PARTIAL AND GLOBAL REPRESENTATIONS OF FINITE GROUPS

MICHELE D'ADDERIO, WILLIAM HAUTEKIET, PAOLO SARACCO, AND JOOST VERCRUYSSE

Abstract. Given a subgroup H of a finite group G, we begin a systematic study of the partial representations of G that restrict to global representations of H. After adapting several results from [DEP00] (which correspond to the case $H = \{1_G\}$), we develop further an effective theory that allows explicit computations. As a case study, we apply our theory to the symmetric group $\mathfrak{S}_n$ and its subgroup $\mathfrak{S}_{n-1}$ of permutations fixing 1: this provides a natural extension of the classical representation theory of $\mathfrak{S}_n$.

Contents

Key words and phrases. Partial representation; partial action; groupoid algebra; globalization; symmetric group.

Introduction

The notion of a *partial action* of a group arose first in the theory of operator algebras as an approach to C^*-algebras generated by partial isometries, allowing the study of their K-theory, ideal structure and representations [Exe94]. In particular, the point of view of crossed products by partial actions of groups was very successful for classifying C^*-algebras. The related notion of a *partial representation* was introduced in [DE05]. Since then, these notions have been studied and applied in a variety of contexts involving operator algebras, dynamical systems, commutative algebras, noncommutative rings and Hopf algebras, among others. See [Bat17] and [Dok19] for two recent surveys.

Our investigation stemmed from the article [DEP00], in which the authors give a first systematic approach to the theory of partial representations of finite groups. Among the main results in [DEP00], there is a proof of an equivalence between the category of partial representations of a finite group G over a field (say $\mathbb{C}$, for instance, although the actual statement is more general) and the category of (usual) representations of the so called *partial group algebra* $\mathbb{C}_{par}G$, which is a unital associative algebra. In [DEP00] it is also shown that $\mathbb{C}_{par}G$ is isomorphic to the groupoid algebra $\mathbb{C}\Gamma(G)$ of a certain groupoid $\Gamma(G)$ associated to G. The algebra $\mathbb{C}_{par}G \cong \mathbb{C}\Gamma(G)$ is proved to be semisimple and a formula is provided in [DEP00], which describes it as a direct product of matrix algebras of the form $M_m(\mathbb{C}K)$ for K varying among the subgroups of G.

It turns out that an explicit computation of such a formula for a given finite group G (or even, more importantly, a description of its irreducible partial representations) seems to be in general out of reach, the problem being mainly that the number of summands $M_m(\mathbb{C}K)$ grows rapidly with $|G|$. Already in [DEP00] it is shown that even for abelian groups, whose global irreducible representations are fairly easy to describe, the computation of their irreducible partial representations quickly becomes way too involved.

Unsatisfied with this state of affairs, we made the following two related observations, that triggered the present work. First of all, by looking at natural examples of partial representations of a finite group G, it is often the case that these partial representations restrict to global (i.e. usual) representations of nontrivial subgroups H of G. Therefore, to understand for example their decomposition into irreducibles, it would be enough to know the irreducibles among all the partial representations of G that restrict to global representations of H, which we call *H-global G-partial representations*. Observe that for H being the trivial subgroup of G, we recover exactly the notion of a partial representation of G.

The second observation is that, if H is relatively large in G, then the problem of describing the irreducible H-global G-partial representations becomes actually tractable and, in our opinion, quite interesting.

In the present article we initiate a systematic study of H-global G-partial representations of finite groups. Our aim is to build an effective theory that allows explicit computations.

With this goal in mind, we start by adapting most of the results in [DEP00] to our more general situation. In details, we define the analogues of the partial group algebra $\mathbb{C}_{par}G$ and of the groupoid $\Gamma(G)$, that we denote by $\mathbb{C}_{par}^{H}G$ and $\Gamma_H(G)$ respectively. Then, we prove theorems whose statements are similar to the aforementioned ones, showing for example that the category of H-global G-partial representations is equivalent to the category of usual representations of the associative unital algebra $\mathbb{C}_{par}^{H}G \cong \mathbb{C}\Gamma_H(G)$ and that such an algebra is semisimple, by providing a formula that exhibits it as a direct product of algebras of the form $M_m(\mathbb{C}K)$ for certain subgroups K of G (which are the isotropy groups of the vertices of $\Gamma_H(G)$, see Theorem 3.22).

After these natural steps, we go deeper into the theory, by achieving the following results:

- we give an explicit construction of all the irreducible H-global G-partial representations in terms of the irreducible representations of the subgroups K appearing in the aforementioned formula;
- we give a formula for the decomposition into (global) representations of H of the restriction to H of the aforementioned H-global G-partial irreducibles;
- we review the notion of a *globalization* (also called *dilation* in the literature, cf. [Aba18]) of a partial representation, presenting its alternative description which allows us to provide an explicit construction of the globalization of our irreducible H-global G-partial representations;
- we prove the existence of an *induced* partial representation of a (global) representation of H to an H-global G-partial representation, giving a *Frobenius reciprocity*.

To make our case for the study of H-global G-partial representations, we apply all the theory that we developed to the irreducible partial representations of the symmetric group $\mathfrak{S}_n$ which restrict to global representations of the subgroup of the permutations that fix 1, which we identify with $\mathfrak{S}_{n-1}$. Observe that in order to understand all the irreducible partial representations of $\mathfrak{S}_n$, according to the formula in [DEP00], we essentially need to understand the irreducible representations of all the subgroups of $\mathfrak{S}_n$: by a well-known theorem of Cayley, this boils down to understand the irreducible representations of every finite group, which is obviously a hopeless task. On the other hand, the irreducible $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations can be described explicitly and they provide a natural extension of the classical representation theory of $\mathfrak{S}_n$. In this vein, we also prove a *branching rule* in this more general setting.

In order to underline the effectiveness of our results, and to make this work accessible to the less categorically inclined reader, all along the article we privilege explicit constructions, while more categorical discussions usually follow, leaving the rest of the text pretty much self-contained.

Concretely, the paper is organized in the following way. In Sections 1.1 and 1.2 we recall some basic definitions. In Section 1.3 we prove that every partial representation has a globalization. In Section 2.1 we give the basic definitions and examples of H-global G-partial representations. In Section 2.2 we give a general construction that provides a large class of examples of H-global G-partial representations. In Section 2.3 we prove some basic identities about partial representations that will be useful in the remaining sections. In Section 3.1 we prove that the category of H-global G-partial representations is equivalent to the category of left modules over the algebra $\mathbb{C}_{par}^H G$. In Section 3.3 we prove that $\mathbb{C}_{par}^H G$ is isomorphic to the groupoid algebra $\mathbb{C}\Gamma_H(G)$. In Section 3.4 we construct all the irreducible H-global G-partial representations. In Section 4.1 we give a formula for the restriction to H of the aforementioned irreducibles. In Section 4.2 we describe the globalization of the aforementioned irreducibles. In Section 4.3 we introduce a partial induced representation of a (global) representation of H, and we prove a Frobenius reciprocity in this setting. In Section 5.1 we apply all our theory to the interesting case $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$, providing a natural extension of the classical representation theory of $\mathfrak{S}_n$. In Section 5.2 we prove a branching rule in this more general setting. Finally in Section 6 we give some further comments and we indicate some possible future directions.

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Though many definitions and results work in a more general setting, in this article all groups are finite and all vector spaces are over $\mathbb{C}$ and finite-dimensional, unless otherwise stated.

1. Preliminary results on partial actions and partial representations

In this section we review some general results for partial actions and partial representations. We refer the reader to [Exe17, Chapters 2 and 3] for further details on these topics. Although essentially all results in this section are known, we present some new characterizations.

1.1. Partial actions.

The following definition is due to Exel.

Definition 1.1 ([Exe98, Definition 1.2])**.** A *partial action* $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of a group G (also called *G-partial action*) on a set X consists of a family of subsets $X_g \subseteq X$ indexed by the elements of G and a family of bijections $\alpha_g \colon X_{g^{-1}} \to X_g$ for each $g \in G$, satisfying the following conditions:

(i) $X_{1_G} = X$ and $\alpha_{1_G} = \mathsf{Id}_X$;
(ii) $\alpha_h^{-1}(X_{g^{-1}} \cap X_h) \subseteq X_{(gh)^{-1}}$ for every $g, h \in G$;
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for every $x \in \alpha_h^{-1}(X_{g^{-1}} \cap X_h)$.

A *morphism* between two partial actions $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $\beta = (\{Y_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ of the same group G is a function $\phi\colon X \to Y$ such that

(a) $\phi(X_g) \subseteq Y_g$ for every $g \in G$, and

(b) for every $x \in X_{g^{-1}}$, $\beta_g(\phi(x)) = \phi(\alpha_g(x))$.

The category of partial actions of G and morphisms between them is denoted as PAct_G. Similarly we denote by Act_G the category of (global) actions of G and their morphisms.

Remark 1.2. It follows easily from the previous definition that $\alpha_g^{-1} = \alpha_{g^{-1}}$ and that, in fact, for all $g, h \in G$

$$\alpha_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}.$$

Clearly a global action of a group G on a set X is in particular a partial action with $X_g = X$ for all $g \in G$. A typical example of a partial action is obtained by restricting a global action to a proper subset.

Definition 1.3 ([Aba03, Example 1.1])**.** Given a global action (Y, β) of a group G, i.e. a homomorphism $\beta\colon G \to \mathsf{Sym}(Y)$ into the symmetric group on Y (i.e. the group of the bijections of Y into itself), and a subset $X \subseteq Y$, we can define the *restriction* $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of (Y, β) to X by setting

(R1) $X_g \coloneqq X \cap \beta_g(X)$ for every $g \in G$, and
(R2) $\alpha_g\colon X_{g^{-1}} \to X_g$, $\alpha_g(x) \coloneqq \beta_g(x)$ for every $g \in G$ and $x \in X_{g^{-1}}$.

It is easy to check that this is indeed a partial action of G on X.

It turns out that every partial action can be obtained in this way from a suitable global action. To state the precise result, we recall another definition.

Definition 1.4. A *globalization* (also called *enveloping action* in [Aba03, Definition 1.2]) of a partial action α of a group G on a set X is a triple (Y, β, φ), in which

(GL1) Y is a set and $\beta\colon G \to \mathsf{Sym}(Y)$ is an action of G on Y;
(GL2) $\varphi\colon X \to Y$ is an injective map;
(GL3) for every $g \in G$, $\varphi(X_g) = \varphi(X) \cap \beta_g(\varphi(X))$;
(GL4) for every $x \in X_{g^{-1}}$, we have $\varphi(\alpha_g(x)) = \beta_g(\varphi(x))$;
(GL5) $Y = \bigcup_{g\in G} \beta_g(\varphi(X))$.

Remark 1.5. Sometimes in the literature the triples not satisfying axiom (GL5) are already called globalizations, while the ones that satisfy it are designated as *admissible* (cf. [Bat17, Definition 4.1]). We prefer our linguistic simplification, as this is the only notion that we will use in this work.

Observe that properties (GL1)-(GL4) amount to say that φ is an isomorphism of partial actions between the initial partial action α and the partial action obtained by restriction of the global action (Y, β) to $\varphi(X) \subseteq Y$. The last axiom (GL5) says that the globalization Y can be identified with the orbit of $\varphi(X)$ under the global action.

The following theorem, in the context of continuous partial group actions on topological spaces (and, in particular, on abstract sets), is due to Abadie [Aba03, Theorem 1.1]. For a proof in the framework under consideration, we refer to [Exe17, Theorem 3.5].

Theorem 1.6. *There exists a globalization for every partial action α of a group G on a set X and it is unique up to isomorphism. It can be explicitly realized as $Y = G \times X/\sim$ where $(g, x) \sim (h, y)$ if and only if $x \in X_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = y$.*

Remark 1.7. After accepting its existence, the uniqueness of the globalization (Y, β, φ) of α follows easily from the following universal property: for every triple (Y', β', φ') satisfying (GL1)-(GL4), the function $\psi\colon Y \to Y'$ given by $\psi(\beta_g(\varphi(x))) \coloneqq \beta'_g(\varphi'(x))$ for all $g \in G$ and $x \in X$ is well defined. In particular, it follows immediately from (GL5) that ψ is uniquely determined by its defining property, that it satisfies $\psi \circ \varphi = \varphi'$ and that it is a morphism of G-sets.

To see why ψ is well defined, observe that if $\beta_g(\varphi(x)) = \beta_h(\varphi(y))$ for some $g, h \in G$ and $x, y \in X$, then $\beta_{h^{-1}}\beta_g(\varphi(x)) = \beta_{h^{-1}g}(\varphi(x)) = \varphi(y)$, which, by (GL4) and (GL2), implies that $\alpha_{h^{-1}g}(x) = y$, so that $\beta'_{h^{-1}}\beta'_g(\varphi'(x)) = \beta'_{h^{-1}g}(\varphi'(x)) = \varphi'(\alpha_{h^{-1}g}(x)) = \varphi'(y)$, which gives $\beta'_g(\varphi'(x)) = \beta'_h(\varphi'(y))$ as we wanted.

Furthermore, assigning to any partial action its globalization provides a functor $\mathsf{PAct}_G \to \mathsf{Act}_G$.

1.2. **Partial representations.** We recall the notion of a partial representation of a group on a vector space, by adapting [DE05, Definition 6.1].

Definition 1.8. A *partial representation* (V, π) of a group G (also called *G-partial representation*) on a vector space V is a map $\pi\colon G \to \mathsf{End}(V)$ such that for all $g, h \in G$

(PR1) $\pi(1_G) = \mathsf{Id}_V$;
(PR2) $\pi(g^{-1})\pi(gh) = \pi(g^{-1})\pi(g)\pi(h)$;
(PR3) $\pi(gh)\pi(h^{-1}) = \pi(g)\pi(h)\pi(h^{-1})$.

We will informally say that π behaves as a group homomorphism "in the presence of a witness".

Let (V, π) and (V', π') be two partial representations of the same group G. A *morphism of partial representations* or a *G-homomorphism* is a linear map $f\colon V \to V'$ such that $f \circ \pi(g) = \pi'(g) \circ f$ for all $g \in G$. The set of all G-homomorphisms from (V, π) to (V', π') is denoted by $\mathsf{Hom}_G(V, V')$. In particular, partial representations of a group form a category that we denote by PRep_G. Similarly we denote by Rep_G the category of (global) representations of G and their morphisms.

Some of the claims in the following lemma already appeared in [DEP00].

Lemma 1.9. *For all $g, h \in G$ we have*

$$\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}g^{-1})\pi(g). \tag{1.1}$$

In particular, the elements $\pi(g)\pi(g^{-1})$ of $\mathsf{End}(V)$ are commuting idempotents, that is to say,

$$\pi(g)\pi(g^{-1})\pi(g)\pi(g^{-1}) = \pi(g)\pi(g^{-1}) \qquad \text{and} \qquad \pi(g)\pi(g^{-1})\pi(h)\pi(h^{-1}) = \pi(h)\pi(h^{-1})\pi(g)\pi(g^{-1}) \tag{1.2}$$

for all $g, h \in G$. Moreover,

$$v \in \pi(g)\pi(g^{-1})(V) \qquad \text{if and only if} \qquad v = \pi(g)\pi(g^{-1})(v). \tag{1.3}$$

Proof. From the defining properties of partial representations, we get

$$\pi(g)\pi(h)\pi(h^{-1}) \overset{\text{(PR3)}}{=} \pi(gh)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}g^{-1}g) \overset{\text{(PR2)}}{=} \pi(gh)\pi(h^{-1}g^{-1})\pi(g).$$

The relation (1.2) on the left is straightforward, while for the one on the right we have

$$\pi(g)\pi(g^{-1})\pi(h)\pi(h^{-1}) \overset{(1.1)}{=} \pi(g)\pi(g^{-1}h)\pi(h^{-1}g)\pi(g^{-1}) \overset{(1.1)}{=} \pi(h)\pi(h^{-1})\pi(g)\pi(g^{-1}).$$

Concerning (1.3), notice that if $v \in \pi(g)\pi(g^{-1})(V)$ then $v = \pi(g)\pi(g^{-1})(w)$ for some $w \in V$ and hence

$$\pi(g)\pi(g^{-1})(v) = \pi(g)\pi(g^{-1})\pi(g)\pi(g^{-1})(w) \overset{(1.2)}{=} \pi(g)\pi(g^{-1})(w) = v.$$

The other implication is obvious. □

It is clear that global (i.e. usual) representations are partial representations. Moreover a partial representation (V, π) of a group G is a global representation if and only if $\pi(g)$ is invertible for all $g \in G$.

A natural example of a partial representation is given by the linearization of a partial action.

Definition 1.10. Given a set X, let $\mathbb{C}[X]$ be the vector space over $\mathbb{C}$ with basis X. Given a partial action $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of a group G on a set X, we define a map $\hat{\alpha}\colon G \to \mathsf{End}(\mathbb{C}[X])$ by setting for all $x \in X$

$$\hat{\alpha}(g)(x) \coloneqq \begin{cases} \alpha_g(x) & \text{if } x \in X_{g^{-1}} \\ 0 & \text{otherwise} \end{cases}$$

and extending by linearity. It is easy to check that $(\mathbb{C}[X], \hat{\alpha})$ defines indeed a partial representation of G on $\mathbb{C}[X]$, which we call the *linearization* of (X, α).

Remark 1.11. Given a set X, for any subset $Y \subseteq X$, let $P_Y\colon \mathbb{C}[X] \to \mathbb{C}[Y]$ be the obvious projection whose kernel is $\mathbb{C}[X \setminus Y]$. Given a partial action $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of a group G on a set X, for all $v \in \mathbb{C}[X]$ we have

$$\hat{\alpha}(g)(v) = \hat{\alpha}(g)(P_{X_{g^{-1}}}(v)).$$

With this construction at hand, we have the following direct result.

Proposition 1.12. *The linearization defines a functor* $\mathbb{C}[-] : \mathsf{PAct}_G \to \mathsf{PRep}_G$ *which allows a right adjoint* $\mathcal{U} : \mathsf{PRep}_G \to \mathsf{PAct}_G$*. This lifts the well-known adjunction given by the linearization* $\mathbb{C}[-] : \mathsf{Set} \to \mathsf{Vec}_{\mathbb{C}}$ *and the underlying functor* $\mathcal{U} : \mathsf{Vec}_{\mathbb{C}} \to \mathsf{Set}$.

Proof. Given a partial representation (V, π) of G one can obtain a partial action $\mathcal{U}(V) = (\{V_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on the set underlying V by setting $V_g := \pi(g)\pi(g^{-1})(V)$ and $\alpha_g := \pi(g)|_{V_{g^{-1}}} : V_{g^{-1}} \to V_g$ for every $g \in G$. One easily checks that this construction, as well as the linearization, constitute well-defined functors. The unit and counit of the adjunction are given by

$$\eta_X : X \to \mathcal{U}(\mathbb{C}[X]), \quad x \mapsto x, \qquad \text{and} \qquad \epsilon_V : \mathbb{C}[\mathcal{U}(V)] \to V, \quad v \mapsto v,$$

for all X in PAct_G and V in PRep_G, respectively. □

Notation 1.13. If there is no risk of confusion, we often denote the vector space generated by a set X simply by $\mathbb{C}X$, without parenthesis. In particular, the group algebra over a group G is denoted indifferently by $\mathbb{C}[G]$ or $\mathbb{C}G$. This is aimed at lightening the notation, for example when considering the groupoid algebra $\mathbb{C}\Gamma_H(G)$ in Section 3.3.

1.3. **Restriction and globalization (or dilation) for partial representations.** As we have seen in Section 1.1, partial actions can be constructed by restricting global actions to subsets, and any partial action has a globalization from which it can be recovered by restriction. Therefore it is natural to ask if similar constructions can be done for partial representations. A minute of thought suggests that a global representation of a group G on a vector space U cannot be simply restricted to a partial representation on any subspace of U: indeed this leads rather to a so-called *geometric partial module* (by the dual of [HV20, Example 2.5], in view of [HV20, Remark 2.3(4)]). In order to obtain a proper partial representation, some additional information is needed. Let us recall the following construction, which arises from [ABV19, Definition 3.1 and Proposition 3.3] (by taking a group algebra as Hopf algebra) and which goes back to [Aba18] for partial representations of groups.

Proposition 1.14. *If* (U, ρ) *is a* G*-global representation and* $T : U \to U$ *is a linear map satisfying* $T^2 = T$ *and*

$$T \circ \rho(g) \circ T \circ \rho(g^{-1}) = \rho(g) \circ T \circ \rho(g^{-1}) \circ T \tag{1.4}$$

for all $g \in G$*, then* $T(U)$ *is a* G*-partial representation by means of the map*

$$\pi \colon G \to \mathsf{End}(T(U)), \;\; \pi(g)(v) = T \circ \rho(g)(v)$$

for all $g \in G$ *and* $v \in T(U)$.

In fact, when analyzing the proof of [ABV19, Proposition 3.3], one can see that it is sufficient to require that the condition (1.4) is satisfied on the image of T, not on the whole of U. This observation motivates us to introduce the following definition.

Definition 1.15. Let (U, ρ) be a global representation of a group G on a vector space U, and let $\varphi \colon V \to U$ and $\tau \colon U \to V$ be two linear maps such that $\tau \circ \varphi = \mathsf{Id}_V$. Consider the map $\pi \colon G \to \mathsf{End}(V)$ defined by

$$\pi(g)(v) := \tau(\rho(g)(\varphi(v))) \tag{1.5}$$

for all $v \in V$, $g \in G$. We say that (V, π) is the *restriction* of the global representation (U, ρ) to V via φ and τ if

(RR1) (V, π) is a partial representation of G;

(RR2) for every $g \in G$ and $v \in V_{g^{-1}} := \pi(g^{-1})\pi(g)(V)$ we have

$$\varphi(\pi(g)(v)) = \rho(g)(\varphi(v)). \tag{1.6}$$

Remark 1.16. If we set $T = \varphi \circ \tau : U \to U$, then one can check that conditions (RR1) and (RR2) are equivalent to the fact that (1.4) holds for all elements in $V \cong \varphi(V) \subseteq U$, as in the discussion preceding the definition.

The first example is of course coming from the restrictions of global actions to subsets. We resort to the notation introduced in Definition 1.10.

Proposition 1.17. *Let* (Y, β) *be a global action of a group* G *on the set* Y*. Let* $X \subseteq Y$ *be a subset and let* (X, α) *be the restriction of* (Y, β) *to* X*. Then the linearization* $(\mathbb{C}[X], \hat{\alpha})$ *of* (X, α) *is the restriction of the linearization* $(\mathbb{C}[Y], \hat{\beta})$ *of* (Y, β) *via the inclusion* $\varphi \colon \mathbb{C}[X] \to \mathbb{C}[Y]$ *and the projection* $\tau := P_X \colon \mathbb{C}[Y] \to \mathbb{C}[X]$.

Proof. Fix $g \in G$. On the one hand, for all $x \in X$,

$$\hat{\alpha}(g)(x) = \alpha_g(P_{X_{g^{-1}}}(x)) = \begin{cases} 0 & \text{if } x \notin X_{g^{-1}} \\ \alpha_g(x) & \text{if } x \in X_{g^{-1}} \end{cases} \overset{(R2)}{=} \begin{cases} 0 & \text{if } x \notin X_{g^{-1}} \\ \beta_g(x) & \text{if } x \in X_{g^{-1}} \end{cases}.$$

On the other hand,

$$\tau(\hat{\beta}(g)(\varphi(x))) = \tau(\beta_g(x)) = \begin{cases} 0 & \text{if } \beta_g(x) \notin X \\ \beta_g(x) & \text{if } \beta_g(x) \in X \end{cases}.$$

However, $\beta_g(x) \in X$ if and only if $x \in X \cap \beta_g^{-1}(X) \overset{(R1)}{=} X_{g^{-1}}$. Therefore,

$$\hat{\alpha}(g) = \tau \circ \hat{\beta}(g) \circ \varphi \qquad \text{for all } g \in G$$

and the map $\pi\colon G \to \mathsf{End}(\mathbb{C}[X])$ given by $\pi(g) \coloneqq \tau \circ \hat{\beta}(g) \circ \varphi$ for all $g \in G$ is a partial representation. Moreover, for every $g \in G$ we have

$$\mathbb{C}[X]_{g^{-1}} = \pi(g^{-1})\pi(g)(\mathbb{C}[X]) = \pi(g^{-1})\left(\mathbb{C}\left[X \cap \beta_g(X)\right]\right) = \mathbb{C}\left[\beta_{g^{-1}}(X) \cap X\right] \overset{(R1)}{=} \mathbb{C}\left[X_{g^{-1}}\right]$$

and for every $x \in X_{g^{-1}}$, $\varphi(\pi(g)(x)) = \beta_g(x) = \hat{\beta}(g)(\varphi(x))$, so that $(\mathbb{C}[X], \pi) = (\mathbb{C}[X], \hat{\alpha})$ is indeed the restriction of $\hat{\beta}$ to $\mathbb{C}[X]$ via φ and τ. ☐

Just as any partial action can be globalized, it has been shown in [Aba18] and [ABV19] that any partial representation can be obtained from a global one by a restriction in the sense of Proposition 1.14. More precisely, for any partial representation (V, π) there exists a global representation U (called its *dilation*) together with a projection T satisfying the conditions as in Proposition 1.14, such that: (1) U is generated by $T(U)$ as a global representation, (2) there exists an isomorphism $\theta : V \cong T(U)$ of partial representations and (3) for any other triple (U', T', θ') with these properties, there exists a unique morphism $\Phi : U' \to U$ such that $T \circ \Phi = \Phi \circ T'$ and $\Phi \circ \theta' = \theta$. Despite the fact that the morphism Φ in this universal property goes in the other direction compared to the universal property of the globalization of a partial action as discussed in Remark 1.7, in [SV21, Theorem 3.13] it was shown that the dilation of a partial representation coincides with its *globalization* in the sense of [SV20, Definition 3.1]. This globalization is a global representation $\overline{V}$ together with an injective morphism (of geometric partial modules) $\varphi : V \to \overline{V}$ that is universal in the following sense: for any other global representation U' admitting a morphism of partial modules $\varphi' : V \to U'$, there is a unique G-homomorphism $\psi : \overline{V} \to U'$ such that $\varphi' = \psi \circ \varphi$. We therefore arrive at the following Theorem 1.18 in the present framework. However, in order to deduce this result from [SV21] along the lines explained above, one needs to prove that the map $\varphi : V \to U$ for a restriction in the sense of Definition 1.15 gives rise to a morphism of geometric partial modules. Since the proof of this fact is rather technical and requires to introduce properly the framework of geometric partial modules, we provide here a direct alternative proof.

Theorem 1.18. *Let (V, π) be a partial representation of a group G. Then there exists, up to isomorphism, a unique quadruple (U, ρ, φ, τ), called the* globalization *of (V, π), which satisfies the following properties:*

- *(GR1) (U, ρ) is a global representation of G;*
- *(GR2) (V, π) is the restriction of (U, ρ) via φ and τ (in the sense of Definition 1.15);*
- *(GR3) for every quadruple $(U', \rho', \varphi', \tau')$ satisfying (GR1) and (GR2) there exists a unique G-homomorphism $\psi\colon U \to U'$ (i.e. ψ is linear and $\psi \circ \rho(g) = \rho'(g) \circ \psi$ for all $g \in G$) such that $\psi \circ \varphi = \varphi'$.*

Assigning to any partial representation its globalization provides a functor $\mathsf{PRep}_G \to \mathsf{Rep}_G$.

Proof. Let (V, π) be a G-partial representation. Set $V_g \coloneqq \pi(g)\pi(g^{-1})(V)$ for all $g \in G$. Consider, in the complex vector space $\mathbb{C}[G] \otimes V$, the subspace Z generated by the vectors

$$\{g \otimes v - h \otimes \pi(h^{-1}g)(v) \mid g, h \in G, v \in V_{g^{-1}h}\} \tag{1.7}$$

and let U be the quotient space $(\mathbb{C}[G] \otimes V)/Z$. Recall that the left multiplication on the first tensorand makes of $\mathbb{C}[G] \otimes V$ a G-global representation. This induces a structure of G-global representation (U, ρ) on U by setting

$$\rho(g)(\overline{h \otimes v}) \coloneqq \overline{gh \otimes v}$$

for all $g,h \in G$ and $v \in V$, where $\bar{t}$ denotes the coset of $t \in \mathbb{C}[G] \otimes V$ in U. To check that ρ is well defined, it is enough to notice that Z is a G-subrepresentation: given $g,h,k \in G$ and $v \in V_{h^{-1}k}$, we have

$$gh \otimes v - gk \otimes \pi(k^{-1}h)(v) = gh \otimes v - gk \otimes \pi(k^{-1}g^{-1}gh)(v) = (gh) \otimes v - (gk) \otimes \pi\left((gk)^{-1}(gh)\right)(v) \in Z$$

which proves the claim.

Clearly the maps $\rho(g)$ are invertible, as $\rho(g)^{-1} = \rho(g^{-1})$, so that ρ gives indeed a global representation of G.

Consider now the map $\varphi\colon V \to U$ defined by $\varphi(v) \coloneqq \overline{1_G \otimes v}$ for all $v \in V$, and the map $\tau : U \to V$ defined by $\tau(\overline{g \otimes v}) \coloneqq \pi(g)(v)$ for all $v \in V$. Observe that the latter is well defined, since the map $\widetilde{\tau}\colon \mathbb{C}[G] \otimes V \to V$ defined by $\widetilde{\tau}(g \otimes v) \coloneqq \pi(g)(v)$ sends the generators of Z to 0: for $h,k \in G$ and $v \in V_{h^{-1}k} = \pi(h^{-1}k)\pi(k^{-1}h)(V)$, we have

$$\widetilde{\tau}(h\otimes v) = \pi(h)(v) \overset{(1.3)}{=} \pi(h)\pi(h^{-1}k)\pi(k^{-1}h)(v) \overset{(1.1)}{=} \pi(k)\pi(k^{-1})\pi(h)(v) \overset{(PR2)}{=} \pi(k)\pi(k^{-1}h)(v) = \widetilde{\tau}(k\otimes\pi(k^{-1}h)(v))$$

and so τ is well defined.

We want to show that (U,ρ,φ,τ) is a globalization of (V,π). We already showed that (U,ρ) is a global representation of G, proving (GR1). Moreover, it is clear that $\tau \circ \varphi = \mathsf{Id}_V$ and, by construction, $\tau(\rho(g)(\varphi(v)) = \pi(g)(v)$. To prove the remaining property of (GR2), observe that for each $v \in V_{g^{-1}}$,

$$\varphi(\pi(g)(v)) = \overline{1_G \otimes \pi(g)(v)} = \overline{g \otimes v} = \rho(g)(\overline{1_G \otimes v}) = \rho(g)(\varphi(v)).$$

Therefore (V,π) is the restriction of (U,ρ) via φ and τ, proving (GR2).

In order to prove (GR3), let $(U',\rho',\varphi',\tau')$ be another quadruple satisfying (GR1) and (GR2). Since

$$U = \sum_{g\in G} \rho(g)(\varphi(V)), \tag{1.8}$$

if a $\psi\colon U \to U'$ with the properties stated in (GR3) exists, then it is uniquely determined by the property $\psi(\varphi(v)) = \varphi'(v)$ for all $v \in V$, which gives $\psi(\overline{1_G \otimes v}) = \psi(\varphi(v)) = \varphi'(v)$, and the property $\psi(\rho(g)(u)) = \rho'(g)(\psi(u))$ for all $u \in U$ and $g \in G$, so that

$$\psi(\overline{g \otimes v}) = \psi(\rho(g)(\overline{1_G \otimes v})) = \psi(\rho(g)(\varphi(v))) = \rho'(g)(\psi(\varphi(v))) = \rho'(g)(\varphi'(v)).$$

This shows the uniqueness of such a map ψ, and it suggests how to define it: we define $\widetilde{\psi}\colon \mathbb{C}[G] \otimes V \to U'$ by setting $\widetilde{\psi}(g \otimes v) \coloneqq \rho'(g)(\varphi'(v))$ for all $g \in G$ and $v \in V$. This map sends the generators of Z to 0: indeed if $g,h \in G$ and $v \in V_{g^{-1}h}$,

$$\widetilde{\psi}(g \otimes v) = \rho'(g)(\varphi'(v)) = \rho'(h)\rho'(h^{-1}g)(\varphi'(v)) \overset{(RR2)}{=} \rho'(h)(\varphi'(\pi'(h^{-1}g)(v))) = \widetilde{\psi}(h \otimes \pi'(h^{-1}g)(v))$$

and hence the map $\psi\colon U \to U'$ given by $\psi(\overline{g \otimes v}) \coloneqq \rho'(g)(\varphi'(v))$ is well defined. Notice that, by construction, ψ is a G-homomorphism and $\psi \circ \varphi = \varphi'$.

Therefore, a globalization exists. Its uniqueness now follows easily from the universal property (GR3). □

Corollary 1.19. *The globalization (U,ρ,φ,τ) of a partial representation (V,π) satisfies in addition:*

(GR3′) $U = \sum_{g\in G} \rho(g)(\varphi(V))$;

(GR4′) for every quadruple $(U',\rho',\varphi',\tau')$ satisfying (GR1) and (GR2), the assignment

$$\psi(\rho(g)(\varphi(v))) \coloneqq \rho'(g)(\varphi'(v)) \qquad \text{for all } g \in G, v \in V$$

gives a well-defined linear map $\psi\colon U \to U'$.

(GR5′) The universal map ψ from (GR4′) satisfies $\tau' \circ \psi = \tau$.

Moreover, any triple (U,ρ,φ,τ) satisfying (GR1), (GR2), (GR3′) and (GR4′) is a globalization.

Proof. Condition (GR3′) tells exactly that U is generated by $\varphi(V)$ as a G-global representation. To see this, it suffices to denote by $\tilde{U}$ the G-subrepresentation of U generated by $\varphi(V)$. Then we can restrict $\tau\colon U \to V$ to a map $\tilde{\tau}\colon \tilde{U} \to V$, and (V,π) is still the restriction of $\tilde{U}$ by φ and $\tilde{\tau}$. Hence by the universal property of U, there exists a map $U \to \tilde{U}$ that is a right inverse for the inclusion. With this, it is not hard to see that $\tilde{U}$ is itself a globalization of (V,π) and hence U and $\tilde{U}$ coincide.

The map ψ from the universal property (GR3) satisfies the identity stated in (GR4′).

To prove (GR5′), first remark that by (GR3′), any element of U can be written as a sum of elements of the form $\rho(g)(\varphi(v))$ and hence it suffices to check the required identity on elements of this form. Since (V,π) is the restriction of U, it follows that $\tau\left(\rho(g)(\varphi(v))\right) = \pi(g)(v)$. On the other hand, using that ψ is a G-homomorphism and that (V,π) is also the restriction of U', we find $\tau'\left(\psi\left(\rho(g)(\varphi(v))\right)\right) = \tau'\left(\rho'(g)\left(\psi(\varphi(v))\right)\right) = \tau'\left(\rho'(g)\left(\varphi'(v)\right)\right) = \pi(g)(v)$.

For the last statement, if (GR3′) and (GR4′) are satisfied, then the map ψ from (GR4′) clearly satisfies the conditions of (GR3). Moreover, ψ is also a unique G-homomorphism, since U is generated by $\varphi(V)$ as a G-representation, and the image of $\varphi(V)$ under ψ is determinded by φ'. ☐

Proposition 1.20. *Let $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of a group G on a set X. Then the linearization $\mathbb{C}[Y]$ of the globalization (Y,β,φ) of α (in the sense of* [Aba03]*) provides the globalization of the linearization $(\mathbb{C}[X],\hat{\alpha})$ of α via the linearization $\hat{\varphi}\colon \mathbb{C}[X]\to\mathbb{C}[Y]$ of φ and the projection $\hat{\tau} := P_X\colon \mathbb{C}[Y]\to\mathbb{C}[X]$. In particular, being the globalization functorial, we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathsf{Act}_G & \xrightarrow{\mathbb{C}[-]} & \mathsf{Rep}_G \\ {\scriptstyle \mathsf{glob}}\uparrow & & \uparrow{\scriptstyle \mathsf{glob}} \\ \mathsf{PAct}_G & \xrightarrow[\mathbb{C}[-]]{} & \mathsf{PRep}_G \end{array} .$$

Proof. The properties (GR1) and (GR2) have been already discussed in Proposition 1.17. By Corollary 1.19, instead of (GR3), we may check (GR3′) and (GR4′). Observe that property (GL5) of (Y,β,φ) implies property (GR3′) of its linearization: it is enough to proceed by double inclusion, after noticing that $\hat{\beta}(g)\left(\hat{\varphi}(x)\right) = \beta_g(\varphi(x))$ for all $x\in X$, $g\in G$. Now in order to check (GR4′), let us show that $Y' := \bigcup_{g\in G}\bigcup_{x\in X}\rho'(g)(\varphi'(x)) \subseteq U'$ together with the restrictions of φ' to X and of $\rho'(g)$ to Y' for all $g\in G$ form a triple satisfying (GL1)-(GL4). Only (GL3) is not immediate. Pick $x\in X_g$ and set $y := \alpha_{g^{-1}}(x)\in X_{g^{-1}}\subseteq \mathbb{C}[X]_{g^{-1}}$. It follows that

$$\varphi'(x) = \varphi'(\alpha_g(y)) = \varphi'(\hat{\alpha}(g)(y)) \overset{(RR2)}{=} \rho'(g)(\varphi'(y)) \in \rho'(g)(\varphi'(X))$$

and hence $\varphi'(X_g)\subseteq \varphi'(X)\cap\rho'(g)(\varphi'(X))$. For the reverse inclusion, pick $z\in X$ such that $\varphi'(z) = \rho'(g)(\varphi'(y))$ for a certain $y\in X$. Then

$$z = \tau'(\varphi'(z)) = \tau'\left(\rho'(g)(\varphi'(y))\right) \overset{(1.5)}{=} \hat{\alpha}(g)(y) = \alpha_g(y)\in X_g$$

and hence $\varphi'(z)\in\varphi'(X_g)$. Therefore, by Remark 1.7, the map $\psi\colon Y\to Y'$ given by $\psi(\beta_g(\varphi(x))) := \rho'(g)(\varphi'(x))$ for all $x\in X$ and $g\in G$ is well defined and its linearization $\hat{\psi}\colon \mathbb{C}[Y]\to U'$ is the map required in (GR4′). This completes the proof that $(\mathbb{C}[Y],\hat{\beta})$ is indeed the globalization of $(\mathbb{C}[X],\hat{\alpha})$. ☐

2. H-global G-partial representations

In this section we introduce H-global G-partial representations and describe some of their general properties.

2.1. Basic definitions and examples.

We introduce some basic notions and examples, setting up the framework in which we are going to work for the rest of the present article.

Definition 2.1. Let H be a subgroup of a group G.

A G-partial action $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a set X is called *H-global* if $X_h = X$ for all $h\in H$.

A G-partial representation (V,π) is *H-global* if the restriction of π to H is a global representation of H.

Example 2.2. Let G be a group and H be a subgroup of G. There are two kinds of H-global G-partial representations that are always available.

(a) Any G-global representation is obviously H-global G-partial.

(b) Let (W,ρ) be a global representation of H. Then we can always construct an H-global G-partial representation $(W,\overline{\rho})$ in the following way: we define

$$\overline{\rho}(g) := \begin{cases} 0 & \text{if } g\in G\setminus H \\ \rho(g) & \text{if } g\in H \end{cases} .$$

It is straightforward to check that this is indeed an H-global G-partial representation.

Before seeing more examples, we introduce one more definition.

Definition 2.3. Let G be a group and let (V, π) be a G-partial representation. We define the *globalizer* $H(V) = H(V, \pi)$ of V to be

$$H(V) \coloneqq \{g \in G \mid \pi(g) \text{ is invertible}\}.$$

Lemma 2.4. *Let (V, π) be a G-partial representation. Then*

$$H(V) = \{h \in G \mid \pi(gh) = \pi(g)\pi(h),\ \forall\, g \in G\} = \{h \in G \mid V_h = V\}\,. \tag{2.1}$$

In particular, the globalizer $H(V)$ is the biggest subgroup of G that acts globally on V via π.

Proof. For all $h \in H(V)$

$$\pi(h)\pi(h)^{-1}\pi(h) = \pi(h) \overset{\text{(PR2)}}{=} \pi(h)\pi(h^{-1})\pi(h)$$

so that $\pi(h)^{-1} = \pi(h^{-1})$. From this it follows that, for all $g \in G$,

$$\pi(gh) = \pi(gh)\pi(1_G) = \pi(gh)\pi(h^{-1})\pi(h) \overset{\text{(PR3)}}{=} \pi(g)\pi(h). \tag{2.2}$$

Conversely, if $\pi(gh) = \pi(g)\pi(h)$ for all $g \in G$, then

$$\mathsf{Id}_V = \pi(1_G) = \pi(h^{-1}h) = \pi(h^{-1})\pi(h)$$

and hence, being V finite-dimensional, $\pi(h)$ is invertible with $\pi(h)^{-1} = \pi(h^{-1})$. This proves the first equality in (2.1). Concerning the second one, if $h \in H(V)$ then $V_h = \pi(h)\pi(h^{-1})(V) = V$. Conversely, if $V = V_h$ then $\pi(h)\pi(h^{-1}) = \mathsf{Id}_V$ and we conclude as above. In particular, $H(V) \subseteq G$ is a subgroup and its maximality follows by definition. □

Remark 2.5. Let (V, π) be a G-partial representation. It is H-global if and only if H is a subgroup of $H(V)$.

It turns out that the globalizer $H(V)$ of a G-partial representation (V, π) is typically nontrivial. This is one of the main motivations to study H-global G-partial representations.

Example 2.6. Let K be a subgroup of a group G and consider the action of G on the left cosets G/K of K by left multiplication. Let $A \subseteq G/K$ be a set of left cosets of K and consider the restriction α of the given action to A, which gives us a G-partial action. Set

$$K_A \coloneqq \{g \in G \mid gA = A\}.$$

It is easy to check that the G-partial action α is H-global if and only if H is a subgroup of K_A.

Consider now the linearization $(\mathbb{C}[A], \hat{\alpha})$ of α. It is clear that the globalizer of this G-partial representation is in fact $H(\mathbb{C}[A]) = K_A$.

The following example is a special case of the previous one; it will be relevant in Section 5.1.

Example 2.7. Let us fix some notation. For any positive integer $n \in \mathbb{N}$, let $[n] := \{1, 2, \ldots, n\}$, and for $I \subseteq [n]$, let

$$\mathfrak{S}_n^I \coloneqq \{\sigma \in \mathfrak{S}_n \mid \sigma(I) = I\},$$

so for example $\mathfrak{S}_n^{\varnothing} = \mathfrak{S}_n^{[n]} = \mathfrak{S}_n$ and $\mathfrak{S}_n^{\{1\}} = \mathfrak{S}_n^{[1]} = \mathfrak{S}_n^{[n]\setminus\{1\}} \cong \mathfrak{S}_1 \times \mathfrak{S}_{n-1} \equiv \mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$.

More generally, for $1 \leq k < n$, consider $\mathfrak{S}_k \times \mathfrak{S}_{n-k} \equiv \mathfrak{S}_n^{[k]} = \mathfrak{S}_n^{[n]\setminus[k]}$. We want to describe the action of $\mathfrak{S}_n$ on the left cosets $\mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k})$.

Observe that, given $\sigma \in \mathfrak{S}_n$ and $I \subseteq [n]$,

$$\sigma\mathfrak{S}_n^I = \{\tau \in \mathfrak{S}_n \mid \tau(I) = \sigma(I)\}$$

therefore, for $\rho \in \mathfrak{S}_n$,

$$\rho \cdot \left(\sigma\mathfrak{S}_n^I\right) = \tau\mathfrak{S}_n^I \qquad \Longleftrightarrow \qquad \rho\sigma(I) = \tau(I),$$

if and only if ρ maps the subset $\sigma(I) \subseteq [n]$ into the subset $\tau(I) \subseteq [n]$ and hence we can identify the action of $\mathfrak{S}_n$ on the left cosets of $\mathfrak{S}_k \times \mathfrak{S}_{n-k} \equiv \mathfrak{S}_n^{[k]}$ with the action of $\mathfrak{S}_n$ on

$$\binom{[n]}{k} \coloneqq \{A \subseteq [n] \mid |A| = k\}$$

via $\sigma\mathfrak{S}_n^{[k]} \leftrightarrow \sigma([k])$. We will freely use this identification in the rest of this article.

Notice that for $k = 1$ we are identifying the action of $\mathfrak{S}_n$ on the cosets $\mathfrak{S}_n/\mathfrak{S}_{n-1}$ with the defining action of $\mathfrak{S}_n$ on the set $\binom{[n]}{1} = [n]$.

We now set

$$Y := \left\{ A \in \binom{[n]}{k} \;\middle|\; 1 \in A \right\} \subsetneq \binom{[n]}{k}$$

and call α the $\mathfrak{S}_n$-partial action on Y that we get by restriction from the $\mathfrak{S}_n$-global action on $\binom{[n]}{k}$. For instance, for $n = 4$ and $k = 2$,

$$\binom{[4]}{2} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}, \qquad Y = \{\{1,2\},\{1,3\},\{1,4\}\}$$

and, for example,

$$Y_{(1,2)} = Y \cap (1,2)\cdot Y = \{\{1,2\},\{1,3\},\{1,4\}\} \cap \{\{1,2\},\{2,3\},\{2,4\}\} = \{\{1,2\}\}$$
$$\text{while} \qquad Y_{(2,3,4)} = \{\{1,2\},\{1,3\},\{1,4\}\} \cap \{\{1,3\},\{1,4\},\{1,2\}\} = Y.$$

In the notation of Example 2.6, it is easy to check that $K_Y = \mathfrak{S}_n^{[1]}$, so that the globalizer of the linearization $(\mathbb{C}[Y], \hat{\alpha})$ of this $\mathfrak{S}_n$-partial action is $\mathfrak{S}_n^{[1]}$.

2.2. **A general construction.** In this section we give a general construction that gives a large class of examples of H-global G-partial representations.

We start with a finite group G and two fixed subgroups H and K of G. Let $G = \coprod_{i=1}^n g_iK$ as disjoint union of left cosets of K. Let $A \subseteq G$ be a union of (H,K)-double cosets, i.e. $hA = A$ for all $h \in H$ and $Ak = A$ for all $k \in K$. Denote by A/K the set of left cosets of K contained in A. Let (W, ρ) be a global representation of K. With these data we want to construct an H-global G-partial representation.

Recall that the induced representation $\mathsf{Ind}_K^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W$ decomposes as a vector space as

$$\mathsf{Ind}_K^G W \cong \bigoplus_{g_iK \in G/K} W^{g_i},$$

where we denoted by W^{g_i} the subspace $\mathbb{C}g_iK \otimes_{\mathbb{C}[K]} W$ corresponding to the coset g_iK. For every $i = 1, \dots, n$, we call $\phi_i \colon W \to W^{g_i}$ the $\mathbb{C}$-linear isomorphism defined by $\phi_i(w) := g_i \otimes_{\mathbb{C}[K]} w$ for every $w \in W$.

Consider the subspace

$$W_A := \bigoplus_{g_iK \in A/K} W^{g_i}.$$

We define a linear map $\pi \colon G \to \mathsf{End}(W_A)$ in the following way: for every $g_iK \in A/K$ and every $x \in W^{g_i}$, there exists a unique $w \in W$ such that $x = \phi_i(w)$. Thus, we set

$$\pi(g)(x) := \begin{cases} \phi_j(\rho(k)(w)) & \text{if } x = \phi_i(w), gg_i = g_jk, k \in K, \text{ and } g_jK \in A/K \\ 0 & \text{otherwise} \end{cases}. \tag{2.3}$$

It is easy to check that this definition does not depend on the choice of the representatives of the left cosets of K.

Lemma 2.8. *The pair (W_A, π) is an H-global G-partial representation.*

Proof. The proof that (W_A, π) is a G-partial representation does not require that A satisfy the property that $hA = A$ for all $h \in H$. This property is used only to show that this G-partial representation is indeed H-global. We leave the tedious but straightforward details to the reader. ☐

Remark 2.9. Observe that the construction of (W_A, π) is in fact the restriction (cf. Definition 1.15) of the global G-representation $\mathsf{Ind}_K^G W$ to the subspace W_A via the obvious inclusion φ and the projection $\tau \colon \mathsf{Ind}_K^G W \to W_A$ whose kernel is

$$\ker \tau = \bigoplus_{g_iK \notin A/K} W^{g_i}.$$

Example 2.10. Using the same notation as before, let W be the trivial representation of K. Then in this case $\mathsf{Ind}_K^G W$ is simply the linearization of the action of G on the left cosets G/K of K by left multiplication. Consider a subset $A \subseteq G$ such that $Ak = A$ for all $k \in K$. Then our construction (W_A, π) in this case corresponds simply to the linearization of the restriction of this action to A/K (cf. Proposition 1.17).

We will see in Section 3.4 that all the irreducible objects in the category of H-global G-partial representations can be built with the construction given in the present section.

2.3. **Some general properties of partial representations.** In this section we discuss some general properties of G-partial representations. In particular we introduce certain orthogonal idempotents that give a general decomposition of such representations. Then we specialize this discussion to the case of H-global G-partial representations, to see what this decomposition looks like in this case.

All over this section we consider a partial representation (V, π) of a finite group G.

Notation 2.11. Set

$$\mathcal{P}_H(G/H) \coloneqq \{A \subseteq G \mid H \subseteq A \text{ and } A \text{ is a union of left cosets of } H\}.$$

In the case $H = \{1_G\}$, we use the notation

$$\mathcal{P}_1(G) \coloneqq \mathcal{P}_{\{1_G\}}(G/\{1_G\}) = \{A \subseteq G \mid 1_G \in A\}.$$

The following definition already appeared in [DEP00], namely at [DEP00, page 510, equation (4)], as well as some of the claims and computations in the forthcoming Lemma 2.14 and 2.16. We report them here for the sake of completeness.

Definition 2.12. For every $A \subseteq G$ define the operator

$$P_A^\pi = P_A := \prod_{g \in A} \pi(g)\pi(g^{-1}) \cdot \prod_{\overline{g} \in G \setminus A} (\mathsf{Id}_V - \pi(\overline{g})\pi(\overline{g}^{-1})).$$

Observe that, thanks to Lemma 1.9, in the definition of P_A^π we do not need to specify the order of the products.

Remark 2.13. Notice that if $1_G \notin A$, i.e. $A \notin \mathcal{P}_1(G)$, then P_A is the zero operator, as it contains the factor $\mathsf{Id}_V - \pi(1_G)\pi(1_G) = 0$.

Lemma 2.14. *For any $A \subseteq G$ and any $g \in G$,*

$$\pi(g)P_A = P_{gA}\pi(g). \tag{2.4}$$

In particular, if $g^{-1} \notin A$, then

$$\pi(g)P_A = 0 = P_A\pi(g^{-1}). \tag{2.5}$$

Proof. The first claim follows immediately from (1.1), i.e. $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}g^{-1})\pi(g)$ for all $g, h \in G$. The second claim follows from the first one and from Remark 2.13. □

Lemma 2.15. *Consider $A \subseteq G$. If $g \in A$, then*

$$\pi(g)\pi(g^{-1})P_A = P_A = P_A\pi(g)\pi(g^{-1}).$$

If $g \notin A$, then

$$\pi(g)\pi(g^{-1})P_A = 0 = P_A\pi(g)\pi(g^{-1}).$$

Proof. It follows immediately from the definition of P_A, Lemma 1.9 and equation (2.5). □

Lemma 2.16. *The set $\{P_A\}_{A \in \mathcal{P}_1(G)}$ is a system of orthogonal idempotents. Moreover, $\sum_{A \in \mathcal{P}_1(G)} P_A = \mathsf{Id}_V$.*

Proof. The fact that every P_A is idempotent follows from Lemma 1.9, and it is easy to see that $P_AP_B = 0 = P_BP_A$ if $A \neq B$. For the last statement, we compute

$$\begin{aligned} \mathsf{Id}_V &= \prod_{g \in G} \mathsf{Id}_V = \prod_{g \in G} (\mathsf{Id}_V - \pi(g)\pi(g^{-1}) + \pi(g)\pi(g^{-1})) \\ &= \sum_{A \subseteq G} \left(\prod_{g \in A} \pi(g)\pi(g^{-1}) \right) \left(\prod_{g \notin A} (\mathsf{Id}_V - \pi(g)\pi(g^{-1})) \right) \\ &= \sum_{A \subseteq G : 1_G \in A} \left(\prod_{g \in A} \pi(g)\pi(g^{-1}) \right) \left(\prod_{g \notin A} (\mathsf{Id}_V - \pi(g)\pi(g^{-1})) \right) = \sum_{A \in \mathcal{P}_1(G)} P_A. \end{aligned}$$

□

As a consequence of the previous lemma, if we set $V^A := P_A^\pi(V)$, then we have the decomposition

$$V = \bigoplus_{A\in\mathcal{P}_1(G)} V^A. \tag{2.6}$$

Notice that, as observed above, if $g^{-1} \notin A$, then $\pi(g)(V^A) = 0$.

Observe also that a G-homomorphism $\varphi \in \mathsf{Hom}_G(V, U)$, i.e. a morphism between the two G-partial representations (V, π) and (U, η), intertwines the action of P_A^π and P_A^η, i.e.

$$P_A^\eta(\varphi(v)) = \varphi(P_A^\pi(v)) \quad \text{for all } v \in V,$$

so that, if we set $V^A \coloneqq P_A^\pi(V)$ and $U^A \coloneqq P_A^\eta(U)$, then we have the decompositions

$$V = \bigoplus_{A\in\mathcal{P}_1(G)} V^A \quad \text{and} \quad U = \bigoplus_{A\in\mathcal{P}_1(G)} U^A$$

and φ is a graded map, i.e. $\varphi(V^A) \subseteq U^A$ for all A.

We want to understand this decomposition in the case when the G-partial representation (V, π) is H-global, for a subgroup H of G, so we assume this from now on.

The following easy observation is so useful that deserves to be called a lemma.

Lemma 2.17. *Let (V, π) be an H-global G-partial representation. Then for all $g \in G$ and $h \in H$ we have*

$$\pi(gh) = \pi(g)\pi(h) \quad \text{and} \quad \pi(hg) = \pi(h)\pi(g). \tag{2.7}$$

Proof. The first identity follows from (2.2), the other one is similar. □

Lemma 2.18. *Let (V, π) be an H-global G-partial representation. Then $P_{Ah}^\pi = P_A^\pi$ for any $A \in \mathcal{P}_1(G)$ and $h \in H$.*

Proof. Using (2.7) this is a straightforward verification. □

Proposition 2.19. *If $A \in \mathcal{P}_1(G)$ such that $P_A \neq 0$, then A is a union of left cosets of H, and $H \subseteq A$.*

Proof. Since the P_A are orthogonal idempotents, if $Ah \neq A$ for some $h \in H$, then $P_{Ah} = P_A$ implies $P_A = 0 = P_{Ah}$. In other words, for P_A to be nonzero we must have $Ah = A$ for all $h \in H$, i.e. A is a union of left cosets of H. Since $1_G \in A$, it is clear that $H \subseteq A$. □

As a consequence of this Proposition, the decomposition (2.6) of an H-global G-partial representation (V, π) simplifies to

$$V = \bigoplus_{A\in\mathcal{P}_H(G/H)} V^A.$$

In particular, if $w \in V^A = P_A(V)$, then for any $g \in G$

$$\pi(g)(w) \;=\; \pi(g)P_A(w) \stackrel{(2.4)}{=} P_{gA}\pi(g)(w),$$

so that

$$\pi(g)(w) \in V^{gA} = P_{gA}(V).$$

The following results are two interesting and useful consequences of Lemma 2.17 that considerably simplify the computations when dealing with H-global G-partial representations and the projections P_A. In particular, the following proposition gives an alternative characterization of H-global G-partial representations.

Proposition 2.20. *A pair (V, π), with V a vector space and $\pi \colon G \to \mathsf{End}(V)$, is an H-global G-partial representation if and only if*

(GPR1) $\pi(1_G) = \mathsf{Id}_V$;
(GPR2) $\pi(\bar{g})\pi(g)\pi(h) = \pi(\bar{g})\pi(gh)$ *for any* $\bar{g}, g, h \in G$ *such that* $\bar{g}g \in H$;
(GPR3) $\pi(g)\pi(h)\pi(\bar{h}) = \pi(gh)\pi(\bar{h})$ *for any* $g, h, \bar{h} \in G$ *such that* $h\bar{h} \in H$.

In particular, for an H-global G-partial representation conditions (PR2) and (PR3) of the definition of a G-partial representation are satisfied by taking as "witness" any element in the same coset.

Proof. If (V,π) satisfies the stated identities, then clearly it is a G-partial representation. To see that it is H-global, just notice that if $g,h\in H$, then we can choose $\bar{g}=1_G=\bar{h}$.

Conversely, by using Lemma 2.17, if (V,π) is an H-global G-partial representation, then for $\bar{g},g,h\in G$ with $\bar{g}g\in H$ we have

$$\begin{aligned}\pi(\bar{g})\pi(g)\pi(h)&=\pi(\bar{g}gg^{-1})\pi(g)\pi(h)\overset{\text{(PR3)}}{=}\pi(\bar{g}g)\pi(g^{-1})\pi(g)\pi(h)\\&\overset{\text{(PR2)}}{=}\pi(\bar{g}g)\pi(g^{-1})\pi(gh)\overset{(2.7)}{=}\pi(\bar{g}gg^{-1})\pi(gh)=\pi(\bar{g})\pi(gh).\end{aligned}$$

The other property is shown in an analogous way. ☐

Proposition 2.21. *Let (V,π) be an H-global G-partial representation. Then the assignment $G\to\mathsf{End}(V):g\mapsto\pi(g)\pi(g^{-1})$ is constant on the left cosets of H in G. In particular, if $\{g_1,\dots,g_r\}$ is a family of representatives of left cosets of H in G, that is, if $G=\coprod_{i=1}^r g_iH$, then*

$$P_A^\pi=\prod_{g_kH\subseteq A}\pi(g_k)\pi(g_k^{-1})\cdot\prod_{g_iH\subseteq G\setminus A}(\mathsf{Id}_V-\pi(g_i)\pi(g_i^{-1}))\tag{2.8}$$

for every $A\in\mathcal{P}_H(G/H)$ and the expression (2.8) does not depend on the chosen representatives.

Proof. In view of Lemma 2.17, we have for any $g\in G,h\in H$

$$\pi(gh)\pi((gh)^{-1})=\pi(gh)\pi(h^{-1}g^{-1})=\pi(g)\pi(h)\pi(h^{-1})\pi(g^{-1})=\pi(g)\pi(g^{-1}),$$

thus proving the first claim. The second claim is now evident, since the elements $\pi(g)\pi(g^{-1})$ are commuting idempotents (Lemma 1.9). ☐

3. Irreducible partial representations

In this section we develop the general theory of H-global G-partial representations, describing in particular its irreducibles.

3.1. **Partial representations as modules.** We will now develop an algebra that plays for H-global G-partial representations the same role as the partial group algebra $\mathbb{C}_{par}G$ does for G-partial representations. Recall ([DEP00, Definition 2.4]) that $\mathbb{C}_{par}G$ is the $\mathbb{C}$-algebra generated by the symbols $\{[g]\mid g\in G\}$ subject to the relations

$$[1_G]=1,\qquad[h^{-1}][h][g]=[h^{-1}][hg],\qquad[g][h^{-1}][h]=[gh^{-1}][h]\qquad\text{for all } g,h\in G.\tag{3.1}$$

Definition 3.1. The $\mathbb{C}$-algebra $\mathbb{C}_{par}^HG$ is generated by the symbols $\{[g]\mid g\in G\}$ subject to the relations

$$[1_G]=1,\qquad[\bar{h}][h][g]=[\bar{h}][hg],\qquad[g][\bar{h}][h]=[g\bar{h}][h],\tag{3.2}$$

for all $g,h,\bar{h}\in G$ such that $\bar{h}h\in H$.

Proposition 3.2. *The algebra $\mathbb{C}_{par}^HG$ is isomorphic to the quotient of $\mathbb{C}_{par}G$ by the ideal generated by $\{[h][h^{-1}]-[1_G]\mid h\in H\}$.*

Proof. For the sake of simplicity, call R the free algebra generated by $\{[g]\mid g\in G\}$, I the ideal of R generated by the relations (3.1) and J the ideal of R generated by the relations (3.2). By double inclusion one observes that $J=\bar{I}\coloneqq I+\langle[h][h^{-1}]-[1_G]\mid h\in H\rangle$. In fact, on the one hand

$$[h][h^{-1}]-[1_G]=\Big(1-[1_G]\Big)[h][h^{-1}]+\Big([1_G][h][h^{-1}]-[1_G][hh^{-1}]\Big)+[1_G]\Big([1_G]-1\Big)\in J$$

for all $h\in H$, because $1_Gh=h\in H$, and hence $\bar{I}\subseteq J$. On the other hand, notice that we can deduce from (3.1) also the following relations for all $g,h,\bar{h}\in G$:

$$[\bar{h}h][h^{-1}\bar{h}^{-1}][\bar{h}][hg]=[\bar{h}h][h^{-1}\bar{h}^{-1}\bar{h}][hg]=[\bar{h}][h][h^{-1}][hg]=[\bar{h}][h][h^{-1}hg]=[\bar{h}][h][g].$$

Hence for all $g,h,\bar{h}\in G$ such that $\bar{h}h\in H$ we have

$$[\bar{h}][hg]-[\bar{h}][h][g]=\Big(1-[\bar{h}h][h^{-1}\bar{h}^{-1}]\Big)[\bar{h}][hg]+\Big([\bar{h}h][h^{-1}\bar{h}^{-1}][\bar{h}][hg]-[\bar{h}][h][g]\Big)\in\bar{I}.$$

For the other relations in (3.2) a similar argument applies, thus proving $J\subseteq\bar{I}$.

Set now $\bar{J}$ to be the ideal generated by $\{[h][h^{-1}] - [1_G] \mid h \in H\}$ in $\mathbb{C}_{par}G$. We clearly have that $J/I = \big(I + \langle [h][h^{-1}] - [1_G] \mid h \in H\rangle\big)/I = \bar{J}$ and hence

$$\mathbb{C}^H_{par}G = R/J \cong \frac{R/I}{J/I} = \mathbb{C}_{par}G \Big/ \bar{J}. \qquad \square$$

In the forthcoming Section 3.3, we will give an important realization of such an abstract algebra as the groupoid algebra of a certain groupoid naturally associated to G and H.

Remark 3.3. Any $\mathbb{C}$-algebra R can be realized as a subalgebra of an algebra of endomorphisms via

$$\lambda\colon R \to \mathsf{End}_{\mathbb{C}}(R), \qquad a \mapsto (\lambda_a\colon b \mapsto ab)\,.$$

If we consider the obvious composition $G \to \mathbb{C}^H_{par}G \to \mathsf{End}_{\mathbb{C}}\left(\mathbb{C}^H_{par}G\right), g \mapsto \lambda_{[g]}$, then it follows from the defining relations of $\mathbb{C}^H_{par}G$ that this defines an H-global G-partial representation of G in the $\mathbb{C}$-vector space $\mathbb{C}^H_{par}G$.

Theorem 3.4. *Given an H-global G-partial representation $\pi : G \to \mathsf{End}(V)$, there exists a unique morphism of $\mathbb{C}$-algebras $\phi_\pi\colon \mathbb{C}^H_{par}G \to \mathsf{End}(V)$ such that $\phi_\pi([g]) = \pi(g)$ for all $g \in G$. This makes V into a left $\mathbb{C}^H_{par}G$-module. Conversely, given any left $\mathbb{C}^H_{par}G$-module V with action $\mu\colon \mathbb{C}^H_{par}G \times V \to V$, the assignment*

$$\pi_\mu\colon G \to \mathsf{End}(V), \qquad g \mapsto [v \mapsto \mu([g], v)]$$

is an H-global G-partial representation. These correspondences induce an isomorphism of categories

$$\Phi\colon \mathsf{PRep}^H_G \;\to\; \mathbb{C}^H_{par}G\text{-}\mathsf{Mod}$$

between the category PRep^H_G of H-global G-partial representations and the category $\mathbb{C}^H_{par}G$-Mod of $\mathbb{C}^H_{par}G$-modules.

Proof. Let (V, π) be an H-global G-partial representation. Consider the unique algebra morphism $\psi_\pi\colon \mathbb{C}_{par}G \to \mathsf{End}(V)$ such that $\psi_\pi([g]) = \pi(g)$ for every $g \in G$. Clearly,

$$\langle [h][h^{-1}] - [1_G] \mid h \in H\rangle \subseteq \ker(\psi_\pi),$$

hence ψ_π induces a well-defined algebra map $\phi_\pi : \mathbb{C}^H_{par}G \to \mathsf{End}(V)$ which is unique with the property that $\phi_\pi([g]) = \pi(g)$ for every $g \in G$. Conversely, assume that V is a $\mathbb{C}^H_{par}G$-module with action $\mu\colon \mathbb{C}^H_{par}G \times V \to V$ and consider the map

$$\pi_\mu\colon G \to \mathsf{End}(V), \qquad g \mapsto [v \mapsto \mu([g], v)]\,.$$

It satisfies $\pi_\mu(1_G)(v) = \mu([1_G], v) = \mu(1, v) = v$ for all $v \in V$, hence $\pi_\mu(1_G) = \mathsf{Id}_V$. Furthermore,

$$\begin{aligned}\pi_\mu(\bar{g})\pi_\mu(g)\pi_\mu(h)(v) &= \mu([\bar{g}], \mu([g], \mu([h], v))) = \mu([\bar{g}][g][h], v)\\ &= \mu([\bar{g}][gh], v) = \pi_\mu(\bar{g})\pi_\mu(gh)(v)\end{aligned}$$

for all $v \in V$ and $\bar{g}, g, h \in G$ such that $\bar{g}g \in H$, thus $\pi_\mu(\bar{g})\pi_\mu(g)\pi_\mu(h) = \pi_\mu(\bar{g})\pi_\mu(gh)$. In an analogous way, one may check that $\pi_\mu(h)\pi_\mu(\bar{g})\pi_\mu(g) = \pi_\mu(h\bar{g})\pi_\mu(g)$, so that π_μ is H-global G-partial by Proposition 2.20. Now, assume that $f\colon V \to W$ is a morphism between the H-global G-partial representations (V, π_V) and (W, π_W). Recall that this means that

$$f(\pi_V(g)(v)) = \pi_W(g)(f(v))$$

for all $v \in V$ and $g \in G$. Using the notations above, we compute

$$f(\phi_{\pi_V}([g])(v)) = f(\pi_V(g)(v)) = \pi_W(g)(f(v)) = \phi_{\pi_W}([g])(f(v))$$

which implies that f is $\mathbb{C}^H_{par}G$-linear. In fact, by definition of the action of $\mathbb{C}^H_{par}G$ on V, f is $\mathbb{C}^H_{par}G$-linear if and only if $f \circ \pi_V(g) = \pi_W(g) \circ f$. Notice also that if (V, π) is an H-global G-partial representation, then

$$\begin{aligned}\pi_{\mu_\pi}(g)(v) &= \mu_\pi([g], v) = \phi_\pi([g])(v) = \pi(g)(v),\\ \mu_{\pi_\mu}([g], v) &= \phi_{\pi_\mu}([g])(v) = \pi_\mu(g)(v) = \mu([g], v),\end{aligned}$$

for all $g \in G$ and $v \in V$, whence $\pi_{\mu_\pi} = \pi$, and $\mu_{\pi_\mu} = \mu$. Therefore, the assignments

$$\begin{array}{ccc}\mathsf{PRep}^H_G & \longleftrightarrow & \mathbb{C}^H_{par}G\text{-}\mathsf{Mod}\\ (V, \pi) & \longleftrightarrow & (V, \mu_\pi)\\ {[f : (V, \pi_V) \to (W, \pi_W)]} & \longleftrightarrow & {[f : (V, \mu_{\pi_V}) \to (W, \mu_{\pi_W})]}\end{array}$$

provide well-defined functors between PRep_G^H and $\mathbb{C}_{par}^H G$-Mod, which form an isomorphism of categories. □

In less categorical terms, Theorem 3.4 says, in particular, that notions like being isomorphic, being irreducible, being a direct sum, for objects in the two categories are in 1-to-1 correspondence via the stated isomorphism. For instance, an object is irreducible in PRep_G^H if and only if the corresponding object in $\mathbb{C}_{par}^H G$-Mod is irreducible.

3.2. $\mathbb{C}_{par}^H G$ as a partial skew group algebra.

Definition 3.5. We say that a group G *acts partially on an algebra* A if there is a partial action $\alpha = (\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on A as in Definition 1.1, such that for any $g \in G$, $A_g = A1_g$, where 1_g is a central idempotent in A, and α_g is multiplicative (cf. [DE05]).

Recall from [DE05, Theorem 6.9] (cf. also [ABV15, Theorem 2.10]) that the partial group algebra can be viewed as a partial skew group algebra $A \rtimes_\alpha G$, associated to a partial action of G on a suitably constructed commutative algebra A. Recall also that the partial skew group algebra is the vector space

$$A \rtimes_\alpha G = \bigoplus_{g\in G} A_g \delta_g,$$

where the elements δ_g are just symbols, with multiplication given by

$$(a_g\delta_g)(b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}.$$

A similar result holds for the partial group algebra $\mathbb{C}_{par}^H(G)$ we constructed above.

To see this, consider for any element $g \in G$ the associated element

$$\varepsilon_g := [g][g^{-1}]$$

in $\mathbb{C}_{par}^H(G)$, which is an idempotent. Then for any element $h \in H$, we find that

$$\varepsilon_{gh} = [gh][h^{-1}g^{-1}] = [gh][1_G][h^{-1}g^{-1}] \overset{\text{Prop. 3.2}}{=} [g][h][h^{-1}][g^{-1}] \overset{(3.2)}{=} [g][g^{-1}] = \varepsilon_g.$$

Consequently, we can associate exactly one idempotent ε_g to each left coset of H. As in the case of the usual partial group algebra, we have the commutativity relation $[g]\varepsilon_h = \varepsilon_{gh}[g]$ and so the idempotents commute among each other (cf. Lemma 1.9). Let A be the subalgebra of $\mathbb{C}_{par}^H(G)$ generated by these idempotents. Then we have the following result.

Proposition 3.6. *The group G acts partially on the algebra A in such a way that the restricted action of H on A is global. Moreover, $\mathbb{C}_{par}^H(G)$ is isomorphic to the associated partial skew group algebra $A \rtimes_\alpha G$.*

Proof. The domains of the partial action of G are given by $A_g = A\varepsilon_g$. The partial isomorphisms are given by

$$\alpha_g(\varepsilon_{g_1}\cdots\varepsilon_{g_n}\varepsilon_{g^{-1}}) = \varepsilon_{gg_1}\cdots\varepsilon_{gg_n}\varepsilon_g.$$

This is easily checked to be a partial action. Since for all $h \in H$ we have $\varepsilon_h = 1$, the domain A_h of h is the whole of A, hence this partial action is H-global. Concerning the second part of the statement, one can check that the maps

$$\mathbb{C}_{par}^H(G) \to A \rtimes_\alpha G, \quad [g_1][g_2]\cdots[g_n] \mapsto \varepsilon_{g_1g_2\cdots g_n}\cdots\varepsilon_{g_1g_2}\varepsilon_{g_1}\delta_{g_1\cdots g_n}$$

and

$$A \rtimes_\alpha G \to \mathbb{C}_{par}^H(G), \quad \varepsilon_g\varepsilon_{g_1}\cdots\varepsilon_{g_k}\delta_g \mapsto \varepsilon_g\varepsilon_{g_1}\cdots\varepsilon_{g_k}[g]$$

are mutual inverse algebra maps. □

3.3. The groupoid algebra $\mathbb{C}\Gamma_H(G)$.

It can be deduced from [DEP00, Theorem 2.6] that G-partial representations correspond to (usual) representations of the groupoid algebra of a certain groupoid $\Gamma(G)$. The idea is to extend this definition from [DEP00] in order to handle H-global G-partial representations. The definitions and results of [DEP00] can be recovered by setting $H = \{1_G\}$.

In this section we will introduce a new groupoid $\Gamma_H(G)$ naturally associated to G and H and show that the universal algebra constructed in Section 3.1 is isomorphic to $\mathbb{C}\Gamma_H(G)$, thus inducing an isomorphism of categories between the category of H-global G-partial representations and that of modules over this groupoid algebra. To this aim, recall the following construction from [Aba04, §2] (cf. [KL04] or [Bat17, page 42] for the equivalent formulation we are using here).

Remark 3.7. Let $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on X and define

$$\mathcal{G} = \mathcal{G}(G, X, \alpha) := \{(x, g) \in X \times G \mid x \in X_{g^{-1}}\}.$$

Then $\mathcal{G}$ is a groupoid with set of objects $\mathcal{G}_0 := X$, source $s(x, g) = x$, range $r(x, g) = \alpha_g(x)$, units $(x, 1_G)$ for $x \in X$ and composition $(x, g) \cdot (y, h) = (y, gh)$ whenever $x = \alpha_h(y)$.

Consider the partial action $\alpha^{\mathcal{P}}$ of G on $X = \mathcal{P}_H(G/H)$ (see Notation 2.11) given by

$$X_g = \{A \in \mathcal{P}_H(G/H) \mid gH \subseteq A\} \qquad \text{and} \qquad \alpha^{\mathcal{P}}_g : X_{g^{-1}} \to X_g, \quad A \mapsto gA,$$

for all $g \in G$.

Definition 3.8. We set $\Gamma_H(G) := \mathcal{G}(G, \mathcal{P}_H(G/H), \alpha^{\mathcal{P}})$, so that the composition is given by

$$(A, g) \cdot (B, h) := \begin{cases} (B, gh) & \text{if } A = hB, \\ \text{not defined} & \text{otherwise} \end{cases}$$

the source map by $s(A, g) = A$, the range map by $r(A, g) = gA$ and the units by $(A, 1_G)$ for $A \in \mathcal{P}_H(G/H)$. The inverse of (A, g) is given by (gA, g^{-1}).

Remark 3.9. In the special case $H = \{1_G\}$ we recover the groupoid $\Gamma(G)$ from [DEP00].

Recall that if x is a vertex of a groupoid Γ, then the *isotropy group* K_x of x is

$$K_x := \{\gamma \in \Gamma \mid s(\gamma) = x = r(\gamma)\}.$$

A groupoid Γ is *connected* if for any two objects x, y of Γ there is an arrow connecting them. We often write $\Gamma = \coprod_i \Gamma_i$, where Γ_i are the connected components of Γ. Despite the fact that $\Gamma_H(G)$ is not connected in general, its connected components (maximal connected subgroupoids) are easily described by the following Lemma 3.10.

Lemma 3.10. *Let $A \in \mathcal{P}_H(G/H)$ and let $K_A := \{g \in G \mid gA = A\}$ be the stabilizer of A in G (i.e. the isotropy group of A in $\Gamma_H(G)$). If $A = \coprod_{j=1}^m K_A t_j$ as disjoint union of right cosets of K_A, then the set of distinct objects in the connected component of $\Gamma_H(G)$ containing A is $\{t_j^{-1}A \mid j = 1, 2, \dots, m\}$.*

Proof. Let Γ be the connected component of the groupoid $\Gamma_H(G)$ containing A. Any other object in Γ is the range $r(A, g^{-1}) = g^{-1}A$ of an element (A, g^{-1}) of $\Gamma_H(G)$ for some $g \in A$. If $g \in A$, then $g = kt_j$ for some $j \in \{1, \dots, m\}$ and some $k \in K_A$ and so $g^{-1}A = t_j^{-1}k^{-1}A = t_j^{-1}A$ is one of the objects of Γ. Moreover, $t_j^{-1}A = t_\ell^{-1}A$ implies $t_\ell t_j^{-1} \in K_A$, so that $K_A t_j = K_A t_\ell$ and hence $j = \ell$. Therefore, the objects $t_j^{-1}A$ are distinct. □

The groupoid algebra $\mathbb{C}\Gamma_H(G)$ corresponding to $\Gamma_H(G)$ has $\Gamma_H(G) \subseteq \mathbb{C}\Gamma_H(G)$ as basis over $\mathbb{C}$ and the product of two elements from $\Gamma_H(G)$ is given by their composition in $\Gamma_H(G)$ and it is set to be 0 when it is not defined.

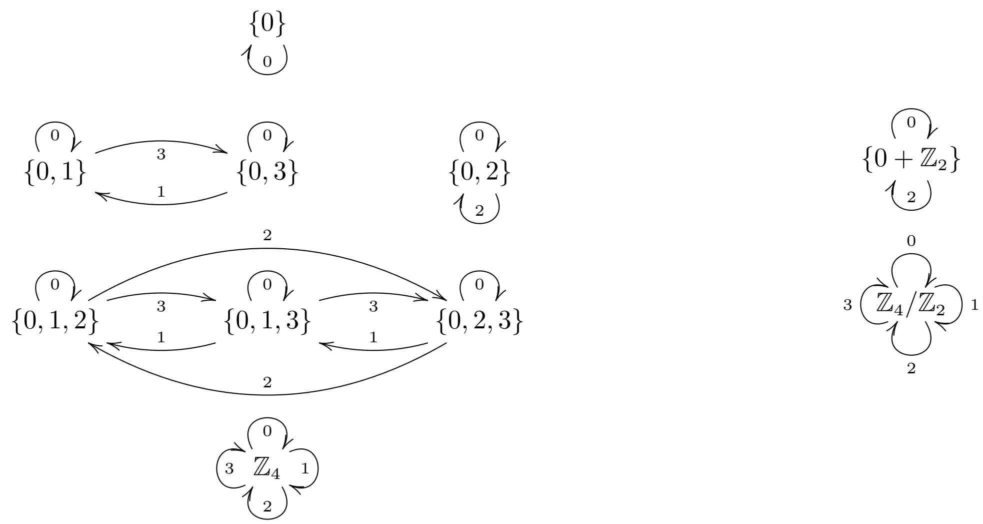
Proposition 3.11. *We have*

$$\dim_{\mathbb{C}}(\mathbb{C}\Gamma_H(G)) = 2^{|G/H|-2}(|G| + |H|). \tag{3.3}$$

Proof. The dimension of $\mathbb{C}\Gamma_H(G)$ over $\mathbb{C}$ is given by $|\Gamma_H(G)|$ which is easily computed as

$$|\Gamma_H(G)| = 2^{|G/H|-2}(|G| - |H|) + 2^{|G/H|-1}|H| = 2^{|G/H|-2}(|G| + |H|).$$ □

Example 3.12. The following picture represents $\Gamma_{\{0\}}\mathbb{Z}_4$ (on the left) and $\Gamma_{\mathbb{Z}_2}\mathbb{Z}_4$ (on the right).



An additional example representing $\Gamma_{\mathfrak{S}_2}\mathfrak{S}_3$ can be found in the forthcoming Example 5.1.

Remark 3.13. Notice that the algebra $\mathbb{C}\Gamma_H(G)$ becomes "easier to handle" (certainly smaller) when $|G/H|$ is small compared to $|G|$. This is one of our main motivations to develop our theory: we will put this into practice in Section 5.1 when we will make explicit computations in the case $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$.

In the following lemma we introduce a fundamental map from G to the space $\mathbb{C}\Gamma_H(G)$: this will give us the link between the H-global G-partial representations and the representations of $\mathbb{C}\Gamma_H(G)$.

Lemma 3.14. *The map $\mu_p\colon G \to \mathbb{C}\Gamma_H(G)$ defined by*

$$\mu_p(g) := \sum_{A \ni g^{-1}} (A, g) \quad \text{for all } g \in G \tag{3.4}$$

satisfies the following properties:

$$\begin{aligned} \mu_p(1_G) &= 1_{\mathbb{C}\Gamma_H(G)}, \\ \mu_p(\bar{g})\mu_p(g)\mu_p(h) &= \mu_p(\bar{g})\mu_p(gh), \\ \mu_p(h)\mu_p(\bar{g})\mu_p(g) &= \mu_p(h\bar{g})\mu_p(g) \end{aligned} \tag{3.5}$$

for any $\bar{g}, g, h \in G$ such that $\bar{g}g \in H$. In particular, the map $L_{\mu_p}\colon G \to \mathsf{End}(\mathbb{C}\Gamma_H(G))$ defined by setting

$$L_{\mu_p}(g)(x) := \mu_p(g) \cdot x \quad \text{for all } g \in G, x \in \mathbb{C}\Gamma_H(G)$$

gives an H-global G-partial representation.

Proof. The first identity stated in (3.5), i.e. $\mu_p(1_G) = \sum_A (A, 1_G) = 1_{\mathbb{C}\Gamma_H(G)}$, is clear.

Observe now that since

$$\left(\sum_{A \ni g^{-1}} (A, g)\right)(B, h) = \begin{cases} (B, gh) & \text{if } (gh)^{-1} \in B \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

we have

$$\mu_p(g)\mu_p(h) = \left(\sum_{A \ni g^{-1}} (A, g)\right)\left(\sum_{B \ni h^{-1}} (B, h)\right) = \sum_{B \supseteq \{(gh)^{-1}, h^{-1}\}} (B, gh). \tag{3.7}$$

Assume that

$$\bar{g}g \in H. \tag{3.8}$$

Then, on the one hand

$$\mu_p(\bar g)\mu_p(g)\mu_p(h) = \sum_{A\ni \bar g^{-1}} (A,\bar g) \sum_{B\ni g^{-1}} (B,g) \sum_{C\ni h^{-1}} (C,h) \overset{(3.7)}{=} \sum_{A\ni \bar g^{-1}} (A,\bar g) \sum_{C\supseteq\{(gh)^{-1},h^{-1}\}} (C,gh)$$
$$\overset{(3.7)}{=} \sum_{C\supseteq\{(\bar g gh)^{-1},(gh)^{-1},h^{-1}\}} (C,\bar g gh) \overset{(3.8)}{=} \sum_{C\supseteq\{(gh)^{-1},h^{-1}\}} (C,\bar g gh).$$

On the other hand

$$\mu_p(\bar g)\mu_p(gh) = \sum_{A\ni \bar g^{-1}} (A,\bar g) \sum_{B\ni (gh)^{-1}} (B,gh) \overset{(3.7)}{=} \sum_{B\supseteq\{(\bar g gh)^{-1},(gh)^{-1}\}} (B,\bar g gh) \overset{(3.8)}{=} \sum_{B\supseteq\{h^{-1},(gh)^{-1}\}} (B,\bar g gh),$$

whence $\mu_p(\bar g)\mu_p(g)\mu_p(h) = \mu_p(\bar g)\mu_p(gh)$ and the second identity in (3.5) is satisfied. The third identity in (3.5) is proved in a similar way. The last statement follows immediately from Proposition 2.20. □

Lemma 3.15. *Let $H \subseteq G$ be a subgroup. The relations*

$$(A,1_G) = \prod_{g\in A} \mu_p(g)\mu_p(g^{-1}) \prod_{\bar g\in G\setminus A} (1_{\mathbb{C}\Gamma_H(G)} - \mu_p(\bar g)\mu_p(\bar g^{-1})) \qquad \text{and} \tag{3.9}$$
$$(A,g') = \mu_p(g') \prod_{g\in A} \mu_p(g)\mu_p(g^{-1}) \prod_{\bar g\in G\setminus A} (1_{\mathbb{C}\Gamma_H(G)} - \mu_p(\bar g)\mu_p(\bar g^{-1}))$$

hold in $\mathbb{C}\Gamma_H(G)$ for all $A \in \mathcal{P}_H(G/H)$ and all $g'^{-1} \in A$.

Proof. All subsets of G are tacitly assumed to be in $\mathcal{P}_H(G/H)$. In light of (3.7) we know that

$$\mu_p(g)\mu_p(g^{-1}) = \sum_{B\ni g} (B,1_G) \tag{3.10}$$

for all $g \in G$. Recalling that

$$(A,1_G)(B,1_G) = \begin{cases} (B,1_G) & \text{if } A = B \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

we compute

$$\begin{aligned} &\mu_p(g_1)\mu_p(g_1^{-1})\mu_p(g_2)\mu_p(g_2^{-1})\cdots\mu_p(g_t)\mu_p(g_t^{-1}) = \\ &\overset{(3.10)}{=} \Big(\sum_{B\ni g_1}(B,1_G)\Big)\Big(\sum_{B\ni g_2}(B,1_G)\Big)\cdots\Big(\sum_{B\ni g_t}(B,1_G)\Big) \\ &\overset{(3.11)}{=} \sum_{B\supseteq\{g_1,\dots,g_t\}} (B,1_G). \end{aligned} \tag{3.12}$$

Summing up,

$$\begin{gathered} \prod_{g\in A} \mu_p(g)\mu_p(g^{-1}) \overset{(3.12)}{=} \sum_{B\supseteq A}(B,1_G) \qquad \text{and} \\ \prod_{\bar g\in G\setminus A} (1_{\mathbb{C}\Gamma_H(G)} - \mu_p(\bar g)\mu_p(\bar g^{-1})) \overset{(3.10)}{=} \prod_{\bar g\in G\setminus A}\Big(\sum_{B\not\ni \bar g}(B,1_G)\Big) \overset{(3.11)}{=} \sum_{B\subseteq A}(B,1_G). \end{gathered} \tag{3.13}$$

Thus,

$$\prod_{g\in A} \mu_p(g)\mu_p(g^{-1}) \prod_{\bar g\in G\setminus A} (1_{\mathbb{C}\Gamma_H(G)} - \mu_p(\bar g)\mu_p(\bar g^{-1})) \overset{(3.13)}{=} \Big(\sum_{B\supseteq A}(B,1_G)\Big)\Big(\sum_{C\subseteq A}(C,1_G)\Big) \overset{(3.11)}{=} (A,1_G). \tag{3.14}$$

Finally, if $g'^{-1} \in A$, then

$$\mu_p(g') \prod_{g\in A} \mu_p(g)\mu_p(g^{-1}) \prod_{\bar g\in G\setminus A} (1_{\mathbb{C}\Gamma_H(G)} - \mu_p(\bar g)\mu_p(\bar g^{-1})) \overset{(3.14)}{=} \Big(\sum_{B\ni g'^{-1}} (B,g')\Big)(A,1_G) \overset{(3.6)}{=} (A,g'). \qquad \square$$

We are now ready to that the algebras $\mathbb{C}\Gamma_H(G)$ and $\mathbb{C}^H_{par}G$ are isomorphic. We will use the map $\mu_p\colon G \to \mathbb{C}\Gamma_H(G)$ from Lemma 3.14.

Theorem 3.16. *The map $\mu_p\colon G \to \mathbb{C}\Gamma_H(G)$ induces an isomorphism of $\mathbb{C}$-algebras*

$$\begin{aligned} \mathbb{C}^H_{par}G &\longleftrightarrow \mathbb{C}\Gamma_H(G)\\ [g] &\longmapsto \sum_{A\ni g^{-1}}(A,g)\\ [g]\cdot[P_A] &\longleftarrow (A,g) \end{aligned}$$

where $[P_A] \coloneqq \prod_{g\in A}[g][g^{-1}]\prod_{\bar{g}\in G\setminus A}(1-[\bar{g}][\bar{g}^{-1}])$.

Proof. Consider the extension of the correspondence (3.4) to the free $\mathbb{C}$-algebra generated by the symbols $\{[g] \mid g \in G\}$, i.e.

$$\hat{\mu}_p([g]) \coloneqq \mu_p(g).$$

It follows from the relations (3.5) that $\hat{\mu}_p$ factors through the quotient defining $\mathbb{C}^H_{par}G$. As a consequence we have a well-defined $\mathbb{C}$-algebra morphism

$$\mu\colon \mathbb{C}^H_{par}G \to \mathbb{C}\Gamma_H(G), \qquad [g] \mapsto \sum_{A\ni g^{-1}}(A,g).$$

In the other direction, consider the assignment

$$\mu^{-1}\colon \mathbb{C}\Gamma_H(G) \to \mathbb{C}^H_{par}G, \qquad (A,g) \mapsto [g]\cdot[P_A].$$

A direct computation shows that, for $g'^{-1} \in A$,

$$\mu(\mu^{-1}((A,g'))) = \mu([g']\cdot[P_A]) = \mu_p(g')\prod_{g\in A}\mu_p(g)\mu_p(g^{-1})\prod_{\bar{g}\in G\setminus A}(1-\mu_p(\bar{g})\mu_p(\bar{g}^{-1})) \overset{(3.9)}{=} (A,g').$$

To prove that the other composition is also the identity, observe that the elements $[P_A]$ in $\mathbb{C}^H_{par}G$ satisfy the same identities as the elements P^π_A associated to an H-global G-partial representation (V,π) that we saw in Section 2.3: indeed, to prove those properties, we only used the analogues (see Proposition 2.20) of the defining relations (3.2).

So, combining Lemma 2.16 with Proposition 2.19, we have

$$\mu^{-1}(\mu([g])) = \mu^{-1}\left(\sum_{A\ni g^{-1}}(A,g)\right) = \sum_{A\ni g^{-1}}[g]\cdot[P_A] \overset{(*)}{=} \sum_{\substack{A\ni g^{-1}\\ A\in\mathcal{P}_H(G/H)}}[g]\cdot[P_A] \overset{(**)}{=} [g]\left(\sum_{A\in\mathcal{P}_H(G/H)}[P_A]\right) = [g],$$

where $(*)$ follows from the fact that $[P_A]=0$ if A is not a union of left cosets of H or if $H \not\subseteq A$ and $(**)$ follows from the fact that $[g]\cdot[P_A]=0$ if $g^{-1}\notin A$ in view of (2.5). ☐

Corollary 3.17. *The H-global G-partial representations of G are in one-to-one correspondence with the (global) representations of $\mathbb{C}\Gamma_H(G)$. Namely, this correspondence is an isomorphism of categories. More precisely, given an algebra homomorphism $\tilde{\pi}\colon \mathbb{C}\Gamma_H(G) \to \mathsf{End}(V)$, this determines an H-global G-partial representation (V,π) with $\pi \coloneqq \tilde{\pi}\circ\mu_p$; conversely, given an H-global G-partial representation (V,π), there exists a unique algebra homomorphism $\tilde{\pi}\colon \mathbb{C}\Gamma_H(G) \to \mathsf{End}(V)$ such that $\pi = \tilde{\pi}\circ\mu_p$.*

Proof. It follows from Theorem 3.4 and Theorem 3.16. ☐

3.4. **Representation theory of $\mathbb{C}\Gamma_H(G)$.** In this section we describe the representation theory of the algebra $\mathbb{C}\Gamma_H(G)$. Some of the results (including their proofs) are natural extensions of results in [DEP00]. The main point of our discussion is to make explicit the general constructions in our specific situation.

In [DEP00, Theorem 3.2] it was shown that $\mathbb{C}\Gamma_{\{1_G\}}(G)$ is a direct product of matrix algebras over the group algebras of the subgroups of G, hence $\mathbb{C}\Gamma_{\{1_G\}}(G)$ is a semisimple algebra. Here we use the same arguments to show that $\mathbb{C}\Gamma_H(G)$ is a semisimple algebra; this time the direct product runs only over certain subgroups.

Definition 3.18. Let K be a finite group and $m \in \mathbb{N}$. Denote by

$$\Gamma_m^K = \{(k,i,j) \mid k \in K; i,j \in \{1,\dots,m\}\}$$

the *trivial groupoid* on the set $\{1,\dots,m\}$ with group K (see, for instance, [Mac87, Example 1.4] for the terminology). Source and range maps are given by

$$s(k,i,j) = j \quad \text{and} \quad r(k,i,j) = i,$$

and the composition law by

$$(k,i,j)\cdot(k',i',j') = \begin{cases}(kk',i,j') & \text{if } j = i',\\ \text{not defined} & \text{if } j \neq i'.\end{cases}$$

The following proposition is proved in [DEP00, Proposition 3.1] (see also [Ste06, Theorem 3.2]).

Proposition 3.19. *Let Γ be a connected groupoid having a finite number m of vertices. Let x be a vertex of Γ and K the isotropy group of x. Then $\Gamma \cong \Gamma_m^K$ and $\mathbb{C}\Gamma \cong M_m(\mathbb{C}[K])$.*

If the number of vertices of Γ is finite (which is the case for $\Gamma_H(G)$ when G/H is finite), then

$$\mathbb{C}\Gamma = \prod_i \mathbb{C}\Gamma_i \cong \prod_i M_{m_i}(\mathbb{C}[K_i]) \cong \prod_{i,j} M_{m_i\cdot n_j(i)}(\mathbb{C}) \tag{3.15}$$

where Γ_i are the connected components of Γ, K_i is the isotropy group of an object of Γ_i, $m_i = |\Gamma_i|$ and where $\mathbb{C}[K_i] \cong \prod_j M_{n_j(i)}(\mathbb{C})$ is the Artin-Wedderburn decomposition of $\mathbb{C}[K_i]$. In particular, $\mathbb{C}\Gamma$ is semisimple.

Now we want to better understand the case of the groupoid $\Gamma_H(G)$, in particular which groups K_i appear and how the numbers m_i are related to them.

Remark 3.20. Let us pick an object $A_i \in \mathcal{P}_H(G/H)$ in a connected component Γ_i of the groupoid $\Gamma_H(G)$ and let $K_i := K_{A_i}$ be its stabilizer. By Lemma 3.10, the number $m_i = m_{A_i}$ of objects in the connected component Γ_i coincides with $|A_i|/|K_i|$. By Proposition 3.19, we have $\mathbb{C}\Gamma_i \cong M_{m_i}(\mathbb{C}[K_i])$.

Notation 3.21. As we have seen in Example 3.12, in general $\Gamma_H(G)$ is not connected. Since any object A of $\Gamma_H(G)$ is a union of cosets of H in G, one can write $\Gamma_H(G)$ as disjoint union of its connected components and distinguish with an index the ones with objects of the same cardinality. Namely, we write $\Gamma_H(G) = \coprod_{i=1}^{[G:H]}\coprod_{j=1}^{c_i}\Gamma_{i,j}$, where $\Gamma_{i,j}$ are connected components of $\Gamma_H(G)$, any object $A_{i,j}$ in $\Gamma_{i,j}$ has cardinality $i\cdot|H|$ for all $j = 1,2,\dots,c_i$ and c_i is exactly the number of connected components whose objects have cardinality $i\cdot|H|$.

The following theorem is now clear.

Theorem 3.22. *Let $\Gamma_H(G) = \coprod_{i=1}^{[G:H]}\coprod_{j=1}^{c_i}\Gamma_{i,j}$ be the decomposition into connected components of $\Gamma_H(G)$, as in Notation 3.21. Let $A_{i,j}$ be an object of $\Gamma_{i,j}$, $K_{i,j} := K_{A_{i,j}}$ and $m_{i,j} := m_{A_{i,j}}$ as in Remark 3.20, for all i and j. Then we have the following algebra isomorphism*

$$\mathbb{C}\Gamma_H(G) \cong \prod_{i=1}^{[G:H]}\prod_{j=1}^{c_i} M_{m_{i,j}}(\mathbb{C}[K_{i,j}]) \cong \prod_{i=1}^{[G:H]}\prod_{j=1}^{c_i}\prod_{k=1}^{e_{i,j}} M_{m_{i,j}\cdot d_k(i,j)}(\mathbb{C}), \tag{3.16}$$

where $\mathbb{C}[K_{i,j}] \cong \prod_{k=1}^{e_{i,j}} M_{d_k(i,j)}(\mathbb{C})$ is the Artin-Wedderburn decomposition of $\mathbb{C}[K_{i,j}]$. In particular, $\mathbb{C}\Gamma_H(G)$ is a semisimple ring.

Example 3.23. Let H be a subgroup of a finite group G such that $[G:H] = 2$. In this case there are only two objects in $\Gamma_H(G)$, of different cardinalities: H and G, so Theorem 3.22 and (3.3) give $\mathbb{C}\Gamma_H(G) \cong \mathbb{C}[H]\times\mathbb{C}[G]$.

See Example 5.1 for another computed example.

In order to understand the representation theory of $\mathbb{C}\Gamma_H(G)$, we outline how the representation theory of a finite-dimensional semisimple associative unital algebra A gets recovered from the representation theory of the algebras eAe for the idempotents $e \in A$.

Let A be a finite-dimensional associative semisimple unital algebra over $\mathbb{C}$, with $A \ncong \mathbb{C}$. So, A is the direct sum of matrix algebras by Wedderburn theory. Let $e \in A$ be a nontrivial idempotent of A, i.e. $0 \neq e \neq 1$ (such an e does exist since $A \ncong \mathbb{C}$). The vector space Ae is an (A, eAe)-bimodule and hence it induces a functor $Ae \otimes_{eAe} - : eAe\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$. For the sake of simplicity, given an eAe-module V we set

$$\mathsf{Ind}_e V := Ae \otimes_{eAe} V.$$

The following theorem follows from standard theory of algebras. A proof is sketched in Appendix A.

Theorem 3.24. *If W is an irreducible eAe-module, then $\mathsf{Ind}_e W$ is an irreducible A-module. Every irreducible A-module V is isomorphic to $\mathsf{Ind}_e W$ for some nontrivial idempotent $e \in A$ and some irreducible eAe-module W.*

We apply this construction to our algebra $\mathbb{C}\Gamma_H(G)$. In this case we have the idempotents $(A, 1_G)$, with $A \in \mathcal{P}_H(G/H)$. Notice that

$$(A, 1_G)\mathbb{C}\Gamma_H(G)(A, 1_G) = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid gA = A\}.$$

So if we set $K = K_A := \{g \in G \mid gA = A\}$ (notice that $A \supseteq K$), then

$$(A, 1_G)\mathbb{C}\Gamma_H(G)(A, 1_G) \cong \mathbb{C}[K]$$

and the assignment $V \mapsto \mathsf{Ind}_A V := \mathsf{Ind}_{(A,1_G)} V = \mathbb{C}\Gamma_H(G)(A, 1_G) \otimes_{\mathbb{C}[K]} V$ induces a functor $\mathsf{Ind}_A : \mathsf{Rep}_K \to \mathsf{PRep}_G^H$.

Now given an irreducible representation (W, ρ) of K, we want to understand the $\mathbb{C}\Gamma_H(G)$-module

$$\mathsf{Ind}_A W := \mathsf{Ind}_{(A,1_G)} W = \mathbb{C}\Gamma_H(G)(A, 1_G) \otimes_{\mathbb{C}[K]} W. \tag{3.17}$$

Observe that

$$\mathbb{C}\Gamma_H(G)(A, 1_G) = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid g^{-1}H \subseteq A\} = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid g^{-1} \in A\} = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid g \in A^{-1}\},$$

so that

$$\mathsf{Ind}_A W = \mathbb{C}\Gamma_H(G)(A, 1_G) \otimes_{\mathbb{C}[K]} W = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid g \in A^{-1}\} \otimes_{\mathbb{C}[K]} W.$$

Notice that $A^{-1} := \{a^{-1} \mid a \in A\}$ is left H-invariant and right K-invariant. In particular, as in Lemma 3.10, we have

$$A^{-1} = \coprod_{i=1}^{m_A} t_i^{-1} K.$$

A computation similar to those performed in the proof of Lemma 3.15 shows that if $g = t_i^{-1}k \in A^{-1}$ then

$$(A, g) = (A, t_i^{-1}k) = (A, t_i^{-1})\,(A, 1_G)(A, k)(A, 1_G) = (A, t_i^{-1}) \cdot k,$$

where we identified $\mathbb{C}[K]$ with $(A, 1)\mathbb{C}\Gamma_H(G)(A, 1)$. Therefore

$$\mathsf{Ind}_A W = \operatorname{span}_{\mathbb{C}}\{(A, g) \mid g \in A^{-1}\} \otimes_{\mathbb{C}[K]} W = \bigoplus_{i=1}^{m_A} \mathbb{C}\,(A, t_i^{-1}) \otimes_{\mathbb{C}[K]} W \tag{3.18}$$

as vector spaces. This suggests a more familiar description of $\mathsf{Ind}_A W$.

Proposition 3.25. *For $A \in \mathcal{P}_H(G/H)$, $K = \{g \in G \mid gA = A\}$ and W any global K-representation, we have that $\mathsf{Ind}_A W \cong W_{A^{-1}}$, where $W_{A^{-1}}$ is the restriction as in Remark 2.9 of the G-global representation $\mathsf{Ind}_K^G W$ to the subspace*

$$\bigoplus_{t_i^{-1}K \in A^{-1}/K} W^{t_i^{-1}} \subseteq \mathsf{Ind}_K^G W,$$

where $W^{t_i^{-1}} = \mathbb{C}t_i^{-1}K \otimes_{\mathbb{C}[K]} W \cong W$ as vector spaces.

Proof. By definition, A^{-1} is left H-invariant and right K-invariant, thus $W_{A^{-1}}$ is an H-global G-partial representation by Lemma 2.8. The assignment $\varphi\colon \mathsf{Ind}_A W \to \mathsf{Ind}_K^G W$ uniquely determined by

$$\bigoplus_{i=1}^{m_A} \mathbb{C}\,(A, t_i^{-1}) \otimes_{\mathbb{C}[K]} W \to \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W, \qquad (A, t_k^{-1}) \otimes_{\mathbb{C}[K]} w \mapsto t_k^{-1} \otimes_{\mathbb{C}[K]} w \tag{3.19}$$

gives a bijection between $\mathsf{Ind}_A W$ and $W_{A^{-1}}$. Furthermore, if we denote by $\pi : G \to \mathsf{End}(W_{A^{-1}})$ and by $\pi' \colon G \to \mathsf{End}\left(\mathsf{Ind}_A W\right)$ the corresponding G-partial representation structures, then on the one hand

$$\varphi\left(\pi'(g)\left((A, t_k^{-1}) \otimes_{\mathbb{C}[K]} w\right)\right) = \varphi\left(\left(\sum_{B \ni g^{-1}} (B, g)(A, t_k^{-1})\right) \otimes_{\mathbb{C}[K]} w\right) = g t_k^{-1} \otimes_{\mathbb{C}[K]} w$$

if and only if $g^{-1} \in t_k^{-1} A$ and 0 otherwise. On the other hand, (2.3) can be rewritten as

$$\pi(g)\left(t_k^{-1} \otimes_{\mathbb{C}[K]} w\right) = \begin{cases} g t_k^{-1} \otimes_{\mathbb{C}[K]} w & g t_k^{-1} K \in A^{-1}/K \\ 0 & \text{otherwise} \end{cases}$$

A straightforward check shows that $g t_k^{-1} K \in A^{-1}/K$ if and only if $g^{-1} \in t_k^{-1} A$, whence $\mathsf{Ind}_A W \cong W_{A^{-1}}$ as H-global G-partial representations. □

Lemma 3.26. *Up to isomorphism, the irreducible H-global G-partial representation $\mathsf{Ind}_A W$ depends only on the connected component of $\Gamma_H(G)$ containing $(A, 1_G)$ and not on the particular vertex chosen.*

Namely, let A_i and A_j be two elements of the same connected component of $\Gamma_H(G)$ and let K_i and K_j be their (isomorphic) stabilizers. If W_i is an irreducible representation of K_i, then there exists an irreducible representation W_j of K_j such that $\mathsf{Ind}_{A_i} W_i \cong \mathsf{Ind}_{A_j} W_j$.

Proof. For the sake of simplicity, set $A \coloneqq A_i$, $W \coloneqq W_i$ and $K \coloneqq K_i$. We already observed in Remark 3.20 that any other element A_j of the connected component of A is of the form $A_j = t_j^{-1} A$. It is clear that $t_j^{-1} K t_j = \{g \in G \mid g t_j^{-1} A = t_j^{-1} A\}$ is the stabilizer of $t_j^{-1} A$, which is therefore isomorphic to the stabilizer K of A. If we consider $W_j = W$ endowed with the map $\rho_j \colon t_j^{-1} K t_j \to \mathsf{End}(W_j)$ defined by $\rho_j(x) \coloneqq \rho(t_j x t_j^{-1})$ for all $x \in t_j^{-1} K t_j$, then (W_j, ρ_j) is clearly an irreducible representation of $t_j^{-1} K t_j$, and it is easy to show that $\mathsf{Ind}_A W$ and $\mathsf{Ind}_{t_j^{-1} A} W_j$ are isomorphic H-global G-partial representations. □

The following theorem is related to [DZ04, Theorem 2.3 and Corollary 2.4].

Theorem 3.27. *Let $\Gamma_H(G) = \coprod_{i=1}^{[G:H]} \coprod_{j=1}^{c_i} \Gamma_{i,j}$ be the decomposition into connected components of $\Gamma_H(G)$, as in Notation 3.21. Fix an object $A_{i,j}$ of $\Gamma_{i,j}$ for all i and j and set again $K_{i,j} \coloneqq K_{A_{i,j}}$ and $m_{i,j} \coloneqq m_{A_{i,j}}$. Then the $\mathsf{Ind}_{A_{i,j}} W$'s, as W runs over all the inequivalent irreducible representations of $K_{i,j}$ for all i, j, form a complete list of all the inequivalent irreducible H-global G-partial representations.*

Proof. From Corollary 3.17 and Theorem 3.24 we know that the $\mathsf{Ind}_{A_{i,j}} W$'s are irreducible H-global G-partial representations. In addition, by resorting to Lemma 3.26 as well, we conclude that any irreducible H-global G-partial representation is isomorphic to one of them. We are left to check that they are inequivalent.

By (3.18) we have $\dim_{\mathbb{C}}\left(\mathsf{Ind}_{A_{i,j}} W\right) = m_{i,j} \dim_{\mathbb{C}}(W)$, whence by Artin-Wedderburn theory

$$\sum_W \dim_{\mathbb{C}}\left(\mathsf{Ind}_{A_{i,j}} W\right)^2 = \sum_W \left(m_{i,j} \dim_{\mathbb{C}}(W)\right)^2 = m_{i,j}^2 \sum_W \dim_{\mathbb{C}}(W)^2 = m_{i,j}^2 |K_{i,j}|$$

where all the sums are over all the inequivalent irreducible representations W of $K_{i,j}$.

Summing over i and j, and comparing with (3.16), this shows that we cannot have redundancy among the irreducible representations that we have found. □

Remark 3.28. It is noteworthy that the case $i = 1$ in Theorem 3.27 corresponds to H-global G-partial representations where any $g \in G \setminus H$ acts as 0 (cf. Example 2.2), while the case $i = [G : H]$ corresponds to the G-global representations.

Example 3.29. Let H be a subgroup of a finite group G such that $[G : H] = 2$. We already saw that $\mathbb{C}\Gamma_H(G) \cong \mathbb{C}[H] \times \mathbb{C}[G]$. The factor $\mathbb{C}[G]$ corresponds to the irreducible G-global representations, while the factor $\mathbb{C}[H]$ corresponds to the irreducible H-global representations where the elements of $G \setminus H$ act as 0 (cf. Example 2.2). Both cases come from our construction (cf. Remark 3.28), so, by Theorem 3.27, we found all the irreducible H-global G-partial representations in this case.

4. Representation theory: restriction, globalization and induction

In this section we discuss some important constructions, like the restriction to H and the globalization of an irreducible H-global G-partial representation, and a partial induction of a global representation of H to an H-global G-partial representation.

4.1. Restriction to H of irreducibles.

Let H be a subgroup of a finite group G, and let (V,π) be an H-global G-partial representation. By definition, the restriction of π to H gives a global representation of H, denoted $\mathsf{Res}_H^G V$. This leads to a functor $\mathsf{Res}_H^G : \mathsf{PRep}_G^H \to \mathsf{Rep}_H$. In this section we describe this restriction for the irreducible H-global G-partial representations that we constructed in Section 3.4.

Recall that to get the H-global G-partial irreducibles, we started with an $A \in \mathcal{P}_H(G/H)$ and we considered an irreducible representation (W,ρ) of the subgroup $K = K_A \coloneqq \{g \in G \mid gA = A\}$ of G. Then the corresponding irreducible H-global G-partial representation was given by $\mathsf{Ind}_A W$, i.e. by the restriction of the G-global representation $\mathsf{Ind}_K^G W$ to the subspace $\bigoplus_{t_i^{-1}K \in A^{-1}/K} W^{t_i^{-1}} \subseteq \mathsf{Ind}_K^G W$ via the obvious inclusion and projection maps. We want to understand the restriction of $\mathsf{Ind}_A W$ to H, whence let us assume that A (and, consequently, K) is fixed all over the subsection. The answer to this problem is given by a well-known formula of Mackey suitably adapted to our "restricted" situation. We just need some more notation.

Notation 4.1 (cf. [Ser77, §7.3])**.** Choose a set S of representatives of (H,K)-double cosets of A^{-1}, i.e. $A^{-1} = \coprod_{s \in S} HsK$. For $s \in S$, let $K_s \coloneqq sKs^{-1} \cap H$, which is a subgroup of H. If we set

$$\rho^s(x) = \rho(s^{-1}xs), \qquad \text{for } x \in K_s,$$

we obtain a representation $\rho^s \colon K_s \to \mathsf{GL}(W)$ of K_s, denoted W_s. Since K_s is a subgroup of H, we can consider the induced representation $\mathsf{Ind}_{K_s}^H W_s$.

The proof of the following theorem is identical to the one of [Ser77, Proposition 22] (which corresponds to the case $A = G$), so it will be omitted.

Theorem 4.2. *The representation $\mathsf{Res}_H^G(\mathsf{Ind}_A W)$ of H is isomorphic to the direct sum of the representations $\mathsf{Ind}_{K_s}^H W_s$, for $s \in S$.*

Example 4.3. In the notation above, suppose that $A = KH$, so that $A^{-1} = HK = H1_G K$ (see Section 5.1 for a concrete example). Then $S = \{1_G\}$, $K_{1_G} = K \cap H$ and ρ^{1_G} is just the restriction of ρ to $K \cap H$, so that

$$\mathsf{Res}_H^G(\mathsf{Ind}_{KH} W) \cong \mathsf{Ind}_{K\cap H}^H(\mathsf{Res}_{K\cap H}^K W).$$

4.2. Globalization of irreducibles.

We proved in Theorem 1.18 that every G-partial representation admits a globalization, unique up to isomorphism. In particular, the globalization construction induces a functor $\mathsf{PRep}_G^H \to \mathsf{Rep}_G$. In this section we will give an explicit description of the globalization of the irreducible H-global G-partial representations.

Let $A \in \mathcal{P}_H(G/H)$ and (W,ρ) be an irreducible representation of the subgroup $K = K_A \coloneqq \{g \in G \mid gA = A\}$ of G. The corresponding irreducible H-global G-partial representation is given by

$$\mathsf{Ind}_A W \cong \bigoplus_{t_i^{-1}K \in A^{-1}/K} \mathbb{C}t_i^{-1} \otimes_{\mathbb{C}[K]} W \subseteq \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W \cong \bigoplus_{g_i K \in G/K} W^{g_i}.$$

Theorem 4.4. *For any K-global representation W, the globalization of $\mathsf{Ind}_A W$ is isomorphic to $\mathsf{Ind}_K^G W$, that is, we have a commutative diagram of functors*

$$\begin{array}{ccc} & \mathsf{Rep}_G & \\ {\scriptstyle \mathsf{Ind}_K^G}\nearrow & & \nwarrow{\scriptstyle \mathsf{glob}} \\ \mathsf{Rep}_K & \xrightarrow[\mathsf{Ind}_A]{} & \mathsf{PRep}_G^H. \end{array}$$

In particular, the globalization of the irreducible H-global G-partial representation $\mathsf{Ind}_A W$ is given by $\mathsf{Ind}_K^G W$ with the obvious inclusion and projection maps.

Proof. Set $U := \mathsf{Ind}_K^G W$ and $V := \mathsf{Ind}_A W$, for the sake of simplicity. Obviously (U,ρ) is a global representation of G and we already observed (cf. Proposition 3.25) that (V,π) is the restriction of (U,ρ) via the inclusion φ and projection τ as in Remark 2.9. This proves properties (GR1) and (GR2) of a globalization.

Instead of proving property (GR3), we will check the properties (GR3′) and (GR4′) of Corollary 1.19. Property (GR3′), i.e. $U = \sum_{g\in G}\rho(g)(V)$, is obvious. In order to check property (GR4′), let $(U',\rho',\varphi',\tau')$ be another quadruple satisfying (GR1) and (GR2). Consider the composition

$$\begin{array}{ccccc} W & \longrightarrow & \mathsf{Ind}_A W & \xrightarrow{\varphi'} & U' \\ w & \longmapsto & (A,1_G)\otimes_{\mathbb{C}[K]} w & \longmapsto & \varphi'\left((A,1_G)\otimes_{\mathbb{C}[K]} w\right) \end{array}.$$

It satisfies

$$\begin{aligned}\varphi'\left((A,1_G)\otimes_{\mathbb{C}[K]}\rho(k)(w)\right) &= \varphi'\left((A,k)\otimes_{\mathbb{C}[K]} w\right) = \varphi'\left(\left(\left(\sum_{B\ni k^{-1}}(B,k)\right)(A,1_G)\right)\otimes_{\mathbb{C}[K]} w\right)\\ &= \varphi'\Big(\pi(k)\Big((A,1_G)\otimes_{\mathbb{C}[K]} w\Big)\Big) \overset{(RR2)}{=} \rho'(k)\Big(\varphi'\Big((A,1_G)\otimes_{\mathbb{C}[K]} w\Big)\Big),\end{aligned}$$

which means that it is a K-homomorphism and so there exists a unique G-homomorphism $\psi\colon U\to U'$ such that

$$\psi(1_G\otimes_{\mathbb{C}[K]} w) = \varphi'\left((A,1_G)\otimes_{\mathbb{C}[K]} w\right), \tag{4.1}$$

by the universal property of $\mathsf{Ind}_K^G W$. In turn, ψ satisfies

$$\begin{aligned}\psi\left(\varphi((A,t_i^{-1})\otimes_{\mathbb{C}[K]} w)\right) &\overset{(3.19)}{=} \psi\left(t_i^{-1}\otimes_{\mathbb{C}[K]} w\right) = \rho'(t_i^{-1})\psi\left(1_G\otimes_{\mathbb{C}[K]} w\right)\\ &\overset{(4.1)}{=} \rho'(t_i^{-1})\varphi'\left((A,1_G)\otimes_{\mathbb{C}[K]} w\right) \overset{(1.6)}{=} \varphi'\left((A,t_i^{-1})\otimes_{\mathbb{C}[K]} w\right)\end{aligned}$$

Being already a G-homomorphism, ψ is the map required in (GR4′), completing the proof. □

4.3. **Induction from H-global to H-global G-partial.** Consider a subgroup H of a finite group G. Given a G-partial representation (V,π), it is clear that the restriction $\mathsf{Res}_H^G(\pi) := \pi_{|H}\colon H\to \mathsf{End}(V)$ gives an H-partial representation, that we denote $\mathsf{Res}_H^G V$. By definition, (V,π) is H-global if and only if $\mathsf{Res}_H^G V$ is a global representation of H.

For global representations, there is the well-known converse construction of the latter: the induction $\mathsf{Ind}_H^G W$ of a global representation W of H, that we already used in this paper. This representation satisfies a universal property, i.e. it is equipped with an H-homomorphism $\eta_W\colon W\to \mathsf{Ind}_H^G W$ such that for any H-homomorphism $f\colon W\to \mathsf{Res}_H^G U$ into a G-global representation U, there exists a unique G-homomorphism $\tilde f\colon \mathsf{Ind}_H^G W\to U$ such that $f = \tilde f\circ \eta_W$, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathsf{Ind}_H^G W & & \\ \uparrow{\scriptstyle \eta_W} & \searrow{\scriptstyle \exists!\tilde f} & \\ W & \xrightarrow[f]{} & \mathsf{Res}_H^G U \equiv U. \end{array}$$

Equivalently, there is a bijection

$$\begin{array}{ccc} \mathsf{Hom}_G\left(\mathsf{Ind}_H^G W, U\right) & \overset{\cong}{\longleftrightarrow} & \mathsf{Hom}_H\left(W, \mathsf{Res}_H^G U\right) \\ f' & \longmapsto & f'\circ\eta_W \\ \tilde f & \longleftarrow & f \end{array}$$

which is known as *Frobenius reciprocity.* In categorical terms, what this says is that Ind_H^G and Res_H^G are adjoint functors.

It is now natural to ask if starting with a global representation W of H, we can construct a *partial induction* which is an H-global G-partial representation satisfying a similar reciprocity. While there are general reasons for the existence of such a correspondence (cf. Remark 4.12), in the spirit of this work we prefer to follow a less categorical approach.

We propose the following definition.

Definition 4.5. The *partial induction* of a global representation W of $H \subseteq G$ to G is an H-global G-partial representation $\mathsf{PInd}_H^G W$ equipped with an H-homomorphism $\eta_W : W \to \mathsf{PInd}_H^G W$, such that for every H-homomorphism $f\colon W \to \mathsf{Res}_H^G U \equiv U$ from W to an H-global G-partial representation U, there exists a unique morphism of G-partial representations $\tilde{f}\colon \mathsf{PInd}_H^G W \to U$ such that $\tilde{f} \circ \eta_W = f$.

In the present section we prove the existence of such a partial induced representation by providing an explicit construction. Notice that, in light of the universal property that defines it, if an induced partial representation exists, then it is necessarily unique up to isomorphism. Therefore, we will refer to it as *the* partial induced representation.

Consider a subgroup H of a finite group G, and let $G = \coprod_{i=1}^r g_i H$ with $g_1 = 1_G$. Given an H-global representation (W, ρ), we define the vector space

$$\overline{\mathsf{PInd}}_H^G W \coloneqq \bigoplus_{A \in \mathcal{P}_H(G/H)} \bigoplus_{g_i H \in A/H} W^{A,i}$$

where the $W^{A,i}$ are vector spaces equipped with linear isomorphisms $\phi_{A,i}\colon W \to W^{A,i}$.

We define $\widetilde{\rho}\colon G \to \mathsf{End}(\overline{\mathsf{PInd}}_H^G W)$ by setting for all $g \in G$, $w \in W$, $A \in \mathcal{P}_H(G/H)$ and $g_i H \subseteq A$

$$\widetilde{\rho}(g)(\phi_{A,i}(w)) \coloneqq \begin{cases} 0 & \text{if } g^{-1}H \not\subseteq A \\ \phi_{gA,j}(\rho(h)(w)) & \text{if } g^{-1}H \subseteq A \text{ and } gg_i = g_j h, \text{ with } h \in H \end{cases}. \tag{4.2}$$

Proposition 4.6. *The pair $(\overline{\mathsf{PInd}}_H^G W, \widetilde{\rho})$ is an H-global G-partial representation.*

Proof. The fact that $\widetilde{\rho}$ is H-global is clear from the definition. The fact that it is G-partial is a tedious but straightforward verification, that we leave to the reader. □

Remark 4.7. For any $A \in \mathcal{P}_H(G/H)$, consider the orthogonal idempotent $P_A \coloneqq P_A^{\widetilde{\rho}}$ as in Definition 2.12. Recall from Proposition 2.21 that

$$P_A = P_A^{\widetilde{\rho}} = \prod_{g_k H \subseteq A} \widetilde{\rho}(g_k)\widetilde{\rho}(g_k^{-1}) \prod_{g_i H \subseteq G \setminus A} \left(\mathsf{Id}_{\overline{\mathsf{PInd}}_H^G W} - \widetilde{\rho}(g_i)\widetilde{\rho}(g_i^{-1}) \right).$$

It can be easily checked that $\bigoplus_{g_i H \subseteq A} W^{A,i} = (\overline{\mathsf{PInd}}_H^G W)^A = P_A(\overline{\mathsf{PInd}}_H^G W)$ (cf. Section 2.3).

Lemma 4.8. *The function*

$$\eta_W : W \to \overline{\mathsf{PInd}}_H^G W, \qquad w \mapsto \sum_{A \in \mathcal{P}_H(G/H)} \phi_{A,1}(w) \tag{4.3}$$

is an H-homorphism. Furthermore, it satisfies

$$\phi_{A,i}(w) = \widetilde{\rho}(g_i) P_{g_i^{-1}A} \eta_W \left(\phi_{g_i^{-1}A,1}^{-1} \widetilde{\rho}\,(g_i^{-1})\,(\phi_{A,i}(w)) \right) \tag{4.4}$$

for every $w \in W$, $A \in \mathcal{P}_H(G/H)$ and all $i = 1, \ldots, r$ such that $g_i H \subseteq A$.

Proof. Let $h \in H$. Then, using (4.2), we compute

$$\begin{aligned} \widetilde{\rho}(h)\,(\eta_W(w)) &= \widetilde{\rho}(h) \left(\sum_{A \in \mathcal{P}_H(G/H)} \phi_{A,1}(w) \right) = \sum_{A \in \mathcal{P}_H(G/H)} \widetilde{\rho}(h)\,(\phi_{A,1}(w)) \\ &= \sum_{A \in \mathcal{P}_H(G/H)} \phi_{hA,1}\,(\rho(h)(w)) = \sum_{B \in \mathcal{P}_H(G/H)} \phi_{B,1}\,(\rho(h)(w)) \\ &= \eta_W\,(\rho(h)(w))\,, \end{aligned}$$

which proves the H-linearity. To show (4.4), using again (4.2), we compute

$$\begin{aligned} \widetilde{\rho}(g_i) P_{g_i^{-1}A} \eta_W \left(\phi_{g_i^{-1}A,1}^{-1} \widetilde{\rho}\,(g_i^{-1})\,(\phi_{A,i}(w)) \right) &= \widetilde{\rho}(g_i) P_{g_i^{-1}A} \eta_W \left(\phi_{g_i^{-1}A,1}^{-1} \left(\phi_{g_i^{-1}A,1}(w) \right) \right) \\ &= \widetilde{\rho}(g_i) P_{g_i^{-1}A} \eta_W\,(w) = \widetilde{\rho}(g_i) \left(\phi_{g_i^{-1}A,1}(w) \right) = \phi_{A,i}(w). \end{aligned}$$

□

We want to show that $(\overline{\mathsf{PInd}}_H^G W, \widetilde{\rho})$ together with η_W given by (4.3) satisfies the universal property of the partial induced representation.

Remark 4.9. By (4.2), for every $h \in H$

$$\widetilde{\rho}(h) \circ \phi_{A,1} = \phi_{hA,1} \circ \rho(h)$$

so that

$$\phi_{hA,1}^{-1} \circ \widetilde{\rho}(h) = \rho(h) \circ \phi_{A,1}^{-1}. \tag{4.5}$$

Let (U, α) be an H-global G-partial representation and let $f \colon W \to \mathsf{Res}_H^G U \equiv U$ be an H-homomorphism. First of all, observe that if $F \colon \overline{\mathsf{PInd}}_H^G W \to U$ is any morphism of G-partial representations such that $F \circ \eta_W = f$, then for every $x = \phi_{A,i}(w)$, where $A \in \mathcal{P}_H(G/H)$ and $g_i H \subseteq A$, using (4.4) we have

$$\begin{aligned} F(x) &= F\left(\widetilde{\rho}(g_i) P^{\widetilde{\rho}}_{g_i^{-1}A} \eta_W\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}\left(g_i^{-1}\right)(x)\right)\right) \\ &= \alpha(g_i) P^{\alpha}_{g_i^{-1}A}\left(F\left(\eta_W\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}\left(g_i^{-1}\right)(x)\right)\right)\right) \\ &= \alpha(g_i) P^{\alpha}_{g_i^{-1}A}\left(f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}\left(g_i^{-1}\right)(x)\right)\right). \end{aligned}$$

Therefore, we define $\hat{f} \colon \overline{\mathsf{PInd}}_H^G W \to U$ by setting, for $x = \phi_{A,i}(w) \in W^{A,i}$,

$$\hat{f}(x) \coloneqq \alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right).$$

Lemma 4.10. *The map $\hat{f}$ is well defined, i.e. it does not depend on the chosen representatives $\{g_1, \ldots, g_r\}$.*

Proof. Given $h \in H$, replacing g_i by $g_i h$ we get

$$\begin{aligned} \alpha(g_i h) P^{\alpha}_{h^{-1}g_i^{-1}A} f\left(\phi^{-1}_{h^{-1}g_i^{-1}A,1} \widetilde{\rho}(h^{-1}g_i^{-1})(x)\right) &= \alpha(g_i)\alpha(h) P^{\alpha}_{h^{-1}g_i^{-1}A} f\left(\phi^{-1}_{h^{-1}g_i^{-1}A,1} \widetilde{\rho}(h^{-1}g_i^{-1})(x)\right) \\ &\overset{(2.4)}{=} \alpha(g_i) P^{\alpha}_{g_i^{-1}A} \alpha(h) f\left(\phi^{-1}_{h^{-1}g_i^{-1}A,1} \widetilde{\rho}(h^{-1}g_i^{-1})(x)\right) \\ &= \alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\rho(h)\phi^{-1}_{h^{-1}g_i^{-1}A,1} \widetilde{\rho}(h^{-1}g_i^{-1})(x)\right) \\ &\overset{(4.5)}{=} \alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(h)\widetilde{\rho}(h^{-1}g_i^{-1})(x)\right) \\ &= \alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right). \end{aligned}$$

□

Lemma 4.11. *The map $\hat{f}$ is a morphism of G-partial representations.*

Proof. Given $g \in G$, let $gg_i = g_j h$ with $h \in H$. Assume initially that $g^{-1} \in A$ and pick $0 \neq x = \phi_{A,i}(w)$ arbitrarily. We have that

(a) $g_i H \subseteq A$, and $g_i^{-1} \in g_i^{-1}A$;
(b) $g_j^{-1} g g_i = h$ and $g^{-1} g_j H = g_i h^{-1} H \subseteq A$;
(c) $g_j^{-1} g_j = g_1$ and $g_j H \subseteq gA$;
(d) $gg_i = g_j h$ and $g^{-1} H \subseteq A$

and therefore

$$\begin{aligned} \alpha(g)\hat{f}(x) &= \alpha(g_j h g_i^{-1})\alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right) = \alpha(g_j)\alpha(h)\alpha(g_i^{-1})\alpha(g_i) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right) \\ &\overset{(a)}{=} \alpha(g_j)\alpha(h) P^{\alpha}_{g_i^{-1}A} f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right) \overset{(2.4)}{=} \alpha(g_j) P^{\alpha}_{hg_i^{-1}A} \alpha(h) f\left(\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right) \\ &= \alpha(g_j) P^{\alpha}_{g_j^{-1}gA} f\left(\rho(h)\phi^{-1}_{g_i^{-1}A,1} \widetilde{\rho}(g_i^{-1})(x)\right) \overset{(4.5)}{=} \alpha(g_j) P^{\alpha}_{g_j^{-1}gA} f\left(\phi^{-1}_{hg_i^{-1}A,1} \widetilde{\rho}(h)\widetilde{\rho}(g_i^{-1})(x)\right) \\ &= \alpha(g_j) P^{\alpha}_{g_j^{-1}gA} f\left(\phi^{-1}_{g_j^{-1}gA,1} \widetilde{\rho}(g_j^{-1}g)(x)\right) = \alpha(g_j) P^{\alpha}_{g_j^{-1}gA} f\left(\phi^{-1}_{g_i^{-1}gA,1} \widetilde{\rho}(g_j^{-1}g)(\phi_{A,i}(w))\right) \end{aligned}$$

$$\overset{(b)}{=} \alpha(g_j)P^{\alpha}_{g_j^{-1}gA} f\left(\phi^{-1}_{g_j^{-1}gA,1}\phi_{g_j^{-1}gA,1}(\rho(h)(w))\right) \overset{(c)}{=} \alpha(g_j)P^{\alpha}_{g_j^{-1}gA} f\left(\phi^{-1}_{g_j^{-1}gA,1}\widetilde{\rho}(g_j^{-1})\phi_{gA,j}(\rho(h)(w))\right)$$
$$= \hat{f}(\phi_{gA,j}(\rho(h)(w))) \overset{(d)}{=} \hat{f}(\widetilde{\rho}(g)(\phi_{A,i}(w))) = \hat{f}(\widetilde{\rho}(g)(x)).$$

If, on the other hand, $g^{-1} \notin A$, then by definition $\widetilde{\rho}(g)(x) = 0$, but also $\alpha(g_j)P^{\alpha}_{g_j^{-1}gA} = 0$, since $g_j^{-1} \notin g_j^{-1}gA$. So $\alpha(g)\hat{f}(x) = 0$ follows from the first few lines of the same computation. □

By construction, $\hat{f}$ is the unique morphism of G-partial representations such that $\hat{f} \circ \eta_W = f$. This proves that $(\overline{\mathsf{PInd}}_H^G W, \tilde{\rho})$ is the partial induction of W, as we wanted. Notice that in particular we established the following *Frobenius reciprocity*:

$$\mathsf{Hom}_G(\mathsf{PInd}_H^G W, U) \cong \mathsf{Hom}_H(W, \mathsf{Res}_H^G U). \tag{4.6}$$

We conclude this section with the following remark, showing some advantages of working with the groupoid algebra $\mathbb{C}\Gamma_H(G)$ and its modules.

Remark 4.12. Assume we are given a finite group G and two subgroups H, K.

(a) We can see $\mathbb{C}\Gamma_H(G)$ as a right $\mathbb{C}[H]$-module using the restriction to H of the map $\mu_p \colon G \to \mathbb{C}\Gamma_H(G)$ defined in (3.4) (cf. Lemma 3.14), which induces a ring homomorphism $\mathbb{C}[H] \to \mathbb{C}\Gamma_H(G)$. Since the restriction functor Res_H^G is exactly the restriction functor associated to this ring homomorphism, it obviously has the adjoint $\mathbb{C}\Gamma_H(G) \otimes_{\mathbb{C}H} -$. Now observe that given a global representation W of H, i.e. a left $\mathbb{C}[H]$-module, we have precisely $\mathsf{PInd}_H^G W \cong \mathbb{C}\Gamma_H(G) \otimes_{\mathbb{C}[H]} W$ as left $\mathbb{C}\Gamma_H(G)$-modules, and hence as H-global G-partial representations. So the existence of PInd_H^G could be deduced by the above argument.

(b) More generally, given a $(\mathbb{C}\Gamma_H(G), \mathbb{C}[K])$-bimodule Q and a left $\mathbb{C}[K]$-module W, $Q \otimes_{\mathbb{C}[K]} W$ is a left $\mathbb{C}\Gamma_H(G)$-module and so an H-global G-partial representation. In the particular situation of Section 2.2, we consider a subset $A \subseteq G$ which is a union of (H, K)-double cosets (e.g. $A = HK \subset G$) and the set A/K of left cosets of K contained in A. Then we can consider the partial action of G on A given by restriction of the global action of G on G by left multiplication. By linearization as in Definition 1.10, $\mathbb{C}[A]$ becomes an H-global G-partial representation and so a left $\mathbb{C}\Gamma_H(G)$-module. It is also a right $\mathbb{C}[K]$-module and the two structures are compatible, whence it is a $(\mathbb{C}\Gamma_H(G), \mathbb{C}[K])$-bimodule. Now,

$$W_A = \bigoplus_{g_i K \in A/K} W^{g_i} \cong \mathbb{C}[A] \otimes_{\mathbb{C}[K]} W,$$

which is then an H-global G-partial representation. This would allow us to recover the general construction performed in Section 2.2 and the induction construction (3.17) from §3.4.

5. An application: $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$

In this section we apply our general theory to the important special case where G is the symmetric group $\mathfrak{S}_n$ and H is the subgroup $\mathfrak{S}_n^{[1]} \cong \mathfrak{S}_1 \times \mathfrak{S}_{n-1} \equiv \mathfrak{S}_{n-1}$ of the permutations fixing 1. This will provide a natural extension of the classical representation theory of $\mathfrak{S}_n$.

5.1. **$\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representation theory.** We use the notation introduced in Example 2.7. Moreover, we use freely classical definitions and results from the representation theory of $\mathfrak{S}_n$ and its combinatorics: for these we refer to the standard [Sta99, Chapter 7].

We start by looking at the algebra $\mathbb{C}\Gamma_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n)$. In order to apply Theorem 3.22 we need to understand the connected components of $\Gamma_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n)$ and the corresponding isotropy groups.

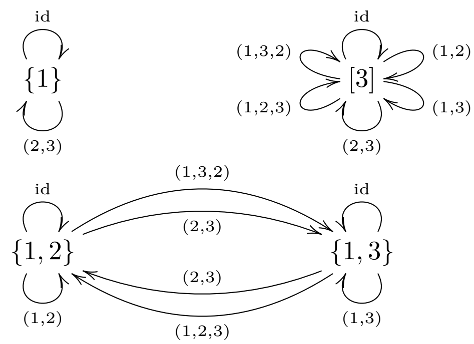
We already identified the action of $\mathfrak{S}_n$ on the cosets $\mathfrak{S}_n/\mathfrak{S}_{n-1}$ with the defining action of $\mathfrak{S}_n$ on the set $\binom{[n]}{1} = [n]$. So under this identification $\mathcal{P}_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n/\mathfrak{S}_{n-1}) \equiv \{A \subseteq [n] \mid 1 \in A\}$. Given $A \in \mathcal{P}_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n/\mathfrak{S}_{n-1})$ of cardinality $k \cdot |\mathfrak{S}_{n-1}| = k \cdot (n-1)!$ with $k \geq 1$, it is clear that its stabilizer is $\mathfrak{S}_n^A \cong \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ (here we identify A with the corresponding subset of $[n]$). Therefore $m_A = |A|/|\mathfrak{S}_n^A| = \binom{n-1}{k-1}$, which is precisely the number of elements $A \in \mathcal{P}_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n/\mathfrak{S}_{n-1})$ of cardinality $k \cdot |\mathfrak{S}_{n-1}| = k \cdot (n-1)!$. Hence for each $1 \leq k \leq n$ there is precisely one connected component. So we can apply Theorem 3.22 to get the formula

$$\mathbb{C}\Gamma_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n) \cong \prod_{k=1}^{n} M_{\binom{n-1}{k-1}}(\mathbb{C}[\mathfrak{S}_k \times \mathfrak{S}_{n-k}]). \tag{5.1}$$

Notice that this formula is in agreement with formula (3.3) of the dimension of $\mathbb{C}\Gamma_{\mathfrak{S}_{n-1}}(\mathfrak{S}_n)$: indeed

$$\begin{aligned}\sum_{k=1}^{n} \binom{n-1}{k-1}^2 k!(n-k)! &= (n-1)! \sum_{k=1}^{n} k \binom{n-1}{k-1} \\ &= (n-1)!(n+1)2^{n-2} \\ &= 2^{n-2}(n! + (n-1)!) \\ &= 2^{|\mathfrak{S}_n/\mathfrak{S}_{n-1}|-2}(|\mathfrak{S}_n| + |\mathfrak{S}_{n-1}|).\end{aligned}$$

Example 5.1. Recall that $\mathfrak{S}_3^{[1]} \equiv \mathfrak{S}_2 = \{\mathrm{id}, (2,3)\}$, $\mathfrak{S}_3/\mathfrak{S}_2 = \{\mathfrak{S}_2, (1,2)\mathfrak{S}_2, (1,3)\mathfrak{S}_2\} \leftrightarrow \{1,2,3\} = [3]$ and that $\sigma \cdot (1,n)\mathfrak{S}_2 = (1,m)\mathfrak{S}_2$ if and only if $\sigma(n) = m$. In order to help intuition and visualization, the following picture represents the groupoid $\Gamma_{\mathfrak{S}_2}(\mathfrak{S}_3)$.



It makes also evident that $\mathbb{C}\Gamma_{\mathfrak{S}_2}(\mathfrak{S}_3) \cong \mathbb{C}[\mathfrak{S}_2] \times \mathbb{C}[\mathfrak{S}_3] \times M_2(\mathbb{C}[\mathfrak{S}_2 \times \mathfrak{S}_1])$, where $\mathfrak{S}_2 \times \mathfrak{S}_1 = \mathfrak{S}_3^{\{3\}} = \{\mathrm{id}, (1,2)\}$.

Now we want to apply Theorem 3.27 to construct all the irreducible $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations.

Recall (cf. [Sta99, Section 7.18]) that the irreducible representations of $\mathfrak{S}_m$ are indexed by the partitions of m: for any $m \geq 1$, let $\{V_\mu\}_{\mu \vdash m}$ be a complete set of inequivalent irreducible representations of $\mathfrak{S}_m$ (where $\mu \vdash m$ means "μ partition of m"). Then, given $[k] \subseteq [n] \equiv \mathfrak{S}_n/\mathfrak{S}_{n-1}$ with $k \geq 1$, a complete set of inequivalent irreducible representations of $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ is $\{V_\lambda \otimes V_\mu\}_{\lambda \vdash k, \mu \vdash n-k}$, so that a complete set of inequivalent irreducible $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations is given by

$$V_{(\lambda,\mu)} \coloneqq \mathsf{Ind}_{[k]}(V_\lambda \otimes V_\mu) \quad \text{ for } 1 \leq k \leq n, \lambda \vdash k, \mu \vdash n-k. \tag{5.2}$$

In particular, by (3.18) as in the proof of Theorem 3.27, we have the formula for the dimension

$$\dim_{\mathbb{C}} V_{(\lambda,\mu)} = \binom{n-1}{k-1} f^\lambda f^\mu, \tag{5.3}$$

where for any partition ν we denote by f^ν the number of standard Young tableaux of shape ν: this is a classical formula for the dimension of V_ν (cf. [Sta99, Section 7.18]).

Now to study $\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_{(\lambda,\mu)}$, we need to better understand the construction of $\mathsf{Ind}_{[k]}(V_\lambda \otimes V_\mu)$.

Given $[k] \subseteq [n] \equiv \mathfrak{S}_n/\mathfrak{S}_{n-1}$ with $k \geq 1$, the corresponding subset of $\mathfrak{S}_n$ is $A_k \coloneqq \{\sigma \in \mathfrak{S}_n \mid \sigma(1) \in [k]\}$ so that $A_k^{-1} = \{\sigma \in \mathfrak{S}_n \mid 1 \in \sigma([k])\}$. Since for every $\sigma \in A_k^{-1}$ we have $\sigma = \tau(1, \sigma^{-1}(1))$, where $\tau = \sigma(1, \sigma^{-1}(1)) \in \mathfrak{S}_n^{[1]}$ and $(1, \sigma^{-1}(1)) \in \mathfrak{S}_n^{[k]}$, it is easy to see that

$$A_k^{-1} = \mathfrak{S}_n^{[1]}\mathfrak{S}_n^{[k]} = \mathfrak{S}_{n-1}(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) = \mathfrak{S}_{n-1}1_{\mathfrak{S}_n}(\mathfrak{S}_k \times \mathfrak{S}_{n-k}),$$

so that $A_k = (\mathfrak{S}_k \times \mathfrak{S}_{n-k})\mathfrak{S}_{n-1}$. Now, $(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \cap \mathfrak{S}_{n-1} = \mathfrak{S}_n^{[k]} \cap \mathfrak{S}_n^{[1]} \cong \mathfrak{S}_1 \times \mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k} \equiv \mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}$, so we deduce from Example 4.3 that

$$\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_{(\lambda,\mu)} = \mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(\mathsf{Ind}_{(\mathfrak{S}_k \times \mathfrak{S}_{n-k})\mathfrak{S}_{n-1}}(V_\lambda \otimes V_\mu)) \cong \mathsf{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}}(\mathsf{Res}^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}_{\mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}}(V_\lambda \otimes V_\mu)).$$

Now we can use the classical formulas for the restriction and the induction of irreducibles of $\mathfrak{S}_n$ to deduce the decomposition into irreducibles (see [Sta99, Sections 7.15 and 7.18] for the missing definitions and results).

Theorem 5.2. *We have*

$$\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_{(\lambda,\mu)} \cong \bigoplus_{\lambda^1\to\lambda} \mathsf{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{k-1}\times\mathfrak{S}_{n-k}} (V_{\lambda^1}\otimes V_\mu) \cong \bigoplus_{\nu\vdash n-1} V_\nu^{\oplus d^\nu_{\lambda\mu}} \tag{5.4}$$

with

$$d^\nu_{\lambda\mu} := \sum_{\lambda^1\to\lambda} c^\nu_{\lambda^1\mu}, \tag{5.5}$$

where $c^\nu_{\lambda^1\mu}$ are the Littlewood-Richardson coefficients*, and $\nu^1\to\nu$ indicates that ν covers ν^1 in the Young lattice (i.e. ν^1 is obtained by removing a corner from ν).*

Remark 5.3. Formula (5.4) reduces to the corresponding Pieri rule when $V_{(\lambda,\mu)}$ is $\mathfrak{S}_n$-global, i.e. when $k=n$ whence $\mu=\varnothing$.

Theorem 5.4. *The globalization of $V_{(\lambda,\mu)}$ is given by*

$$\mathsf{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}(V_\lambda\otimes V_\mu)\cong\bigoplus_{\rho\vdash n} V_\rho^{\oplus c^\rho_{\lambda\mu}}, \tag{5.6}$$

where $c^\rho_{\lambda\mu}$ are the Littlewood-Richardson coefficients.

Proof. The result follows from Theorem 4.4 and classical formulas (see [Sta99, Section 7.18]). □

Finally, we want to apply the Frobenius reciprocity (4.6) to get formulas for the partial induction to $\mathfrak{S}_n$ of an irreducible representation of $\mathfrak{S}_{n-1}$. Given $\nu\vdash n-1$ and the corresponding irreducible V_ν of $\mathfrak{S}_{n-1}$, we want to find a formula for $\mathsf{PInd}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}V_\nu$. Given $\lambda\vdash k$ and $\mu\vdash n-k$ with $1\le k\le n$, by Frobenius reciprocity we have

$$\mathsf{Hom}_{\mathfrak{S}_n}(\mathsf{PInd}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}V_\nu, V_{(\lambda,\mu)})\cong \mathsf{Hom}_{\mathfrak{S}_{n-1}}(V_\nu,\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}V_{(\lambda,\mu)}).$$

Combining this with (5.4) we get immediately the following theorem.

Theorem 5.5. *We have*

$$\mathsf{PInd}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}V_\nu = \bigoplus_{\lambda,\mu:|\lambda|+|\mu|=n} V_{(\lambda,\mu)}^{\oplus d^\nu_{\lambda\mu}}, \tag{5.7}$$

where $d^\nu_{\lambda\mu}$ is defined in (5.5).

In the next section we describe a situation that does not occur in general for any G and H, but that is typical of towers of groups, like it is the case for the symmetric groups.

5.2. **Branching rules.** Consider one of the irreducible $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations $V_{(\lambda,\mu)}$, where $\lambda\vdash k$ and $\mu\vdash n-k$ and where $k\ge 1$. Consider the subgroup $\mathfrak{S}'_{n-1}:=\mathfrak{S}_n^{\{n\}}\equiv\mathfrak{S}_{n-1}\times\mathfrak{S}_1\subset\mathfrak{S}_n$. We want to describe the decomposition of the $\mathfrak{S}'_{n-1}$-partial representation $\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}'_{n-1}}V_{(\lambda,\mu)}$ as a sum of irreducibles.

To lighten the notation, we let $G:=\mathfrak{S}_n$, $H:=\mathfrak{S}_{n-1}$, $G':=\mathfrak{S}'_{n-1}$, $H':=G'\cap H=\mathfrak{S}'_{n-2}$, $K:=\mathfrak{S}_k\times\mathfrak{S}_{n-k}\subseteq G$ and $K':=G'\cap K\equiv\mathfrak{S}_k\times\mathfrak{S}_{n-k-1}$.

The case $k=n$ corresponds to $\mathfrak{S}_n$-global representations, so it gives the well-known branching rule

$$\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}'_{n-1}}V_{(\lambda,\varnothing)}\cong\bigoplus_{\lambda^1\to\lambda}V_{(\lambda^1,\varnothing)}.$$

The case $k=1$ corresponds to the $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations where any $\sigma\in\mathfrak{S}_n\setminus\mathfrak{S}_{n-1}$ acts as 0 (see Remark 3.28). It is easy to see that

$$\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}'_{n-1}}V_{((1),\mu)}=\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}'_{n-1}}(\mathsf{Ind}_{[1]}(V_{(1)}\otimes V_\mu))\cong\bigoplus_{\mu^1\to\mu}\mathsf{Ind}_{[1]}(V_{(1)}\otimes V_{\mu^1})=\bigoplus_{\mu^1\to\mu}V_{((1),\mu^1)}.$$

From now on we assume $n>k>1$. By setting $\mathfrak{S}'_{n-2}:=\mathfrak{S}'_{n-1}\cap\mathfrak{S}_{n-1}=\mathfrak{S}_n^{\{1\}}\cap\mathfrak{S}_n^{\{n\}}$, it is clear that these will be actually irreducible $\mathfrak{S}'_{n-2}$-global $\mathfrak{S}'_{n-1}$-partial representations.

We already identified in Example 2.7 the action of G on $X := G/K$ by left multiplication as the action on $\binom{[n]}{k}$. Under this identification, there are two G'-orbits in X: the set $X' := G'K/K$ of k-sets not containing n and the set $\widetilde{X} := G'(2,n)K/K$ of k-sets containing n.

In our identification (i.e. $\sigma\mathfrak{S}_n^{[k]} \leftrightarrow \sigma([k])$), the set $Y := HK/K \subset X = \binom{[n]}{k}$ corresponds to

$$Y = \left\{\sigma\mathfrak{S}_n^{[k]} \mid \sigma \in \mathfrak{S}_n^{[1]}\right\} = \left\{ A \in \binom{[n]}{k} \;\middle|\; 1 \in A \right\}$$

so that

$$Y \cap X' = H'K/K = \left\{ A \in \binom{[n]}{k} \;\middle|\; 1 \in A, n \notin A \right\}$$

and

$$Y \cap \widetilde{X} = H'(2,n)K/K = \left\{ A \in \binom{[n]}{k} \;\middle|\; 1 \in A, n \in A \right\}.$$

So if (W,ρ) is an irreducible representation of K, then the H-global G-partial representation

$$\mathsf{Ind}_{KH} W \cong \bigoplus_{gK \in HK/K} W^{gK}$$

decomposes, as G'-partial representation, as the direct sum of two subrepresentations

$$\bigoplus_{gK \in H'K/K} W^{gK} \oplus \bigoplus_{gK \in H'(2,n)K/K} W^{gK}. \tag{5.8}$$

We begin by considering the restriction to G' of the representation on the left in (5.8). The assignment

$$\varphi\colon H'K/K \to H'K'/K', \qquad gK \mapsto gK',$$

is a well-defined isomorphism of G'-partial actions, so that we have an isomorphism of G'-partial representations

$$\bigoplus_{gK \in H'K/K} W^{gK} \cong \bigoplus_{gK' \in H'K'/K'} W^{gK'} = \mathsf{Ind}_{K'H'}(\mathsf{Res}^K_{K'}(W)).$$

Applying this to $W = V_\lambda \otimes V_\mu$ gives

$$\bigoplus_{gK \in H'K/K} W^{gK} \cong \bigoplus_{\mu^1 \to \mu} \mathsf{Ind}_{K'H'}(V_\lambda \otimes V_{\mu^1}) = \bigoplus_{\mu^1 \to \mu} V_{(\lambda,\mu^1)}.$$

Concerning the representation on the right in (5.8), the argument is similar: the map

$$\varphi'\colon H'(2,n)K/K \to H'K'_{(2,n)}/K'_{(2,n)}, \qquad g(2,n)K \mapsto gK'_{(2,n)},$$

(where we recall that $K'_{(2,n)} := (2,n)K'(2,n)$ as in Notation 4.1) is an isomorphism of G'-partial actions, so that we have a G'-isomorphism

$$\bigoplus_{g(2,n)K \in H'(2,n)K/K} W^{g(2,n)K} \cong \bigoplus_{gK' \in H'K'_{(2,n)}/K'_{(2,n)}} W_{(2,n)}^{gK'_{(2,n)}} = \mathsf{Ind}_{K'_{(2,n)}H'}(\mathsf{Res}^{K_{(2,n)}}_{K'_{(2,n)}} W_{(2,n)}),$$

where $W_{(2,n)}$ denotes the representation $\rho^{(2,n)}\colon K_{(2,n)} \to \mathsf{GL}(W)$, $x \mapsto \rho\big((2,n)x(2,n)\big)$, again as in Notation 4.1. Applying this to $W = V_\lambda \otimes V_\mu$ gives

$$\mathsf{Res}^{K_{(2,n)}}_{K'_{(2,n)}} W_{(2,n)} \cong \mathsf{Res}^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}_{\mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}}(V_\lambda \otimes V_\mu) \cong \bigoplus_{\lambda^1 \to \lambda} V_{\lambda^1} \otimes V_\mu$$

so that

$$\bigoplus_{g(2,n)K \in H'(2,n)K/K} W^{g(2,n)K} \cong \bigoplus_{\lambda^1 \to \lambda} \mathsf{Ind}_{K'_{(2,n)}H'}(V_{\lambda^1} \otimes V_\mu) = \bigoplus_{\lambda^1 \to \lambda} V_{(\lambda^1,\mu)}.$$

Finally we get the following branching rule.

Theorem 5.6. *We have*

$$\mathsf{Res}^{\mathfrak{S}_n}_{\mathfrak{S}'_{n-1}} V_{(\lambda,\mu)} \cong \bigoplus_{\lambda^1 \to \lambda} V_{(\lambda^1,\mu)} \oplus \bigoplus_{\mu^1 \to \mu} V_{(\lambda,\mu^1)}. \tag{5.9}$$

6. Comments and future directions

One thing that we did not discuss here is the character theory. In fact there is a general character theory (over $\mathbb{C}$) for finite groupoids [IR19], which can be applied to $\Gamma_H(G)$. This would allow us to recover the character theory of H-global G-partial representations from the one of the subgroups $K_{i,j}$ appearing in Theorem 3.22.

Another thing that we left out is a possible approach to H-global G-partial representations via inverse semigroups (cf. [Exe98]), the latter admitting a general theory (cf. [Ste06, Ste08, Ste16]) that can certainly be deployed in this direction. We leave these aspects to future work.

Moreover, there are several questions that arise naturally from this work.

First of all, one can look at examples other than $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1} \equiv \mathfrak{S}_1 \times \mathfrak{S}_{n-1}$. It should be noted that already the cases $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-2} \equiv \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_{n-2}$ or $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_2 \times \mathfrak{S}_{n-2}$ seem to be too complicated to compute explicitly. On the other hand, for example, it would be interesting to look into other Coxeter groups.

On the combinatorial side, a combinatorics could be developed for $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations analogous to the classical one for $\mathfrak{S}_n$-global representations.

Also, we wonder what properties of G are determined by the H-global G-partial representations when H is a characteristic subgroup of G, e.g. the derived group or the center of G. For example, it follows from [DEP00, Theorem 4.4] that the isomorphism class of G/G' is determined by the partial representations of G/G', where G' is the derived subgroup of G. What can we say about G by knowing the G'-global G-partial representations?

At a more speculative level, it would be worthwhile to see how the H-global G-partial representations are related to the Hecke algebra $\mathsf{End}_G(\mathbb{C}[G/H]) \equiv \mathbb{C}[G]^{H\times H}$, especially in the case when (G, H) is a Gelfand pair.

Also, as the definition of an H-global G-partial representation makes sense also in infinite contexts, it would be interesting to look into infinite situations, like for example infinite compact groups, or Lie groups.

Even more generally, one could look to the case of Hopf algebras, where similar notions can be defined, and they would be interesting to study. For example fixing a Hopf subalgebra H of a given Hopf algebra, in some cases this might produce computable and interesting H-global partial representations.

Appendix A. Irreducibles of semisimple algebras

In this appendix we outline how the representation theory of a finite-dimensional associative semisimple unital algebra A gets recovered from the representation theory of the algebras eAe for the idempotents $e \in A$. Along the way, we sketch a proof of Theorem 3.24.

Let A be a finite-dimensional associative semisimple unital algebra over $\mathbb{C}$, with $A \not\cong \mathbb{C}$. So, A is the direct sum of matrix algebras by Wedderburn theory. Let $e \in A$ be a nontrivial idempotent of A, i.e. $0 \neq e \neq 1$ (such an e does exist since $A \not\cong \mathbb{C}$).

Given an eAe-module W we define the A-module $\mathsf{Ind}_e W$ by setting

$$\mathsf{Ind}_e W := Ae \otimes_{eAe} W.$$

Viceversa, given an A-module V, we define the eAe-module $\mathsf{Res}_e V$ by setting

$$\mathsf{Res}_e V := eV.$$

Observe that for any eAe-module W we have natural isomorphisms

$$\mathsf{Res}_e(\mathsf{Ind}_e W) = e(Ae \otimes_{eAe} W) \cong eAe \otimes_{eAe} W \cong W.$$

Moreover

$$Ae\,\mathsf{Ind}_e W = Ae(Ae \otimes_{eAe} W) = Ae(eAe) \otimes_{eAe} W = Ae \otimes_{eAe} W = \mathsf{Ind}_e W.$$

Proposition A.1. *If V is an A-module such that $\mathsf{Res}_e V = eV$ is an irreducible eAe-module and $AeV = V$, then V is irreducible.*

Proof. If $V = V_1 \oplus V_2$ as A-modules, then $eV = eV_1 \oplus eV_2$; but eV is irreducible, so without loss of generality $eV_2 = 0$. Now

$$V = AeV = AeV_1 \oplus AeV_2 = AeV_1 \subseteq V_1,$$

which implies $V = V_1$. So V is irreducible. □

Putting together the previous observation we get the following corollary, which is part of Theorem 3.24.

Corollary A.2. *If W is an irreducible eAe-module, then $\mathsf{Ind}_e W$ is irreducible.*

Proposition A.3. *If V is an irreducible A-module and $\mathsf{Res}_e V = eV \neq 0$, then $\mathsf{Res}_e V = eV$ is an irreducible eAe-module.*

Proof. For any $0 \neq v \in eV$ we have

$$eAev = eAv = eV$$

as $Av = V$ since V is irreducible. □

From the previous results we can easily deduce the following theorem, which is the remaining part of Theorem 3.24.

Theorem A.4. *Every irreducible A-module V is isomorphic to $\mathsf{Ind}_e W$ for some nontrivial idempotent $e \in A$ and some irreducible eAe-module W.*

Proof. Let V be an irreducible A-module. Since A is semisimple, there exists a nontrivial idempotent $e \in A$ such that $eV \neq 0$ (otherwise 1, which is a sum of nontrivial idempotents, acts as 0). Then eV is an irreducible eAe-module by Proposition A.3 and $\mathsf{Ind}_e(eV)$ is an irreducible A-module by Corollary A.2. So we have a map of A-modules

$$\mathsf{Ind}_e(eV) = Ae \otimes_{eAe} eV \to V, \quad ae \otimes ev \mapsto aev,$$

which is clearly nonzero and hence an isomorphism by Schur's lemma. □

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Université Libre de Bruxelles (ULB), Département de Mathématique, Boulevard du Triomphe, B-1050 Bruxelles, Belgium & Università di Pisa, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
Email address: michele.dadderio@ulb.be

Université Libre de Bruxelles (ULB), Département de Mathématique, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
Email address: william.hautekiet@ulb.be

Université Libre de Bruxelles (ULB), Département de Mathématique, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
Email address: paolo.saracco@ulb.be

Université Libre de Bruxelles (ULB), Département de Mathématique, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
Email address: joost.vercruysse@ulb.be