometric partial comodules

- the following pushouts





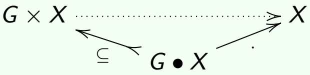
coincide (up to isomorphism).

We denote by gPCom^H the category of geometric partial comodules over H and their morphisms.

Examples of geometric partial comodules

Example

A partial action of a group G on a set X



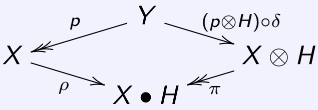
is a geometric partial G-comodule in $\mathrm{Set}^{\mathrm{op}}$.

- Partial actions of groups/monoids on sets ($\mathcal{C} = \mathrm{Set}^{\mathrm{op}}$)
- Partial actions of (topological) groups/monoids on topological spaces ($\mathcal{C} = \mathrm{Top}^{\mathrm{op}}$)
- Partial modules over Hopf algebras ($\mathcal{C} = \mathrm{Vect}_{\Bbbk}^{\mathrm{op}}$)
- Partial comodules over Hopf algebras ($\mathcal{C} = \mathrm{Vect}_{\Bbbk}$)
- Partial comodule algebras over Hopf algebras ($\mathcal{C} = \mathrm{Alg}_{\Bbbk}$)
- ...

Induced geometric partial comodules

Definition [Hu-Vercruysse, 2018]

Let (Y, δ) be an H-comodule and $p : Y \to X$ an epi in $\mathcal{C}$. The pushout



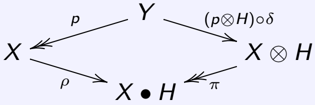
makes of X a gpc_H and of p a morphism of gpcs_H. We refer to this as the induced partial comodule structure from Y to X.

Globalization for geometric partial comodules

Definition

Given a gpc_H $(X, X \bullet H, \pi, \rho)$, a globalization for X is an H-comodule (Y, δ) with an epimorphism $p : Y \to X$ in $\mathcal{C}$ such that

- the diagram



  commutes and it is a pushout square in $\mathcal{C}$;
- Y is universal among all comodules admitting a morphism of gpcs_H to X: if (Z, δ') is global and $p' : Z \to X$ is of gpcs_H, then there exists a unique morphism of coms_H $\eta : Z \to Y$ such that $p \circ \eta = p'$.

X is globalizable if a globalization for X exists and we denote by gPCom^H_{gl} the full subcategory of gPCom^H of the globalizable partial comodules.

When are gpcs globalizable?

Theorem [S.-Vercruysse]

Let H be a coalgebra in the monoidal category $\mathcal{C}$. Then a geometric partial comodule $(X, X \bullet H, \pi, \rho)$ is globalizable iff

- the equalizer

$$(Y_X, \delta) \overset{\kappa}{\longrightarrow} (X \otimes H, X \otimes \Delta) \underset{(\pi\otimes H)\circ(X\otimes\Delta)}{\overset{\rho\otimes H}{\rightrightarrows}} (X \bullet H \otimes H, X \bullet H \otimes \Delta)$$

  exists in Com^H;

- the commuting diagram

$$\begin{array}{ccccc} & & Y_X & & \\ & \overset{\epsilon := (X\otimes\varepsilon)\circ\kappa}{\swarrow} & & \overset{\kappa}{\searrow} & \\ X & & & & X \otimes H \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\ & & X \bullet H & & \end{array}$$

  is a pushout diagram in $\mathcal{C}$.

Under these equivalent conditions Y_X is the globalization of X.

Outline

Partial actions of monoids and groups

Set $\mathcal{C} = \mathrm{Set}^{\mathrm{op}}$. A gpc_H is a partial action of a monoid H.

Corollary [Megrelishvili-Schröder, 2004]

If X is a partial action of H, then $Y_X = (X \times M)/R$ is the globalization of X, where $R \subseteq (X \times M) \times (X \times M)$ is the equivalence relation generated by

$$\left\{ \left((x \cdot m, n), (x, mn) \right) \mid m, n \in M, x \in X_m \right\}.$$

In particular, we have $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$ for every monoid H.

Y_X is the coequalizer of $X \bullet H \times H \underset{(\pi \times H)\circ(X \times \mu)}{\overset{\rho \times H}{\rightrightarrows}} X \times H$.

Corollary [Abadie, 2003]

For every group G, $\mathrm{gPCom}^G_{gl} = \mathrm{gPCom}^G$.

Geometric partial comodules in abelian categories

Let $\mathcal{C}$ be any abelian monoidal category.

Theorem [S.-Vercruysse]

If H is a coalgebra in $\mathcal{C}$ such that Com^H admits equalizers and $\mathrm{Com}^H \to \mathcal{C}$ preserves them, then $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$.

- Partial modules over Hopf algebras are globalizable and the globalization coincides with their standard dilation [Alves-Batista-Vercruysse]
- Partial representations of finite groups are globalizable [D'Adderio-Hautekiet-S.-Vercruysse]
- Partial comodules over Hopf algebras are globalizable
- Partial comodule algebras over Hopf algebras are globalizable, but their globalization is not their enveloping coaction [Alves-Batista]

Many thanks

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