Globalization for Geometric Partial Comodules

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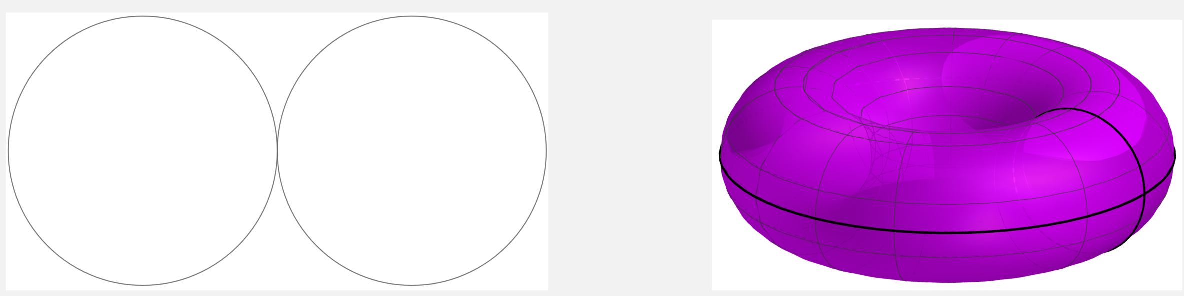
Abstract

The study of partial symmetries (such as partial dynamical systems, partial (co)actions, partial comodule algebras) is a recent field in continuous expansion, whose origins can be traced back to the study of C^*-algebras generated by partial isometries.
One of the central questions in the study of partial symmetries is the existence and uniqueness of a so-called globalization (or enveloping action).
We propose here a unified approach to globalization in a categorical setting and we provide a procedure to construct globalizations in concrete cases of interest.
Our approach relies on the notion of geometric partial comodules.

1. Introduction

In the framework of partial actions of groups, any global action of a group G on a set induces a partial action of the group on any subset by restriction. Globalizing a given partial action means to find a (universal) global G-set such that the initial partial action can be realized as the restriction of this global one.

Example



Example

The Möbius group acts partially on the complex plane. The corresponding globalization is the Riemann sphere.

The importance of globalization is testified by the many results existing in the literature:

1999 Abadie: Globalization for partial actions of (topological) groups on sets, topological spaces, C^*-algebras

2004 Kellendonk-Lawson: Globalization for partial actions of (topological) groups on sets, topological spaces

2004 Megrelishvili-Schröder: Partial actions of (topological) monoids and globalization

2005 Dokuchaev-Exel: Globalization for partial actions of groups on algebras

2010 Alves-Batista: Enveloping (co)actions of partial (co)module algebras

2015 Alves-Batista-Vercruysse: Globalization for partial modules over Hopf algebras

2018 Khrypchenko-Novikov: globalization problem in the universal algebra setting

However, these are always based on some *ad hoc* constructions, depending on the nature of the objects carrying the partial action.

2. Geometric Partial Comodules

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal category with pushouts. Let (H, Δ, ε) be a coalgebra in $\mathcal{C}$.

Definition

A *geometric partial comodule* is a cospan $X \xrightarrow{\rho} X \bullet H \xleftarrow{\pi} X \otimes H$ such that

- the following cospans coincide (up to isomorphism):

$$X \xrightarrow{\rho} X \bullet H \xrightarrow{X\bullet\varepsilon} X_\varepsilon \leftarrow X \quad (\text{pushout of } X\bullet H \xleftarrow{\pi} X\otimes H \xrightarrow{X\otimes\varepsilon} X) \qquad \text{and} \qquad X \rightarrow X \leftarrow X$$

- the following cospans coincide (up to isomorphism):

$$X \xrightarrow{\rho} X \bullet H \xrightarrow{\rho\bullet H} (X\bullet H)\bullet H \leftarrow (X\bullet H)\otimes H \xleftarrow{\pi\otimes H} X\otimes H\otimes H \quad (\text{pushout of } X\bullet H \xleftarrow{\pi} X\otimes H \xrightarrow{\rho\otimes H} (X\bullet H)\otimes H)$$

$$X \xrightarrow{\rho} X \bullet H \xrightarrow{X\bullet\Delta} X\bullet(H\bullet H) \leftarrow (X\bullet H)\otimes H \xleftarrow{\pi\otimes H} X\otimes H\otimes H \quad (\text{pushout of } X\bullet H \xleftarrow{\pi} X\otimes H \xrightarrow{(\pi\otimes H)(X\otimes\Delta)} (X\bullet H)\otimes H)$$

gPCom^H is the category of geometric partial comodules over H and their morphisms.

3. Main Results [6, 7]

Definition

Given a partial comodule X, a *globalization* for X is a global comodule (Y, δ) with a morphism $p : Y \to X$ in $\mathcal{C}$ such that

- $$\begin{array}{ccccc} Y & \xrightarrow{\delta} & Y\otimes H & \xrightarrow{p\otimes H} & X\otimes H \\ \downarrow p & & & & \downarrow \pi \\ X & & \xrightarrow{\rho} & & X\bullet H \end{array}$$ commutes (i.e. p is a morphism of partial comodules) and it is a pushout square in $\mathcal{C}$;
- Y is universal with respect to this property, i.e. there is a bijective correspondence Com$^H(Z, Y) \to$ gPCom$^H(\mathcal{I}(Z), X), \eta \mapsto p \circ \eta$.

We say that X is *globalizable* if a globalization for X exists and we denote by gPCom$^H_{gl}$ the full subcategory of gPCom^H composed by the globalizable partial comodules.

Theorem

Let H be a coalgebra in a monoidal category $\mathcal{C}$ with pushouts. Then a geometric partial H-comodule X is globalizable if and only if

- the equalizer $(Y, \delta) \xrightarrow{\kappa} (X\otimes H, X\otimes\Delta) \underset{(\pi\otimes H)(X\otimes\Delta)}{\overset{\rho\otimes H}{\rightrightarrows}} (X\bullet H\otimes H, X\bullet H\otimes\Delta)$ exists in Com^H.
- $$\begin{array}{ccc} Y & \xrightarrow{\kappa} & X\otimes H \\ \downarrow (X\otimes\varepsilon)\kappa & & \downarrow \pi \\ X & \xrightarrow{\rho} & X\bullet H \end{array}$$ is a pushout diagram in $\mathcal{C}$.

Moreover, if these conditions hold, then the morphism $\epsilon = (X\otimes\varepsilon)\circ\kappa : Y \to X$ is an epimorphism in $\mathcal{C}$, $\kappa = (\epsilon\otimes H)\circ\delta$ and (Y, ϵ) is the globalization of X.

Proposition

Let $\mathcal{C}$ be an abelian monoidal category. If H is a coalgebra in $\mathcal{C}$ which is left flat (i.e., Com$^H \to \mathcal{C}$ preserves equalizers), then gPCom$^H_{gl}$ = gPCom^H.

4. Applications [7, 8]

- Partial actions of monoids are globalizable [5]
- Partial actions of groups are globalizable [1]
- Topological partial actions of topological groups are globalizable [1] (even if arbitrary geometric partial modules over topological monoids are not)
- Partial modules over Hopf algebras are globalizable and the globalization is their standard dilation [3]
- Partial reps of finite groups are globalizable [4]
- Partial comods over Hopf algebras are globalizable [7]
- Partial comodule algebras over Hopf algebras are globalizable [8], but their globalization is not always their enveloping coaction [2]
- Right Hopf partial comodules over a bialgebra B (i.e., geometric partial comodules over B in Mod$_B$) are globalizable [7]
- *Fundamental Theorem for Hopf partial comodules.* Let H be a Hopf algebra. The category gHPCom^H_H is equivalent to the category whose objects are pairs (V, N) composed by a vector space V and an H-submodule N of $V\otimes H$ such that $(V\otimes\Delta(H))\cap(N\otimes H) = 0$, and whose morphisms $(V, N) \to (V', N')$ are given by linear maps $f : V \to V'$ such that $(f\otimes H)(N) \subset N'$.

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