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# Smash and crossed product decompositions of universal enveloping algebras and Lie-Rinehart algebra connections

Paolo Saracco

ULB - Université Libre de Bruxelles

Fonds de la Recherche Scientifique - FNRS

Hopf algebras, monoidal categories and related topics

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(based on a joint work with X. Bekaert and N. Kowalzig)

# The Setting, the Known, and the Question

- Let $\pi\colon P \to M$ be a principal $H$-bundle.
- The associated short exact sequence of vector bundles

$$0 \to VP \to TP \to \pi^* TM \to 0$$

induces a short exact sequence of Lie algebroids

$$(\dagger) \qquad 0 \to \frac{VP}{H} \to \frac{TP}{H} \to TM \to 0$$

over $M$, called the Atiyah sequence of the principal bundle.

- An Ehresmann connection is a chosen splitting of $(\dagger)$ as sequence of vector bundles.

**Question**

- How does the existence of connections reflect on the associated algebra of differential operators $\mathcal{D}^H(P)$?
- Can we characterize connections on Lie algebroids in terms of their universal enveloping algebras?

# Sources of inspiration

**(A)** The universal enveloping algebra of a semi-direct sum $\mathfrak{g} = \mathfrak{n} \ni\!\!\!\!+ \mathfrak{h}$ is isomorphic to the smash product of $U(\mathfrak{n})$ and $U(\mathfrak{h})$,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) \# U(\mathfrak{h}),$$

as associative algebras.

**(B)** The universal enveloping algebra of a Lie algebra extension

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

is isomorphic to the crossed product of $U(\mathfrak{n})$ and $U(\mathfrak{h})$,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) * U(\mathfrak{h}),$$

as associative algebras.

**(C)** If $\mathfrak{n} \subseteq \mathfrak{g}$ is a Lie subalgebra, then $U(\mathfrak{g})$ is free over $U(\mathfrak{n})$. More precisely,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) \otimes S(\mathfrak{h})$$

as a left $U(\mathfrak{n})$-module.

# Lie algebroids, Lie-Rinehart algebras . . .

**Definition**

A Lie algebroid is a vector bundle $E \to M$ together with a Lie bracket on its space of sections $\Gamma(E)$ and a vector bundle morphism $\rho\colon E \to TM$ satisfying the Leibniz rule

$$[X, f \cdot Y] = f \cdot [X, Y] + \rho(X)(f) \cdot Y$$

for all $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$.

**Definition**

A Lie-Rinehart algebra over the commutative algebra $A$ is a Lie algebra $\mathfrak{h}$ together with an $A$-module structure $A \to \mathrm{End}(\mathfrak{h})$ and a Lie algebra map $\omega\colon \mathfrak{h} \to \mathrm{Der(A)}$ such that $\omega$ is $A$-linear and the Leibniz rule

$$[X, f \cdot Y] = f \cdot [X, Y] + \omega(X)(f) \cdot Y$$

is satisfied for all $X, Y \in \mathfrak{h}$, $f \in A$.

**Remark**

The global sections $\Gamma(E)$ of a Lie algebroid $E$ form a Lie-Rinehart algebra over $C^\infty(M)$, which is finitely generated and projective as $C^\infty(M)$-module.

# ... and their universal enveloping algebras

**Definition**

The universal enveloping algebra of a Lie-Rinehart algebra $(A, \mathfrak{h}, \omega)$ is a $\Bbbk$-algebra $\mathcal{U}(\mathfrak{h})$ endowed with a morphism $\iota_A \colon A \to \mathcal{U}(\mathfrak{h})$ of $\Bbbk$-algebras and a morphism $\iota_{\mathfrak{h}} \colon \mathfrak{h} \to \mathcal{U}(\mathfrak{h})$ of Lie algebras such that

$$\iota_{\mathfrak{h}}(aX) = \iota_A(a)\,\iota_{\mathfrak{h}}(X) \quad \text{and} \quad \iota_{\mathfrak{h}}(X)\,\iota_A(a) - \iota_A(a)\,\iota_{\mathfrak{h}}(X) = \iota_A\big(\omega(X)(a)\big)$$

for all $a \in A$, $X \in \mathfrak{h}$, and which is universal with respect to this property.

**Remark** [S.]

There is a more compact version of this universal property in terms of a functor $\mathcal{L}_A$ from $A$-rings to Lie-Rinehart algebras over $A$.

The universal enveloping algebra of a Lie algebroid is the universal enveloping algebra of the associated Lie-Rinehart algebra.

**Example**

The universal enveloping algebra of the Lie algebroid $TM \to M$ is the algebra of differential operators on $M$.

# Rephrasing the question

Given a short exact sequence of Lie-Rinehart algebras

$$(\dagger) \qquad 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

which are finitely generated and projective as $A$-modules, an Ehresmann connection is a chosen splitting of $(\dagger)$ as sequence of $A$-modules.

**Question**

Can we describe Ehresmann connections in terms of the involved universal enveloping algebras?

**Definition**

A (left) Hopf algebroid is a pair $(A, \mathcal{H})$ of $\Bbbk$-algebras such that

- $\mathcal{H}$ is an $A^{e} = A \otimes A^{\mathrm{op}}$-ring via an algebra map $\mathsf{s} \otimes \mathsf{t} = \eta \colon A^{e} \to \mathcal{H}$;
- ${}_{\eta}\mathcal{H}$ admits an $A$-coring structure $(\mathcal{H}, \Delta \colon \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}, \varepsilon \colon \mathcal{H} \to A)$;
- $\Delta$ is multiplicative, i.e. $\Delta(u)\Delta(v)$ makes sense and equals $\Delta(uv)$;
- $\varepsilon$ is a left character, i.e. $\varepsilon(u\,\mathsf{s}\varepsilon(v)) = \varepsilon(uv) = \varepsilon(u\,\mathsf{t}\varepsilon(v))$;
- $\beta \colon \mathcal{H} \otimes_{A^{\mathrm{op}}} \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$, $u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)} v$ is bijective.

# A Blattner-Cohen-Montgomery theorem

**Theorem** [Bekaert-Kowalzig-S.]

Let $(U, A)$ and $(V, A)$ be left Hopf algebroids. Let $U \xrightarrow{\pi} V$ be a surjective morphism of left Hopf algebroids with left Hopf kernel $B$ and which also admits an $A$-coring section $\gamma \colon V \to U$.

Assume that $U_{\triangleleft}$ is projective as an $A$-module and that

$$\gamma(1_V) = 1_U, \quad \gamma(a \blacktriangleright v) = a \blacktriangleright \gamma(v), \quad \gamma(v \blacktriangleleft a) = \gamma(v) \blacktriangleleft a.$$

Then there exists a Hopf 2-cocycle $\sigma \colon U \otimes_{A^{\mathrm{op}}} U \to B$ and an isomorphism of $A^{\mathrm{e}}$-rings and of right $V$-comodule algebras

$$\Phi \colon B \#_{\sigma} V \to U, \quad b \# v \mapsto b\, \gamma(v).$$

# Applications . . .

Given a L-R algebra $(A, \mathfrak{g}, \omega)$, $\mathcal{U}(\mathfrak{g})$ is a Hopf algebroid.

**Theorem** [Bekaert-Kowalzig-S.]

If $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$ is a ses of projective L-R algebras, then

$$\mathcal{U}(\mathfrak{g}) \;\simeq\; U(\mathfrak{n}) \,\#_\sigma\, \mathcal{U}(\mathfrak{h}),$$

as $A$-rings and right $\mathcal{U}(\mathfrak{h})$-comodule algebras. In particular,

- If $\mathfrak{g} \simeq \mathfrak{n} \ni \mathfrak{h}$ is a semi-direct sum, then

$$\mathcal{U}(\mathfrak{n} \ni \mathfrak{h}) \;\simeq\; U(\mathfrak{n}) \,\#\, \mathcal{U}(\mathfrak{h}).$$

- If $\mathfrak{g} \simeq \mathfrak{n} \ni_\tau \mathfrak{h}$ is a curved semi-direct sum, then

$$\mathcal{U}(\mathfrak{n} \ni_\tau \mathfrak{h}) \;\simeq\; U(\mathfrak{n}) \,\#_\sigma\, \mathcal{U}(\mathfrak{h}).$$

**Key:** The construction of a symmetrization map $\mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

# . . . and related results

**Theorem** [Bekaert-Kowalzig-S.]
Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an inclusion of Lie-Rinehart algebras which are projective as left $A$-modules.

Suppose that the quotient $A$-module $\mathfrak{g}/\mathfrak{h}$ is projective, too.

Then we have an isomorphism

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_A \mathcal{S}(\mathfrak{g}/\mathfrak{h})$$

as left $\mathcal{U}(\mathfrak{h})$-modules. In particular, $\mathcal{U}(\mathfrak{g})$ is projective over $\mathcal{U}(\mathfrak{h})$.

# Back to the geometric example

**Example** [Bekaert-Kowalzig-S.]
Consider a principal $H$-bundle of total space $P$ and base $M$.
Recall that an Ehresmann connection on $P$ is equivalent to a splitting of the Atiyah sequence

$$0 \to \frac{VP}{H} \to \frac{TP}{H} \to TM \to 0.$$

Up to a technical condition, Ehresmann connections on $P$ correspond bijectively to factorisations

$$\mathcal{D}^H(P) \;\cong\; \mathcal{V}^H(P) \,\#_\sigma\, \mathcal{D}(M)$$

of the algebra generated by the invariant vector fields on $P$.

Here, $\mathcal{V}^H(P)$ and $\mathcal{D}(M)$ are generated by invariant vector fields tangentials to the fibres and by differential operators on the base, respectively.

# The end

Thank you