On a correspondence between ideal coideals and coideal subrings of Hopf algebroids

Paolo Saracco

ULB - Université Libre de Bruxelles

Fonds de la Recherche Scientifique - FNRS

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A soft (?!) introduction

- Let G be a (discrete, linear, Lie) group.
- Let $\mathcal{R}_{\Bbbk}(G)$ be the algebra of "representative" functions on G

$\mathrm{Fun}(G,\Bbbk)$	$\Bbbk[G]$	$\mathcal{R}_{\Bbbk}(G)$
(finite, discrete)	(linear)	(Lie)

- $\mathcal{R}_{\Bbbk}(G)$ is a Hopf algebra with $\varepsilon(f) = f(1_G)$, $S(f) = f \circ (-)^{-1}$ and

$$\Delta(f) = \sum f_1 \otimes f_2 \quad \Longleftrightarrow \quad \sum f_1(x) f_2(y) = f(xy), \quad \forall\, x, y \in G.$$

- If $I \subseteq \mathcal{R}_{\Bbbk}(G)$ is an ideal, then $\mathcal{R}_{\Bbbk}(G)/I$ "is a subspace of" G.
  If I is also a coideal, $\mathcal{R}_{\Bbbk}(G)/I$ "is a submonoid of" G.
  If I is also compatible with S, then $\mathcal{R}_{\Bbbk}(G)/I$ "is a subgroup of" G.
- If $B \subseteq \mathcal{R}_{\Bbbk}(G)$ is a subalgebra, then B "is a quotient space of" G.
  If B is also a subcoalgebra, B "is a quotient monoid of" G.
  If B is also a sub-Hopf algebra, then B "is a quotient group of" G.

Where do we want to go?

Let H be a Hopf algebra over a field $\Bbbk$ (a bialgebra would suffice).

- If $I \subseteq H$ is a coideal, then $\pi\colon H \to \frac{H}{I}$ is a coalgebra map and

$$^{\mathrm{co}\frac{H}{I}}H := \{h \in H \mid \pi(h_1) \otimes h_2 = 1 \otimes h\}.$$

- If $B \subseteq H$ is a subalgebra,

$$B^+ := B \cap \ker(\varepsilon).$$

Fact [T]

$$(\star) \qquad \begin{array}{ccc} \left\{ \begin{array}{c} \text{left ideal} \\ \text{coideals of } H \end{array} \right\} & \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{right coideal} \\ \text{subalgebras of } H \end{array} \right\} \\ I & \longmapsto & ^{\mathrm{co}\frac{H}{I}}H \\ HB^+ & \longleftarrow\!\shortmid & B \end{array}$$

[T] M. Takeuchi, *Relative Hopf modules–equivalences and freeness criteria*. J. Algebra 60 (1979), no. 2, 452–471.

Why do we want to go there?

- [T] Suppose that H is commutative. The correspondence $(\star)$ is a bijection between sub-Hopf algebras of H and normal Hopf ideals of H. This gives algebraic proofs of a number of results on affine $\Bbbk$-groups, among which that affine abelian groups form an abelian category.
- [N] Suppose that H is cocommutative (i.e., a formal group scheme). The correspondence $(\star)$ is a bijection between sub-Hopf algebras of H and left ideal coideals of H.
- [S] The correspondence $(\star)$ is a bijection between normal Hopf subalgebras B such that H_B is faithfully flat and normal Hopf ideals I such that $H^{H/I}$ is faithfully coflat. This suggests the "correct" definition of s.e.s. of quantum groups.
- $(\star)$ can be extended to Hopf-Galois extensions of $\Bbbk$ different from H. For a classical Galois extension $K/\Bbbk$ with Galois group G and for $H := \mathcal{F}un(G, \Bbbk)$, it is the well-known Galois correspondence.

[N] K. Newman, *A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras*. J. Algebra 36 (1975), no. 1, 1–15.

[S] H.-J. Schneider, *Some remarks on exact sequences of quantum groups*. Comm. Algebra 21 (1993), no. 9, 3337–3357.

[T] M. Takeuchi, *A correspondence between Hopf ideals and sub-Hopf algebras*. Manuscripta Math. 7 (1972), 251–270.

Why (and what are) Hopf algebroids?

What

A (left) Hopf algebroid is a pair $(A, \mathcal{H})$ of $\Bbbk$-algebras such that

- $\mathcal{H}$ is an $A^{\mathrm{e}} = A \otimes A^{\mathrm{o}}$-ring via an algebra map $\mathrm{s} \otimes \mathrm{t} = \eta \colon A^{\mathrm{e}} \to \mathcal{H}$;
- ${}_{\eta}\mathcal{H}$ admits an A-coring structure $(\mathcal{H}, \Delta \colon \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}, \varepsilon \colon \mathcal{H} \to A)$;
- Δ is multiplicative, i.e. $\Delta(u)\Delta(v)$ makes sense and equals $\Delta(uv)$;
- ε is a left character, i.e. $\varepsilon(u\,\mathrm{s}\varepsilon(v)) = \varepsilon(uv) = \varepsilon(u\,\mathrm{t}\varepsilon(v))$;
- $\beta \colon \mathcal{H} \otimes_{A^{\mathrm{o}}} \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$, $u \otimes_{A^{\mathrm{o}}} v \mapsto u_{(1)} \otimes_A u_{(2)} v$ is bijective.

Why (sloppily)

- Affine groupoid schemes are commutative Hopf algebroids.
- Algebras of differential operators are cocommutative Hopf algebroids.
- Hopf algebroids are quantum groupoids.

Examples

Commutative

- $(A, A \otimes A)$ is a Hopf algebroid w.r.t. $s(a) = a \otimes 1, \quad t(a) = 1 \otimes a,$
$\Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b), \quad \varepsilon(a \otimes b) = ab, \quad S(a \otimes b) = b \otimes a.$
It corresponds to the groupoid of pairs $(X, X \times X)$ on the set X.

- $\big(A, (A \otimes A)[T]\big)$ is a Hopf algebroid w.r.t. the structure above and
$$\Delta(T) = T \otimes_A 1 + 1 \otimes_A T, \qquad \varepsilon(T) = 0, \qquad S(T) = -T.$$
It corresponds to the additive groupoid $(X, X \times G \times X)$ where X is a set, G an abelian group, and $(x, a, y) \circ (y, b, z) = (x, a + b, z)$.

Cocommutative

The Weyl algebra $\Bbbk\langle x, p\rangle / \langle px - xp - 1\rangle$ is a Hopf algebroid over $A := \Bbbk[x]$:
$\Delta(p) = p \otimes_A 1 + 1 \otimes_A p, \quad \varepsilon(p) = 0, \quad \beta^{-1}(p \otimes_A 1) = p \otimes_A 1 - 1 \otimes_A p.$

Quantum

- The pair $(A, A \otimes A^{o})$ is a Hopf algebroid as above.
- If U is a Hopf algebra and A is a left U-module algebra, then the Connes-Moscovici bialgebroid $A \odot U \odot A$ is a Hopf algebroid over A.

Coideal subrings and ideal coideals

Let $(A, \mathcal{H})$ be a bialgebroid.

- If $I \subseteq \mathcal{H}$ is a coideal, then $\pi \colon \mathcal{H} \to \frac{\mathcal{H}}{I}$ is an A-coring map and

$$^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H} := \{h \in \mathcal{H} \mid \pi(h_1) \otimes_A h_2 = \pi(1) \otimes_A h\}.$$

- If $B \subseteq \mathcal{H}$ is a subalgebra,

$$B^+ := B \cap \ker(\varepsilon).$$

Proposition

Under the assumption that ${}_s\mathcal{H}$ is A-flat, we have well-defined inclusion-preserving correspondences

$$\left\{ \begin{array}{c} \text{left ideal} \\ \text{coideals of } \mathcal{H} \end{array} \right\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{right } \mathcal{H}\text{-comodule} \\ A^{\mathrm{o}}\text{-subrings of } \mathcal{H} \text{ via } t \end{array} \right\}$$

$$I \longmapsto {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}$$

$$\mathcal{H}B^+ \longleftarrow B$$

The canonical inclusions and the Galois connection

Theorem

Let $(A, \mathcal{H})$ be a bialgebroid such that ${}_s\mathcal{H}$ is A-flat.

- If B is a right $\mathcal{H}$-comodule A^{o}-subring via t of $\mathcal{H}$, then we have an inclusion $\eta_B \colon B \subseteq {}^{\mathrm{co}\frac{\mathcal{H}}{\mathcal{H}B^+}}\mathcal{H} = \Psi\Phi(B)$. Moreover, $\Phi\Psi\Phi(B) = \Phi(B)$.
- If I is a left ideal coideal in $\mathcal{H}$, then we have an inclusion $\epsilon_I \colon \Psi\Phi(I) = \mathcal{H}\left({}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}\right)^+ \subseteq I$. Moreover, $\Psi\Phi\Psi(I) = \Psi(I)$.

In other words, Φ and Ψ form a *monotone Galois connection* (or, equivalently, an adjunction) between the two lattices and we have that

$$\mathcal{H}B^+ \subseteq I \quad \Longleftrightarrow \quad B \subseteq {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}.$$

The Hopf algebroid case

Let us fix a Hopf algebroid $(A, \mathcal{H})$ such that ${}_s\mathcal{H}$ is A-flat.

Proposition

Let B be a right $\mathcal{H}$-comodule A^{o}-subring via t of $\mathcal{H}$ such that

- $B \longrightarrow \mathcal{H} \underset{x \mapsto x \otimes_B 1}{\overset{x \mapsto 1 \otimes_B x}{\rightrightarrows}} \mathcal{H} \otimes_B \mathcal{H}$ is an equalizer,
- $\beta^{-1}(B \otimes_A 1)$ lies in the image of $B \otimes_{A^o} \mathcal{H}$ inside $\mathcal{H} \otimes_{A^o} \mathcal{H}$.

Then $B = {}^{\mathrm{co}\frac{\mathcal{H}}{\mathcal{H}B^+}}\mathcal{H}$, that is $\Psi\Phi(B) = B$.

Proposition

Let $I \subseteq \mathcal{H}$ be a left ideal coideal such that

- $\mathcal{H} \square^{\frac{\mathcal{H}}{I}} \mathcal{H} \underset{\mathcal{H} \otimes_A \varepsilon}{\overset{\varepsilon \otimes_A \mathcal{H}}{\rightrightarrows}} \mathcal{H} \longrightarrow \mathcal{H}/I$ is a coequalizer,
- $\mathcal{H} \otimes_{A^o} {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}$ injects into $\mathcal{H} \otimes_{A^o} \mathcal{H}$.

Then $I = \mathcal{H}({}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H})^+$, that is to say, $\Phi\Psi(I) = I$.

Purity

Purity

Given a ring R and a morphism of left (resp. right) R-modules $M \to N$, this is left (resp. right) pure if and only if

$$P \otimes_R M \to P \otimes_R N \quad (\text{resp. } M \otimes_R P \to N \otimes_R P)$$

is injective for every right (resp. left) R-module P.

Examples

- If R is a von Neumann regular ring (e.g. if R is a field), then every morphism is pure.
- If $M \to N$ is split injective, then it is pure.
- A ring extension $R \to S$ is pure as a morphism of left (resp. right) R-modules if and only if $- \otimes_R S$ (resp. $S \otimes_R -$) is a faithful functor.

The main result

Theorem

Let $(A, \mathcal{H})$ be a Hopf algebroid such that ${}_s\mathcal{H}$ is A-flat. We have a well-defined inclusion-preserving bijective correspondence

$$\left\{ \begin{array}{c} \text{left ideal coideals } I \text{ for which} \\ \mathcal{H}\square^{\frac{\mathcal{H}}{I}}\mathcal{H} \rightrightarrows \mathcal{H} \to \mathcal{H}/I \text{ is a} \\ \text{coequalizer and } {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H} \subseteq \mathcal{H} \\ \text{is left and right } A^{\mathrm{o}}\text{-pure} \end{array} \right\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{right } \mathcal{H}\text{-comodule } A^{\mathrm{o}}\text{-subrings } B \text{ for} \\ \text{which } B \to \mathcal{H} \rightrightarrows \mathcal{H} \otimes_B \mathcal{H} \text{ is an} \\ \text{equalizer, } B \subseteq \mathcal{H} \text{ is left and right} \\ A^{\mathrm{o}}\text{-pure, } \beta^{-1}(B \otimes_A 1) \subseteq B \otimes_{A^{\mathrm{o}}} \mathcal{H} \end{array} \right\}$$

$$I \longmapsto {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}$$

$$\mathcal{H}B^+ \longleftarrow\!\!\!\!\dashv B$$

The commutative case

- A *commutative Hopf algebroid* is a cogroupoid object in the category of commutative algebras or, equivalently, an affine groupoid scheme (i.e. a representable presheaf of groupoids on $\mathsf{Aff}_{\Bbbk}$).
- It consists of a pair of commutative $\Bbbk$-algebras $(A, \mathcal{H})$ together with a diagram of algebra maps

$$A \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{H} \ (\varepsilon\colon \mathcal{H} \to A,\ \mathcal{S}\colon \mathcal{H} \to \mathcal{H}) \xrightarrow{\ \Delta\ } \mathcal{H} \otimes_A \mathcal{H}$$

satisfying the duals of the groupoid conditions.

- the inverse of the Hopf-Galois map $\beta\colon u \otimes_A v \mapsto u_1 \otimes_A u_2 v$ is

$$\beta^{-1}\colon \mathcal{H} \otimes_A \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}, \qquad u \otimes_A v \mapsto u_1 \otimes_A \mathcal{S}(u_2) v.$$

- If $(A, \mathcal{H})$ is a commutative Hopf algebroid then $\mathscr{G} := (\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$ is the associated groupoid scheme:

$$\mathscr{G}(R) = \Big(\mathsf{CAlg}_{\Bbbk}(A, R), \mathsf{CAlg}_{\Bbbk}(\mathcal{H}, R)\Big)$$

Hopf ideals and subgroupoids

- An ideal $I \subseteq \mathcal{H}$ in a commutative Hopf algebroid $(A, \mathcal{H})$ is called a (*wide*) *Hopf ideal* if

$$\varepsilon(I) = 0, \qquad \Delta(I) \subseteq \mathrm{im}(\mathcal{H} \otimes_A I + I \otimes_A \mathcal{H}), \qquad \mathcal{S}(I) \subseteq I.$$

- $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I})$ is a (*wide closed*) *subgroupoid* of $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$ if $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I})$ is a groupoid itself and $\pi^* : \mathscr{G}_{\mathcal{H}/I} \hookrightarrow \mathscr{G}_{\mathcal{H}}$ induces a morphism of groupoids $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I}) \to (\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$, that is, if and only if I is a Hopf ideal.

Example (The isotropy Hopf algebroid)

- The ideal $\langle s(a) - t(a) \mid a \in A \rangle$ is a Hopf ideal in $\mathcal{H}$.
- $\mathcal{H}_{(i)} := \mathcal{H}/\langle s - t \rangle$ is a commutative Hopf A-algebra.
- The presheaf of groupoids $\left(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}}^{(i)}\right)$ obtained by taking the isotropy groupoid of $\big(\mathscr{G}_A(R), \mathscr{G}_{\mathcal{H}}(R)\big)$ for all R in $\mathsf{CAlg}_{\Bbbk}$ is represented by $\mathcal{H}_{(i)}$:

$$\left(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}}^{(i)}\right) \cong \left(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}_{(i)}}\right).$$

Normal Hopf ideals and normal subgroupoids

- $\mathcal{H}_{(i)}$ is a right $\mathcal{H}$-comodule algebra with coaction defined by

$$\delta_{\mathcal{H}_{(i)}} \colon \mathcal{H}_{(i)} \to \mathcal{H}_{(i)} \otimes_A \mathcal{H}, \qquad \overline{h} \mapsto \overline{h_2} \otimes_A \mathcal{S}(h_1)h_3.$$

- A Hopf ideal I of $\mathcal{H}$ is said to be *normal* if $\langle s-t \rangle \subseteq I$ and for all $\overline{x}$ in $I_{(i)} := I/\langle s-t \rangle$, we have

$$\delta_{\mathcal{H}_{(i)}}(\overline{x}) = \overline{x_2} \otimes_A \mathcal{S}(x_1)x_3 \ \in \ \mathrm{im}\left(I_{(i)} \otimes_A \mathcal{H}\right).$$

- I is a normal Hopf ideal of $(A, \mathcal{H})$ if and only if $\left(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I}\right)$ is a normal subgroupoid of $\left(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}}\right)$.

The main result in the commutative setting

Theorem

If $(A,\mathcal{H})$ is a commutative Hopf algebroid such that ${}_s\mathcal{H}$ is flat, then we have a well-defined inclusion-preserving bijective correspondence

$$\left\{\begin{array}{c}\text{normal Hopf ideals } I \text{ in } \mathcal{H} \text{ s.t.}\\ \mathcal{H}\square^{\frac{\mathcal{H}}{I}}\mathcal{H} \rightrightarrows \mathcal{H} \to \mathcal{H}/I \text{ is a}\\ \text{coeq. and } \mathcal{H} \text{ is pure over } {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{sub-Hopf algebroids } \mathcal{K}\subseteq\mathcal{H}\\ \text{such that } \mathcal{H} \text{ is pure over } \mathcal{K}\end{array}\right\}$$

$$I \longmapsto {}^{\mathrm{co}\frac{\mathcal{H}}{I}}\mathcal{H}$$

$$\mathcal{H}\mathcal{K}^+ \longleftarrow\!\shortmid\; \mathcal{K}$$

Proposition

Let $\Bbbk=\overline{\Bbbk}$ and $(A,\mathcal{K})$ be a sub-Hopf algebroid of $(A,\mathcal{H})$ such that $\mathcal{H}$ is pure over $\mathcal{K}$. Then $\Theta\colon \mathsf{CAlg}_{\Bbbk}(\mathcal{H},\Bbbk)\to\mathsf{CAlg}_{\Bbbk}(\mathcal{K},\Bbbk)$ is surjective.
In the setting of the theorem there are canonical isomorphisms

$$\mathscr{G}_{\mathcal{H}}(\Bbbk)/\mathscr{G}_{\mathcal{H}/\mathcal{H}\mathcal{K}^+}(\Bbbk) \;\cong\; \mathscr{G}_{\mathcal{K}}(\Bbbk) \quad \text{and} \quad \mathscr{G}_{\mathcal{H}}(\Bbbk)/\mathscr{G}_{\mathcal{H}/I}(\Bbbk)\cong\mathscr{G}_{{}^{\mathrm{co}\mathcal{H}/I}\mathcal{H}}(\Bbbk)\,.$$

The Hopf algebra case

For Hopf algebras over a field $\Bbbk$, our main result becomes

Theorem

There is an inclusion-preserving bijective correspondence

$$\left\{ \begin{array}{c} \text{left ideal coideals } I \text{ s.t.} \\ H \square^{\frac{H}{I}} H \rightrightarrows H \to H/I \\ \text{is a coequalizer} \end{array} \right\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{right coideal subalgebras } B \\ \text{s.t. } B \to H \rightrightarrows H \otimes_B H \\ \text{is an equalizer} \end{array} \right\}$$

$$I \longmapsto {}^{\mathrm{co}\frac{H}{I}}H$$

$$HB^+ \longleftarrow\!\shortmid B$$

This extends the known cases such as Takeuchi, Newman, Schneider.

Example

$\mathbb{C}[x, y]$ is a right coideal subalgebra of $H := \mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$ which satisfies the condition above, but over which H is not faithfully flat.

The end

Thank you