Partial Representation Theory and the Globalization Question

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My memories. . .

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At the origins of partial actions

End of XIX century. Given a Lipschitz vector field X on $U \subseteq \mathbb{R}^n$ open,

$$f'(t) = X(f(t)), \qquad f(0) = x_0,$$

has a unique solution for every x_0 in U.

If $\{\phi_t \mid t \in \mathbb{R}\}$ is the flow of X, then each ϕ_t is a diffeo between open subsets of U and $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ when they make sense.

This is the core of the notion of a partial dynamical system.

Beginning of the 90s. R. Exel formalizes the notion of partial action of a group and a related crossed product construction which allows to realize a C^*-algebra with a circle action as partial crossed products.

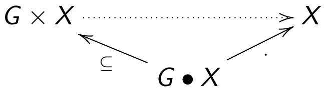
It turns out that many graded C^*-algebras can be described analogously.

Since 2000, partial actions are studied in purely algebraic terms: partial (co)representations of groups and Hopf algebras and of more general objects, Morita and Galois theory, the globalization question, . . .

Partial actions of groups

Definition [Exel, 1998]

A partial action of a group G on a set X is a partial function



satisfying

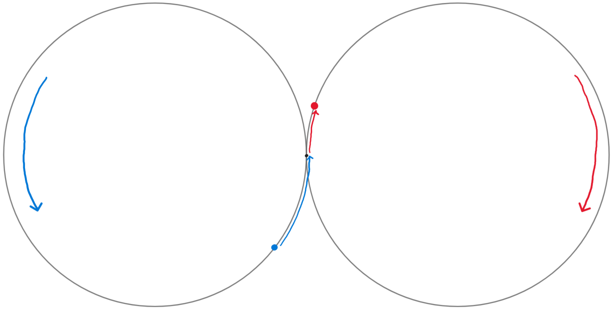
- unitality : $\exists\ 1_G \cdot x$ for every x and $1_G \cdot x = x$;
- partial associativity : $\exists\ g \cdot (h \cdot x)$ iff $\exists\ gh \cdot x$, $h \cdot x$ and $g \cdot (h \cdot x) = gh \cdot x$.

Example 1

The Möbius group acts only partially on the complex plane $\mathbb{C}$.

Example 2

$G = S^1 \times S^1$ acts partially on



Globalization

Fact

Any restriction of a global action $G \times Y \to Y$ to a (non-stable) subspace $X \subseteq Y$ induces a partial action $G \bullet X \to X$.

Definition

A globalization for $G \bullet X \dot{\to} X$ is

- a universal G-set Y
- an injection $\epsilon : X \rightarrowtail Y$

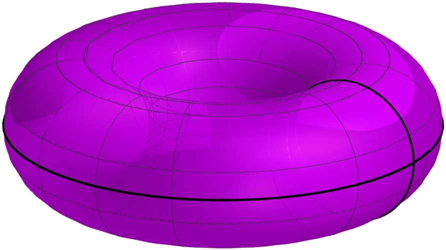
such that the partial action on X is provided by restriction.

Example 1

The Möbius group acts globally on the Riemann sphere, which is the globalization of the partial action of the Möbius group on $\mathbb{C}$.

Example 2

$G = S^1 \times S^1$ acts globally on



Does the globalization exist ?

Theorem [Abadie, 2003]

Every partial action of a (topological) group on a set or topological space admits a globalization.

Since 2003, numerous *ad hoc* globalization constructions :

2004 Kellendonk, Lawson – Partial actions of (topological) groups

2004 Megrelishvili, Schröder – Partial actions of (topological) monoids

2005 Dokuchaev, Exel – Partial group action on unital rings

2010 Alves, Batista – Partial (co)module algebras

2012 Bagio, Paques – Partial groupoid actions

2013 Alvares, Alves, Batista – Partial Hopf module categories

2015 Castro & al. – Partial actions of weak Hopf algebras

2018 Nystedt – Partial category actions on sets and topological spaces

2019 Castro, Quadros – Partial (co)module coalgebras

. . .

Definition [Hu-Vercruysse, 2018]

A geometric partial comodule over a coalgebra H in a monoidal category $\mathcal{C}$ (gpc, for short) is a cospan in $\mathcal{C}$

$$\begin{array}{ccc} X & \dashrightarrow & X \otimes H \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X \bullet H & \end{array}$$

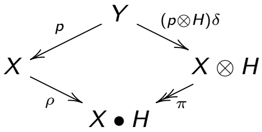
satisfying

- counitality $(X \bullet \varepsilon) \circ \rho \simeq \mathrm{id}_X$ and
- the geometric partial coassociativity $(\rho \bullet \mathrm{id}_H) \circ \rho \simeq (\mathrm{id}_X \bullet \Delta) \circ \rho$.

Definition [S.-Vercruysse]

Given a gpc $(X, X \bullet H, \pi, \rho)$, a globalization for X is an H-comodule (Y, δ) with a morphism $p : Y \to X$ in $\mathcal{C}$ such that

- the following diagram is a pushout square in $\mathcal{C}$



- Y is universal in the sense that the following is bijective

$$\mathrm{Com}^H(Z, Y) \to \mathrm{gPCom}^H(\mathcal{I}(Z), X), \qquad f \mapsto p \circ f.$$

When are gpcs globalizable ?

Theorem [S.-Vercruysse, 2020]

A geometric partial comodule $(X, X \bullet H, \pi, \rho)$ is globalizable iff

(a) A certain equalizer exists in Com^H:

$$\left(Y_X, \delta\right) \xrightarrow{\kappa} \left(X \otimes H, \mathrm{id}_X \otimes \Delta\right) \underset{(\pi \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)}{\overset{\rho \otimes \mathrm{id}_H}{\rightrightarrows}} \left(X \bullet H \otimes H, \mathrm{id}_{X \bullet H} \otimes \Delta\right)$$

(b) The gpc structure on X is induced by the global one on Y_X.
That is, the following is a pushout diagram in $\mathcal{C}$

$$\begin{array}{ccccc} & & Y_X & & \\ & \swarrow^{(X \otimes \varepsilon)\kappa} & & \searrow^{\kappa} & \\ X & & & & X \otimes H \\ & \searrow_{\rho} & & \swarrow_{\pi} & \\ & & X \bullet H & & \end{array}$$

Under these conditions Y_X is the globalization of X.

Applications

How effective is the new globalization theorem ?

- Partial actions of groups and monoids on sets ($\mathcal{C} = \mathsf{Set}^{\mathrm{op}}$)
[Abadie, '03; Kellendonk-Lawson, '04; Megrelishvili-Schröder, '04]

- Partial actions of topological groups on topological spaces ($\mathcal{C} = \mathsf{Top}^{\mathrm{op}}$)
[Abadie, '03]

- Partial comodule algebras over bialgebras and Hopf algebras ($\mathcal{C} = \mathsf{Alg}_{\Bbbk}$)
[Alves-Batista, '10]

- Partial modules over Hopf algebras and partial representations of finite groups ($\mathcal{C} = {\mathsf{Vect}_{\Bbbk}}^{\mathrm{op}}$)
[Alves-Batista-Vercruysse, '19; D'Adderio-Hautekiet-S.-Vercruysse, '22]

- Partial comodules over Hopf algebras ($\mathcal{C} = \mathsf{Vect}_{\Bbbk}$)
[Batista-Hautekiet-S.-Vercruysse, '23]

- In $\mathsf{Top}^{\mathsf{op}}$, $\mathsf{Alg}_{\Bbbk}$ and $\mathsf{CAlg}_{\Bbbk}$ a general globalization does not exist.

A different perspective: from $\mathbb{C}G$ to $\mathbb{C}_{par}G$

Any partial representation of a group G is a module over a groupoid algebra $\mathbb{C}_{par}G$, which is semisimple for G finite and distinguishes finite abelian groups [Dokuchaev-Exel-Piccione, 2000].

Partial modules over a Hopf algebra H are modules over a Hopf algebroid H_{par} [Alves-Batista-Vercruysse, 2015].

Regular partial H-comodules are comodules over a Hopf coalgebroid H^{par} [Alves-Batista-Castro-Quadros-Vercruysse, 2021].

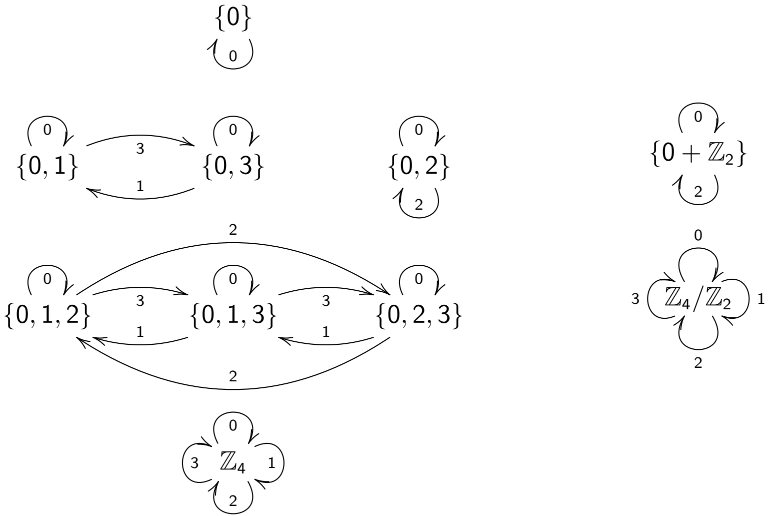
Theorem [D'Adderio-Hautekiet-S.-Vercruysse, 2022]

For $H \subseteq G$ finite groups, H-global G-partial representations are modules over a groupoid algebra $\mathbb{C}_{par}^{H}G$, which is semisimple.

- $\rightsquigarrow$ construction of all the irreducible H-global G-partial representations,
- $\rightsquigarrow$ H-rep decomposition of the restriction of the H-global G-partial irreps,
- $\rightsquigarrow$ Frobenius reciprocity,
- $\rightsquigarrow$ branching rule for $\mathfrak{S}_{n-1}$-global $\mathfrak{S}_n$-partial representations.

Groupoids comparison

$\Gamma_{\{0\}}\mathbb{Z}_4$ (on the left) and $\Gamma_{\mathbb{Z}_2}\mathbb{Z}_4$ (on the right).



Open questions, some first results

Some open questions:

(a) What properties do H_{par} and H^{par} have in common with H ?
E.g.: if H is a semisimple Hopf algebra, is H_{par} semisimple as well ?

(b) Can we classify partial (co)modules, at least in favourable cases ?

General construction [Batista-Hautekiet-S.-Vercruysse, 2023]

- Take $e \in H$ such that $e^2 = e$ and $ee_{(1)} \otimes e_{(2)} = e_{(1)}e \otimes e_{(2)}$,
- Let $A \subseteq H$ be the right coideal subalgebra generated by e,
- $\bar{H} := H/HA^+$ is a coalg and He is an ($\bar{H}$-global, H-partial)-comod,
- For any right $\bar{H}$-comodule W, $W \,\square^{\bar{H}} He$ is a right partial H-comodule.

Theorem [Batista-Hautekiet-S.-Vercruysse, 2023]

Every 1-dim partial comodule over a Hopf algebra H with invertible antipode is of the form $W \,\square^{\bar{H}} He$ for some 1-dim right $\bar{H}$-comodule W.

More results, open conjectures

Theorem [Batista-Hautekiet-S.-Vercruysse, 2023]

If $H = kG$ for a finite group G, then the partial comodule $W \,\square^{k(G/K)} kGe$ is simple if and only if W is a simple (i.e. 1-dim) $k(G/K)$-comodule.

$W \,\square^{k(G/K)} kGe$ and $W' \,\square^{k(G/K')} kGe'$ are isomorphic iff $K = K'$, $W \cong W'$ and there exists a multiplicative character ν of K such that the algebra automorphism $kK \to kK$, $g \mapsto \nu(g)g$, maps e to e'.

Conjecture A [Alves-Batista-Vercruysse, 2019]

If H is a finite dimensional semi-simple Hopf algebra, then H_{par} is also semi-simple and finite dimensional.

Conjecture B [Batista-Hautekiet-S.-Vercruysse, 2023]

If H is finite-dimensional and W is a simple right $\bar{H}$-comodule, then $W \,\square^{\bar{H}} He$ is a simple partial H-comodule.

Conjecture C [Batista-Hautekiet-S.-Vercruysse, 2023]

Every finite-dimensional simple partial H-comodule is of the form $W \,\square^{\bar{H}} He$ for some idempotent e and some simple right $\bar{H}$-comodule W.

Supporting evidence

Examples [Batista-Hautekiet-S.-Vercruysse, 2023]

- G finite abelian group;

- $G = S_3$, we have

$$(\mathbb{C}S_3^*)_{par} \cong \mathbb{C}^{18} \times M_2(\mathbb{C})^2 \times M_5(\mathbb{C});$$

- G is D_8, the dihedral group of 8 elements, or Q_8, the quaternion group. It turns out that

$$(\mathbb{C}D_8^*)_{par} \cong \mathbb{C}^{35} \times M_2(\mathbb{C})^2 \times M_3(\mathbb{C})^7 \times M_5(\mathbb{C}) \times M_7(\mathbb{C}),$$
$$(\mathbb{C}Q_8^*)_{par} \cong \mathbb{C}^{19} \times M_2(\mathbb{C})^6 \times M_3(\mathbb{C})^7 \times M_5(\mathbb{C}) \times M_7(\mathbb{C});$$

- For the Kac-Paljutkin algebra H_8

$$(H_8)_{par} \cong \mathbb{C}^{23} \times M_2(\mathbb{C})^5 \times M_3(\mathbb{C})^7 \times M_5(\mathbb{C}) \times M_7(\mathbb{C}).$$

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