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Smash and crossed product decompositions of universal enveloping algebras and Lie-Rinehart algebra connections

Paolo Saracco

ULB - Université Libre de Bruxelles

Fonds de la Recherche Scientifique - FNRS

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(based on a joint work with X. Bekaert and N. Kowalzig)

The Setting, the Known, and the Question

- Let $\pi\colon P \to M$ be a principal H-bundle.
- The associated short exact sequence of vector bundles

$$0 \to VP \to TP \to \pi^* TM \to 0$$

induces a short exact sequence of Lie algebroids

$$(\dagger) \qquad 0 \to \frac{VP}{H} \to \frac{TP}{H} \to TM \to 0$$

over M, called the Atiyah sequence of the principal bundle.
- An Ehresmann connection is a chosen splitting of $(\dagger)$ as sequence of vector bundles.

Question

- How does the existence of connections reflect on the associated algebra of differential operators $\mathcal{D}^H(P)$?
- Can we characterize Ehresmann connections in this way?

Sources of inspiration

(A) The universal enveloping algebra of a semi-direct sum $\mathfrak{g} = \mathfrak{n} \niplus \mathfrak{h}$ of Lie algebras is isomorphic to the smash product of $U(\mathfrak{n})$ and $U(\mathfrak{h})$,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) \# U(\mathfrak{h}),$$

as associative algebras.

(B) The universal enveloping algebra of a Lie algebra extension

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

is isomorphic to the crossed product of $U(\mathfrak{n})$ and $U(\mathfrak{h})$,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) * U(\mathfrak{h}),$$

as associative algebras.

(C) If $\mathfrak{n} \subseteq \mathfrak{g}$ is a Lie subalgebra, then $U(\mathfrak{g})$ is free over $U(\mathfrak{n})$. More precisely,

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}) \otimes S(\mathfrak{h})$$

as a left $U(\mathfrak{n})$-module.

Lie algebroids, Lie-Rinehart algebras . . .

Definition

A Lie algebroid is a vector bundle $E \to M$ together with a Lie bracket on its space of sections $\Gamma(E)$ and a vector bundle morphism $\rho \colon E \to TM$ satisfying the Leibniz rule

$$[X, f \cdot Y] = f \cdot [X, Y] + \rho(X)(f) \cdot Y$$

for all $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$.

Definition

A Lie-Rinehart algebra over the commutative algebra A is a Lie algebra $\mathfrak{h}$ together with an A-module structure $A \to \mathrm{End}(\mathfrak{h})$ and a Lie algebra map $\omega \colon \mathfrak{h} \to \mathrm{Der(A)}$ such that ω is A-linear and the Leibniz rule

$$[X, f \cdot Y] = f \cdot [X, Y] + \omega(X)(f) \cdot Y$$

is satisfied for all $X, Y \in \mathfrak{h}$, $f \in A$.

Remark

The global sections $\Gamma(E)$ of a Lie algebroid E form a Lie-Rinehart algebra over $C^\infty(M)$, which is finitely generated and projective as $C^\infty(M)$-module.

. . . and their universal enveloping algebras

Definition

The universal enveloping algebra of a Lie-Rinehart algebra $(A, \mathfrak{h}, \omega)$ is a $\Bbbk$-algebra $\mathcal{U}_A(\mathfrak{h})$ endowed with a morphism $\iota_A\colon A \to \mathcal{U}_A(\mathfrak{h})$ of $\Bbbk$-algebras and a morphism $\iota_{\mathfrak{h}}\colon \mathfrak{h} \to \mathcal{U}_A(\mathfrak{h})$ of Lie algebras such that

$$\iota_{\mathfrak{h}}(aX) = \iota_A(a)\,\iota_{\mathfrak{h}}(X) \quad \text{and} \quad \iota_{\mathfrak{h}}(X)\,\iota_A(a) - \iota_A(a)\,\iota_{\mathfrak{h}}(X) = \iota_A\big(\omega(X)(a)\big)$$

for all $a \in A$, $X \in \mathfrak{h}$, and which is universal with respect to this property.

Remark [S.]

There is a more compact version of this universal property in terms of a functor $\mathcal{L}_A$ from A-rings to Lie-Rinehart algebras over A.

The universal enveloping algebra of a Lie algebroid is the universal enveloping algebra of the associated Lie-Rinehart algebra.

Example

The universal enveloping algebra of the Lie algebroid $TM \to M$ is the algebra of differential operators on M.

Rephrasing the question

Given a short exact sequence of Lie-Rinehart algebras

$$(\dagger) \qquad 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

which are finitely generated and projective as A-modules, an Ehresmann connection is a chosen splitting of $(\dagger)$ as sequence of A-modules.

- If the splitting $\gamma \colon \mathfrak{h} \to \mathfrak{g}$ is Lie-Rinehart, then $\mathfrak{g} \cong \mathfrak{n} \niplus \mathfrak{h}$ where

$$\big[(m, X), (n, Y)\big] = \big([m, n] + [\gamma(X), n] - [\gamma(Y), m], [X, Y]\big)$$

- If the splitting $\gamma \colon \mathfrak{h} \to \mathfrak{g}$ is just A-linear, then $\mathfrak{g} \cong \mathfrak{n} \niplus_{\tau} \mathfrak{h}$ where

$$\tau \colon \mathfrak{h} \wedge_A \mathfrak{h} \to \mathfrak{n}, \quad (X, Y) \mapsto \big[\gamma(X), \gamma(Y)\big] - \gamma[X, Y] \qquad \text{and}$$

$$\big[(m, X), (n, Y)\big] = \big([m, n] + [\gamma(X), n] - [\gamma(Y), m] + \tau(X, Y), [X, Y]\big)$$

Question

Can we describe Ehresmann connections in terms of the involved universal enveloping algebras?

Hopf algebroids

Definition

A (left) Hopf algebroid is a pair $(A, \mathcal{H})$ of $\Bbbk$-algebras such that

- $\mathcal{H}$ is an $A^e = A \otimes A^{\mathrm{op}}$-ring via an algebra map $\mathsf{s} \otimes \mathsf{t} = \eta \colon A^e \to \mathcal{H}$. We write $a \triangleright x \triangleleft b = s(a)t(b)x$ and $a \blacktriangleright x \blacktriangleleft b = xs(b)t(a)$.
- ${}_{\eta}\mathcal{H}$ admits an A-coring structure $(\mathcal{H}, \Delta \colon \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}, \varepsilon \colon \mathcal{H} \to A)$;
- Δ is multiplicative, i.e. $\Delta(u)\Delta(v)$ makes sense and equals $\Delta(uv)$;
- ε is a left character, i.e. $\varepsilon(u\,\mathsf{s}\varepsilon(v)) = \varepsilon(uv) = \varepsilon(u\,\mathsf{t}\varepsilon(v))$;
- $\beta \colon \mathcal{H} \otimes_{A^{\mathrm{op}}} \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$, $u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)}v$ is bijective. We write $\beta^{-1}(u \otimes_A 1) = u_{(+)} \otimes_{A^{\mathrm{op}}} u_{(-)}$.

Examples

- Affine groupoid schemes are commutative Hopf algebroids.
- For any algebra A, $(A, A \otimes A^{\mathrm{op}})$ is a Hopf algebroid with $\eta = \mathrm{Id}$,

$$\Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b), \qquad \varepsilon(a \otimes b) = ab.$$

- For a Lie-Rinehart algebra $(A, \mathfrak{g}, \omega)$, $\mathcal{U}_A(\mathfrak{g})$ is a Hopf algebroid with

$$\Delta(X) = X \otimes_A 1 + 1 \otimes_A X, \qquad \varepsilon(X) = 0,$$
$$\beta^{-1}(X \otimes_A 1) = X \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} X.$$

A Blattner-Cohen-Montgomery theorem

Theorem [Bekaert-Kowalzig-S.]

Let (U, A) and (V, A) be left Hopf algebroids. Let $U \xrightarrow{\pi} V$ be a surjective morphism of left Hopf algebroids with left Hopf kernel

$$B := \left\{ u \in U \mid u_{(1)} \otimes_A \pi(u_{(2)}) = u \otimes_A 1_V \right\}.$$

We have a well-defined left U-action $u \triangleright b := u_{(+)} b u_{(-)}$ on B.

Assume that π admits an A-coring section $\gamma \colon V \to U$, that $U_{\triangleleft}$ is projective as an A-module and that

$$\gamma(1_V) = 1_U, \quad \gamma(a \blacktriangleright v) = a \blacktriangleright \gamma(v), \quad \gamma(v \blacktriangleleft a) = \gamma(v) \blacktriangleleft a.$$

Then there exists a Hopf 2-cocycle $\sigma \colon V \otimes_{A^{\mathrm{op}}} V \to B$ and an isomorphism of A^{e}-rings and of right V-comodule algebras

$$\Phi \colon B \,\#_\sigma\, V \to U, \quad b \,\#\, v \mapsto b\,\gamma(v).$$

NB: In $B \,\#_\sigma\, V$, $(b \otimes_A u)(b' \otimes_A v) := b(u_{(1)} \triangleright b')\sigma(u_{(2)}, v_{(1)}) \otimes_A u_{(3)} v_{(2)}$.

Applications ...

Theorem [Bekaert-Kowalzig-S.]

If $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 0$ is a ses of projective Lie-Rinehart algebras, then $U_A(\mathfrak{n})$ is the left Hopf kernel of the induced epimorphism $\mathcal{U}_A(\mathfrak{g}) \to \mathcal{U}_A(\mathfrak{h})$, which also admits an A-coring splitting. Hence

$$\mathcal{U}_A(\mathfrak{g}) \;\simeq\; U_A(\mathfrak{n}) \,\#_\sigma\, \mathcal{U}_A(\mathfrak{h}),$$

as A-rings and right $\mathcal{U}_A(\mathfrak{h})$-comodule algebras. In particular,

- If $\mathfrak{g} \simeq \mathfrak{n} \ni \mathfrak{h}$ is a semi-direct sum, then

$$\mathcal{U}_A(\mathfrak{n} \ni \mathfrak{h}) \;\simeq\; U_A(\mathfrak{n}) \,\#\, \mathcal{U}_A(\mathfrak{h}).$$

- If $\mathfrak{g} \simeq \mathfrak{n} \ni_\tau \mathfrak{h}$ is a curved semi-direct sum, then

$$\mathcal{U}_A(\mathfrak{n} \ni_\tau \mathfrak{h}) \;\simeq\; U_A(\mathfrak{n}) \,\#_\sigma\, \mathcal{U}_A(\mathfrak{h}).$$

Key: The construction of a symmetrization map $\mathcal{S}_A(\mathfrak{g}) \to \mathcal{U}_A(\mathfrak{g})$.

. . . and related results

Theorem [Bekaert-Kowalzig-S.]
Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an inclusion of Lie-Rinehart algebras which are projective as left A-modules.

Suppose that the quotient A-module $\mathfrak{g}/\mathfrak{h}$ is projective, too.

Then we have an isomorphism

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_A \mathcal{S}(\mathfrak{g}/\mathfrak{h})$$

as left $\mathcal{U}(\mathfrak{h})$-modules. In particular, $\mathcal{U}(\mathfrak{g})$ is projective over $\mathcal{U}(\mathfrak{h})$.

Back to the geometric example

Example [Bekaert-Kowalzig-S.]
Consider a principal H-bundle of total space P and base M.
Recall that an Ehresmann connection on P is equivalent to a splitting of the Atiyah sequence

$$0 \to \frac{VP}{H} \to \frac{TP}{H} \to TM \to 0.$$

Up to a technical condition, Ehresmann connections on P correspond bijectively to factorisations

$$\mathcal{D}^H(P) \cong \mathcal{V}^H(P) \#_\sigma \mathcal{D}(M)$$

of the algebra generated by the invariant vector fields on P.

Here, $\mathcal{V}^H(P)$ and $\mathcal{D}(M)$ are generated by invariant vector fields tangentials to the fibres and by differential operators on the base, respectively.

Moving forward

Let $H \subseteq G$ be a Lie subgroup of the Lie group G (i.e., a Klein geometry) with associated Lie algebras $\mathfrak{h} \subseteq \mathfrak{g}$.

A Cartan geometry of type (G, H) on a manifold M is a principal H-bundle of total space P and base M, together with a $\mathfrak{g}$-valued H-equivariant 1-form $\varpi \colon TP \to P \times \mathfrak{g}$ (the Cartan connection) inducing an isomorphism $TP/H \cong P \times_H \mathfrak{g}$ which extends the canonical $VP/H \cong P \times_H \mathfrak{h}$.

Remark

Given a Cartan geometry (P, ϖ) on M modelled on (G, H), one can consider the principal G-bundle $Q \coloneqq P \times_H G \to M$. Ehresmann connections on Q induce all the Cartan connections on P.

Question

How can we describe Cartan connections (and their relationship with Ehresmann connections) in terms of the associated universal enveloping algebras?

Transformation Lie-Rinehart algebras and their UEAs

Let $(A, \mathfrak{g}, \omega)$ be a Lie-Rinehart algebra. Suppose that R is a commutative algebra together with an action of $\mathfrak{g}$ by derivations and an algebra morphism $A \to R$ which are compatible.

Then $R \otimes_A \mathfrak{g}$ becomes a Lie-Rinehart algebra over R with respect to

$$\left[r \otimes_A X, r' \otimes_A Y\right] = rr' \otimes_A [X, Y] - r' Y(r) \otimes_A X + rX(r') \otimes_A Y$$

which we call transformation L-R algebra and we denote by $R \rtimes_A \mathfrak{g}$.

Theorem [Bekaert-Kowalzig-S.]

The smash product $R \# \mathcal{U}_A(\mathfrak{g})$ is a cocommutative Hopf algebroid and

$$\mathcal{U}_R(R \rtimes_A \mathfrak{g}) \cong R \# \mathcal{U}_A(\mathfrak{g})$$

as cocommutative Hopf algebroids over R.

Example

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then $\mathfrak{X}(G) \cong C^\infty(G) \rtimes \mathfrak{g}$ (by Maurer-Cartan). Thus,

$$\mathcal{D}(G) \cong C^\infty(G) \# U(\mathfrak{g}).$$

The end

Thank you