

EVERYBODY KNOWS WHAT A NORMAL GABI-ALGEBRA IS

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ABSTRACT. Let A be a k -algebra over a commutative ring k . By the renowned Tannaka-Kreĭn reconstruction, liftings of the monoidal structure from ${}_k\mathcal{M}$ to ${}_A\mathcal{M}$ correspond to bialgebra structures on A and liftings of the closed monoidal structure correspond to Hopf algebra structures on A . In this paper, we determine conditions on A that correspond to liftings of the closed structure alone, i.e. without considering the monoidal one, which lead to the notion of what we call a *gabi-algebra*. First, we tackle the question from the general perspective of monads, then we focus on the set-theoretic and the linear setting. Our main and most surprising result is that a *normal gabi-algebra*, that is an algebra A whose category of modules is (associative and unital normal) closed with closed forgetful functor to ${}_k\mathcal{M}$, is automatically a Hopf algebra (thus justifying our title).

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1. INTRODUCTION

The starting point of the present paper is the Tannaka-Kreĭn reconstruction theory for Hopf algebras (or quantum groups), where one aims at reconstructing algebraic structures and properties of (co)algebras from categorical structures and properties of their categories of (co)modules. Such reconstruction theorems have been widely studied at different levels of generality: see e.g. [Sc1, U] for the case of Hopf algebras, [Mj, Sa1] for the case of coquasi-bialgebras (with preantipode), [Sc2, Sc3] for bialgebroids and Hopf algebroids, [BFVV] for oplax Hopf algebras, [BLV] for Hopf monads and [V] for an overview.

More precisely, it is well-known that there is a bijective correspondence between bialgebra structures on an algebra A over a commutative ring k and monoidal structures on its category of left modules ${}_A\mathcal{M}$ which are lifted along the forgetful functor $\omega: {}_A\mathcal{M} \rightarrow {}_k\mathcal{M}$, that is to say, monoidal structures on ${}_A\mathcal{M}$ for which ω is a strict monoidal functor (see [P]). Moreover, the category of modules over a bialgebra A is always a *right closed monoidal* category, in the sense that for every left A -module N , the endofunctor $- \otimes N: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$ has a right adjoint given by ${}_A\mathcal{M}(.A \otimes .N, -)$, where $.A \otimes .N$ is an A -bimodule with respect to $x \cdot (y \otimes n) \cdot z = x_1yz \otimes x_2 \cdot n$ for all $x, y, z \in A$ and $n \in N$ (see e.g. [Sc3]). In fact, it is also *left closed monoidal*, meaning that the endofunctor $N \otimes -$ has a right adjoint given by ${}_A\mathcal{M}(.N \otimes .A, -)$. Hence, ${}_A\mathcal{M}$ is always a *closed monoidal* category. Remark, however, that the right closed structure (i.e. the “right internal homs”) in ${}_A\mathcal{M}$ differs in general from that in ${}_k\mathcal{M}$. In fact, the forgetful functor ω lifts not only the monoidal, but the full right closed monoidal structure if and only if also ${}_k\mathcal{M}(N, -)$ (endowed with some A -module structure) provides a right adjoint to $- \otimes N: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$. If this is the case, then the two right adjoints are related via the canonical isomorphism induced by $.A \otimes N \rightarrow .A \otimes .N$, $x \otimes n \mapsto x_1 \otimes x_2 \cdot n$. For $N = A$, this map is exactly the so-called *Galois* or *fusion* map of the bialgebra A , whose bijectivity corresponds to the existence of an antipode for A . This makes it clear why lifting the right closed monoidal structure is equivalent to having a Hopf algebra structure on A . Furthermore, a similar argument tells that the left closed structure is lifted along ω if and only if the bialgebra has an *opantipode*. Thus, A is a Hopf algebra with bijective antipode if and only if both left and right closure lift along ω .

It is noteworthy that any one of the earlier mentioned cases of reconstruction theorems follows the above structure, where first a bialgebra-type structure is obtained from lifting the monoidal structure and then additional Hopf-like structures arise from closeness or rigidity of this monoidal structure. However, in this paper, we change perspective and we focus on closed structures as starting point. Indeed, closed structures on a category can be defined independently of monoidal ones: see for instance [EK, St2, UVZ]. In a nutshell, a closed category is a category \mathcal{C} together with a distinguished object $\mathbf{1}$ and a bifunctor $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$, which can be understood as the prototype of internal homs, satisfying suitable compatibilities. From this perspective, we may equally look at ${}_k\mathcal{M}$ as a closed category for which the endofunctor ${}_k\mathcal{M}(V, -): {}_k\mathcal{M} \rightarrow {}_k\mathcal{M}$ has a left adjoint for every object V . Therefore, it is natural to wonder what we can say about an algebra A for which the category ${}_A\mathcal{M}$ of left A -modules is closed (without necessarily being monoidal) in such a way that the forgetful functor $\omega: {}_A\mathcal{M} \rightarrow {}_k\mathcal{M}$ is a closed functor, and only after analyse the case in which also the monoidal structure is lifted.

Despite being a perfectly natural and relevant question, the closed side of reconstruction theory seems to have been mostly neglected in the literature so far. This paper is aimed at filling this apparent gap by studying algebras A over the commutative ring k for which the forgetful functor ω lifts the closed structure but not necessarily the monoidal one. We will mainly consider two different levels of strictness: skew- and (unital and associative) normal-closed structures. In case the endofunctors $[X, -]$ on a closed category \mathcal{C} have left adjoint, which we then suggestively denote as $- \otimes X$, then the closed structure is skew-closed if and only if the associated monoidal product \otimes is skew-monoidal (see [Sz, §2]). Unital and associative normality of the closed structure then correspond respectively to (strong) unitality and associativity of the monoidal product. An important difference between monoidal and closed structures is that the normality conditions can no longer be expressed as *internal* properties in the category \mathcal{C} , but they need to be expressed *externally* in the category of sets (or in any category over which \mathcal{C} is enriched, if one considers other than Set-enriched categories). This is a first aspect that makes the theory of lifting closed structures different from the usual Tannaka theory. Indeed, when lifting a monoidal product together with the associativity and unitality constraints along a faithful functor ω , then the lifted structure is strong monoidal whenever the initial monoidal structure was. In the closed case, where normality is an external property, the normality of the lifted structure is an additional requirement.

It turns out that, for a given k -algebra A , lifting the skew-closed structure from ${}_k\mathcal{M}$ to ${}_A\mathcal{M}$ corresponds to the existence of algebra maps $\delta: A \rightarrow A \otimes A^{\text{op}}$ and $\varepsilon: A \rightarrow k$ satisfying appropriate conditions. Namely,

Theorem (Theorem 3.12). *Let A be an algebra over a commutative ring k . Then the closed structure of ${}_k\mathcal{M}$ lifts to a skew-closed structure on ${}_A\mathcal{M}$ if and only if:*

- (1) *there exists an algebra map $\varepsilon: A \rightarrow k$ and*
- (2) *there exists an algebra map $\delta: A \rightarrow A \otimes A^{\text{op}}$, $\delta(a) = a_+ \otimes a_-$ (summation understood), such that for all $a \in A$*

$$\begin{aligned} a_+ \varepsilon(a_-) &= a \\ a_+ a_- &= \varepsilon(a) 1_A \\ a_{++} \otimes a_{-+} \otimes a_{--} a_{+-} &= a_+ \otimes a_- \otimes 1 \end{aligned}$$

In this case, A acts on ${}_k\mathcal{M}(M, V)$ as $(a.f)(m) = a_+ f(a_- m)$. ■

We call an algebra satisfying the equivalent conditions of the above Theorem a *gabi-algebra*⁽¹⁾ and we show how a quite unexpected source of examples is provided by certain one-sided Hopf algebras in the sense of [GNT], i.e. bialgebras with a morphism which is just a one-sided convolution inverse of the identity. In particular, modules over a one-sided Hopf algebra are skew-closed and monoidal in such a way that the forgetful functor is closed and monoidal, but they do not form a closed monoidal category. Furthermore, inspired by the existing structure on the module category of a gabi-algebra and by the

⁽¹⁾ We prefer to keep the origin of the name gabi-algebra somewhat mysterious. Being an algebra with the same additional structure maps as a bialgebra, although satisfying other compatibility conditions (in particular, a different coassociativity condition), one could think of it as a “generalized associative bialgebra”. Another hint, however, might be [Bö1], where the above axioms were given for the first time.

prevailing examples, we also wonder when a gabi-algebra is a (one-sided) Hopf algebra and we provide necessary and sufficient conditions in § 5. Our most striking result in this direction is the following, which justifies the title of the paper in view of [Be].

Theorem (Theorem 5.12). *Let A be an algebra. Then there is a bijective correspondence between normal gabi-algebra structures on A and Hopf algebra structures on A . ■*

Concretely, our paper is organized as follows. In § 2 we recall the basics of (skew-)closed and (skew-)monoidal categories and the duality between them. § 3 is devoted to paving the way towards gabi-algebras. In § 3.1 and § 3.2 we address the question of lifting a skew-closed structure in the monadic setting, from a skew-closed category \mathcal{C} to the Eilenberg-Moore category \mathcal{C}^T of algebras for a monad T on \mathcal{C} ; they also contain our first main results, Theorem 3.4 and Proposition 3.7, providing necessary and sufficient conditions for the lifting. In § 3.3 we take advantage of the generality offered by the monadic perspective to fully answer the question in the set-theoretic setting and in § 3.4 we apply our machinery to tackle the linear setting, which is leading to the definition of gabi-algebras. With § 4, we finally introduce gabi-algebras, we study some elementary properties of them and we provide a few concrete examples. Finally, in § 5 we address the question of determining which additional conditions make a gabi-algebra into a Hopf algebra. There is also an appendix, Appendix A, where we prove a lifting theorem for internal homs fairly more general than the corresponding part in Theorem 3.4 and that represents, in fact, the core of the proof of the lifting property in Theorem 3.4.

As a matter of notation, k denotes a fixed commutative ring (unless stated otherwise). If A is a k -algebra, we denote by 1_A (or simply 1) both its unit and the k -linear map $k \rightarrow A$ giving the k -algebra structure. The multiplication of A might be denoted by m , \cdot or simple juxtaposition. For a generic category \mathcal{C} , the external hom-set between two objects X, Y is denoted by $\mathcal{C}(X, Y)$, and the identity morphism of an object X may be denoted by id_X or X . The square brackets $[-, -]$ always denote an internal hom of a given closed structure and $\mathbf{1}$ a distinguished object (either the closed or the monoidal unit).

2. CLOSED AND MONOIDAL CATEGORIES

In this section we review various flavours of closed categories and monoidal categories of different laxities.

2.1. Left skew-closed categories. First we turn to a suitably lax version of closed categories, following [St2, UVZ], which has the unquestionable advantage of involving only conditions internal to \mathcal{C} .

Definition 2.1. A (left) skew-closed category is a tuple $(\mathcal{C}, [-, -], \mathbf{1}, \Gamma, i, j)$, where

- (a) $\mathbf{1} \in \mathcal{C}$ is an object
- (b) $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor
- (c) $i: [\mathbf{1}, -] \Rightarrow -$ is a natural transformation
- (d) $j: \mathbf{1} \rightrightarrows [-, -]$ is a dinatural transformation (see, e.g., [Mc, Chapter IX, §4])
- (e) Γ is a family of morphisms $\Gamma_{Y,Z}^X: [Y, Z] \rightarrow [[X, Y], [X, Z]]$ for X, Y, Z objects in \mathcal{C} , which is natural in the lower indices and dinatural in the upper index.

These data are subject to the following axioms:

$$\begin{array}{ccc}
\mathbf{1} \xrightarrow{j_1} [\mathbf{1}, \mathbf{1}] & & [X, Y] \xrightarrow{\Gamma_{X,Y}^X} [[X, X], [X, Y]] \\
\parallel & \searrow & \parallel \\
\mathbf{1} & \downarrow i_1 & [X, Y] \xleftarrow{i_{[X,Y]}} [\mathbf{1}, [X, Y]] \\
& & \downarrow [j_X, [X, Y]]
\end{array}, \quad
\begin{array}{ccc}
\mathbf{1} \xrightarrow{j_Y} [Y, Y] & & [X, Y] \xrightarrow{\Gamma_{X,Y}^1} [[\mathbf{1}, X], [\mathbf{1}, Y]] \\
\searrow j_{[X,Y]} & \downarrow \Gamma_{Y,Y}^X & \searrow [i_X, Y] \\
[X, Y], [X, Y] & & [[\mathbf{1}, X], Y] \\
& & \downarrow [[\mathbf{1}, X], i_Y]
\end{array},$$

$$\begin{array}{ccc}
[W, X] & \xrightarrow{\Gamma_{W,X}^U} & [[U, W], [U, X]] \\
\downarrow \Gamma_{W,X}^V & & \downarrow \Gamma_{[U,W],[U,X]}^{[U,V]} \\
[V, W], [V, X] & \xrightarrow{[[V,W], \Gamma_{V,X}^U]} & [[U, V], [U, W]], [[U, V], [U, X]] \\
& & \downarrow \Gamma_{[U,V],[U,X]}^U \\
& & [[V, W], [[U, V], [U, X]]]
\end{array}$$

A skew-closed category is said to be

(N1) *left normal* if and only if

$$\hat{j}_{X,Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(\mathbf{1}, [X, Y]), \quad f \mapsto [f, Y] \circ j_Y$$

is a natural bijection;

(N2) *right normal* if and only if i is a natural isomorphism;

(N3) *associative normal* if the canonical morphism

$$\hat{\Gamma}_{U,X,Y,Z} : \int^{V \in \mathcal{C}} \mathcal{C}(U, [V, Z]) \times \mathcal{C}(X, [Y, V]) \rightarrow \mathcal{C}(U, [X, [Y, Z]]),$$

defined component-wise as

$$\hat{\Gamma}_{U,X,Y,Z}(f, g) : U \xrightarrow{f} [V, Z] \xrightarrow{\Gamma_{V,Z}^Y} [[Y, V], [Y, Z]] \xrightarrow{[g, [Y, Z]]} [X, [Y, Z]]$$

for all $f \in \mathcal{C}(U, [V, Z])$, $g \in \mathcal{C}(X, [Y, V])$, is a bijection.

A skew-closed category satisfying left and right normality conditions will be called *unital normal*. A skew-closed category satisfying all three normality conditions will be called *normal-closed*.

Remark 2.2. (1) The adjective *left* in front of the term *skew-closed category* in Definition 2.1 shall be interpreted as the adjective *left* in *left skew-monoidal category* (we recall this notion in Definition 2.8, for the convenience of the unaccustomed reader). In fact, as we will recall in Theorem 2.12, left skew-closed structures are adjoints to left skew-monoidal ones and vice-versa. Moreover, under this correspondence, normality for the skew-closed structure corresponds to normality for the skew-monoidal one.

Analogously, one could speak about *right* skew-closed structures and these would be adjoints to *right* skew-monoidal ones. The interested reader may refer to [UVZ, §4] for additional details. Henceforth, and as far as we are concerned, the left-hand side case would be enough and so we will often omit to specify it.

- (2) The term *closed category* is used in different ways in the literature, for a skew-closed category satisfying none, some or all of the normality conditions. Following [UVZ], we agree that the term *closed category* should be reserved for a skew-closed category that satisfies all three normality conditions, in light of the duality Theorem 2.12. However, in order to avoid confusion, we will call such a category “normal-closed” in this paper, and avoid to speak about “closed category” without any prefix.
- (3) In contrast to coherences of monoidal categories, left and associative normality for skew-closed categories are conditions *external* to \mathcal{C} , in the sense that they involve the fact that certain functions in the category of sets are bijections. They cannot be expressed by only using morphisms of \mathcal{C} . This fact is the strongest feature of closed categories and it plays a crucial role in § 5. Already at this stage it can be seen that a functor that preserves the closed structure, even in a strict way, will not necessarily preserve the normality conditions, because they are external. ∇

Our main examples of skew-closed categories are the following ones.

Example 2.3. The category **Set** of sets and functions is skew-closed. The inner hom functor is given by $[A, B] = \mathbf{Set}(A, B)$. The unit object is the unit object of the monoidal structure, i.e. a fixed one-element set $*$. The natural transformation $i_A: \mathbf{Set}(*, A) \rightarrow A$ is the isomorphism given by $i_A(f) = f(*)$, and the dinatural transformation $j_A: * \rightarrow \mathbf{Set}(A, A)$ picks out the identity, i.e. $j_A(*) = \text{id}_A$. Finally, the transformation $\Gamma_{B,C}^A$ is given by post-composition, meaning

$$\Gamma_{B,C}^A: \mathbf{Set}(B, C) \rightarrow \mathbf{Set}(\mathbf{Set}(A, B), \mathbf{Set}(A, C)), \quad f \mapsto (g \mapsto f \circ g). \quad \triangle$$

Example 2.4. Let k be a commutative ring with unit 1. Its category of, say, left modules ${}_k\mathcal{M}$ is skew-closed. The inner hom is given by $[M, N] = {}_k\mathcal{M}(M, N)$, on which k acts as $(k.f)(m) = kf(m)$. The unit object is k . The natural transformation $i_M: {}_k\mathcal{M}(k, M) \rightarrow M$ is the isomorphism given by $i_M(f) = f(1)$. The dinatural transformation j_M is again ‘picking out the identity’, i.e. it is the unique k -module map with $j_M(1) = \text{id}_M$. Lastly, the transformation $\Gamma_{N,P}^M$ is again given by post-composition, $\Gamma_{N,P}^M(f) = f \circ -$. \triangle

The categories in Example 2.3 and Example 2.4 are both normal-closed, in fact. The interested reader may check this directly, but it follows more easily from the fact that they are closed monoidal (a notion that we will recall shortly in § 2.2), in conjunction with the forthcoming Theorem 2.12. An example of a non normal-closed category is the following.

Example 2.5. Let k be a commutative ring and R be a (not necessarily commutative) k -algebra. Consider the category ${}_R\mathcal{M}$ of left R -modules. For any $M, N \in {}_R\mathcal{M}$, define $[M, N] = {}_k\mathcal{M}(M, N)$ with the following R -action.

$$(r.f)(m) = r \cdot f(m)$$

for all $f \in {}_k\mathcal{M}(M, N)$, $r \in R$ and $m \in M$. The unit object is given by the regular module R . The structure maps are given by

$$\begin{aligned} i_M: {}_k\mathcal{M}(R, M) &\rightarrow M, & f &\mapsto f(1_R), \\ j_M: R &\rightarrow {}_k\mathcal{M}(M, M), & r &\mapsto \{m \mapsto r \cdot m\} \end{aligned}$$

and Γ is again given by post-composition. This is an associative normal but not unital normal closed structure. Again, the interested reader may check this directly, but it would be easier to observe that the following chain of bijections

$$\begin{aligned} {}_R\mathcal{M}(.M, {}_k\mathcal{M}(N, .P)) &\cong {}_R\mathcal{M}(.M, {}_R\mathcal{M}(.R \otimes N, .P)) \cong {}_R\mathcal{M}((.R \otimes N) \otimes_R .M, .P) \\ &\cong {}_R\mathcal{M}(.M \otimes N, .P) \end{aligned}$$

makes it clear that the given skew-closed structure $[-, -]$ corresponds, by adjunction, to the skew-monoidal structure \otimes with regular left R -action on the left-hand side tensor factor, which is an associative normal skew-monoidal structure, but not unital normal (see the forthcoming Example 2.10). \triangle

Functors between skew-closed categories can be required to preserve the skew-closed structure in a coherent way. This leads to the following definition.

Definition 2.6. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between skew-closed categories is *closed* if there is a morphism $F_0: \mathbf{1}_{\mathcal{D}} \rightarrow F\mathbf{1}_{\mathcal{C}}$ and a natural transformation $F_2(X, Y): F[X, Y]_{\mathcal{C}} \rightarrow [FX, FY]_{\mathcal{D}}$ satisfying

$$\begin{array}{ccc} \mathbf{1}_{\mathcal{D}} & \xrightarrow{F_0} & F\mathbf{1}_{\mathcal{C}} \\ j_{FX} \downarrow & & \downarrow Fj_X \\ [FX, FX]_{\mathcal{D}} & \xleftarrow{F_2(X, X)} & F[X, X]_{\mathcal{C}} \\ & & \uparrow i_{FX} \\ FX & \xleftarrow{[1_{\mathcal{D}}, FX]_{\mathcal{D}}} & [1_{\mathcal{D}}, FX]_{\mathcal{D}} \\ Fi_X \uparrow & & \uparrow [F_0, FX]_{\mathcal{D}} \\ F[1_{\mathcal{C}}, X]_{\mathcal{C}} & \xrightarrow{F_2(1_{\mathcal{C}}, X)} & [F1_{\mathcal{C}}, FX]_{\mathcal{D}} \end{array}$$

$$\begin{array}{ccccc} F[X, Y]_{\mathcal{C}} & \xrightarrow{F\Gamma_{X, Y}^Z} & F[[Z, X]_{\mathcal{C}}, [Z, Y]_{\mathcal{C}}]_{\mathcal{C}} & \xrightarrow{F_2([Z, X], [Z, Y])} & [F[Z, X]_{\mathcal{C}}, F[Z, Y]_{\mathcal{C}}]_{\mathcal{D}} \\ F_2(X, Y) \downarrow & & & & \downarrow [\text{id}, F_2(Z, Y)] \\ [FX, FY]_{\mathcal{D}} & \xrightarrow{\Gamma_{FX, FY}^{FZ}} & [[FZ, FX]_{\mathcal{D}}, [FZ, FY]_{\mathcal{D}}]_{\mathcal{D}} & \xrightarrow{[F_2(Z, X), \text{id}]} & [F[Z, X]_{\mathcal{C}}, [FZ, FY]_{\mathcal{D}}]_{\mathcal{D}} \end{array}$$

If all of these are isomorphisms (identities), then F is called *strong (strict) closed*.

Example 2.7. The identity functor on a skew-closed category is strict closed. The composition of closed functors is closed: if F, G are composable closed functors, then FG is closed with $(FG)_0 = FG_0 \circ F_0$ and $(FG)_2(X, Y) = F_2(GX, GY) \circ FG_2(X, Y)$. \triangle

2.2. Skew-monoidal categories. The notion of a skew-monoidal category is dual to that of a skew-closed category, in a sense to be made precise in Theorem 2.12.

Definition 2.8. Following [Sz], a (left) skew-monoidal category is a tuple $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ where \mathcal{C} is a category, $\mathbf{1}$ is an object of \mathcal{C} , $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor, and

$$\begin{aligned} \alpha_{X,Y,Z}: (X \otimes Y) \otimes Z &\rightarrow X \otimes (Y \otimes Z), \\ \lambda_X: \mathbf{1} \otimes X &\rightarrow X, \quad \rho_X: X \rightarrow X \otimes \mathbf{1}, \end{aligned} \tag{2.1}$$

are natural transformations subject to the following axioms:

$$\begin{array}{ccc} & (W \otimes (X \otimes Y)) \otimes Z & \\ & \swarrow \alpha \otimes Z & \searrow \alpha \\ ((W \otimes X) \otimes Y) \otimes Z & & W \otimes ((X \otimes Y) \otimes Z) \\ \alpha \downarrow & & \downarrow W \otimes \alpha \\ (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha} & W \otimes (X \otimes (Y \otimes Z)) \\ \\ X \otimes Y \xrightarrow{\rho} (X \otimes Y) \otimes \mathbf{1} & & (\mathbf{1} \otimes X) \otimes Y \xrightarrow{\alpha} \mathbf{1} \otimes (X \otimes Y) \\ \searrow X \otimes \rho \quad \downarrow \alpha & & \lambda \otimes Y \downarrow \quad \swarrow \lambda \\ & X \otimes (Y \otimes \mathbf{1}) & X \otimes Y \\ \\ X \otimes Y \xrightarrow{\rho \otimes Y} (X \otimes \mathbf{1}) \otimes Y & & \mathbf{1} \\ \parallel & \downarrow \alpha & \rho \downarrow \\ X \otimes Y \xleftarrow{X \otimes \lambda} X \otimes (\mathbf{1} \otimes Y) & & \mathbf{1} \otimes \mathbf{1} \xrightarrow{\lambda} \mathbf{1} \end{array}$$

A skew-monoidal category is

- (1) *left normal* if and only if λ is an isomorphism,
- (2) *right normal* if and only if ρ is an isomorphism,
- (3) *associative normal* if and only if α is an isomorphism.

A *monoidal category* is a skew-monoidal category satisfying all three normality conditions.

Remark 2.9. A *right* skew-monoidal category would be defined similarly, but with all structure morphisms (2.1) reversed. Since we will be interested mainly in left skew-monoidal categories, we will often omit to specify it. ∇

Example 2.10. Let k be a commutative ring with unit and let R be a k -algebra. The category ${}_R\mathcal{M}$ of left R -modules is an associative normal skew-monoidal category with respect to the tensor product \otimes over k . The left R -module structure on the tensor product $M \otimes N$ of two left R -modules is given by the left regular action on the left tensorand: $r \cdot (m \otimes n) = (r \cdot m) \otimes n$ for all $r \in R$, $m \in M$, $n \in N$. The natural transformations are

$$\begin{aligned} \alpha_{M,N,P}: (M \otimes N) \otimes P &\xrightarrow{\cong} M \otimes (N \otimes P), & (m \otimes n) \otimes p &\mapsto m \otimes (n \otimes p), \\ \lambda_M: R \otimes M &\rightarrow M, & r \otimes m &\mapsto r \cdot m, \\ \rho_M: M &\rightarrow M \otimes R, & m &\mapsto m \otimes 1_R. \end{aligned} \tag{2.2}$$

\triangle

2.3. Skew-closed skew-monoidal categories. Let now \mathcal{C} be a skew-closed category, with the usual notation, and assume that for each X in \mathcal{C} there is an adjunction $L_X \dashv [X, -]$. Denote by coev^X and ev^X the unit and counit of each adjunction. Since $[-, -]$ is a bifunctor, there is a unique way to assign to each arrow $f: X \rightarrow Y$ and each object Z of \mathcal{C} an arrow $L_f: L_X Z \rightarrow L_Y Z$ of \mathcal{C} so that $(X, Y) \mapsto L_X Y$ becomes a bifunctor for which the bijection $\mathcal{C}(L_X Y, Z) \cong \mathcal{C}(Y, [X, Z])$ is natural in all three variables (cf. the symmetric version of [Mc, Theorem IV.7.3]). The action of L on morphisms $f: X \rightarrow Y$ in the lower index is given by

$$L_f Z: L_X Z \xrightarrow{L_X \text{coev}_Z^Y} L_X[Y, L_Y Z] \xrightarrow{L_X[f, L_Y Z]} L_X[X, L_Y Z] \xrightarrow{\text{ev}_{L_Y Z}^X} L_Y Z .$$

For later use we record the following property of the adjunctions above (see [Mc, §IX.4, page 216]).

Lemma 2.11. *The evaluations and coevaluations ev^X and coev^X are dinatural in X .*

Proof. We show it for the evaluation, the coevaluation being completely analogous. One computes for $f: X \rightarrow Y$

$$\begin{aligned} \text{ev}_Z^Y \circ L_f[Y, Z] &= \text{ev}_Z^Y \circ \text{ev}_{L_Y[Y, Z]}^X \circ L_X[f, L_Y[Y, Z]] \circ L_X \text{coev}_{[Y, Z]}^Y \\ &\stackrel{(*)}{=} \text{ev}_Z^X \circ L_X[f, Z] \circ L_X[Y, \text{ev}_Z^Y] \circ L_X \text{coev}_{[Y, Z]}^Y \\ &\stackrel{(\star)}{=} \text{ev}_Z^X \circ L_X[f, Z] \end{aligned}$$

using $(*)$ naturality of ev^X , and (\star) one of the triangles of the adjunction $L_X \dashv [X, -]$. \square

Transporting the closed structure through the adjunction imparts on \mathcal{C} the structure of a skew-monoidal category [St2, Proposition 18]. Let us describe here how the structure is obtained. We have the natural transformation

$$p_{X, Y, Z}: [L_Y X, Z] \xrightarrow{\Gamma_{L_Y X, Z}^Y} [[Y, L_Y X], [Y, Z]] \xrightarrow{[\text{coev}_{X, [Y, Z]}^Y]} [X, [Y, Z]] . \quad (2.3)$$

From this, one constructs the natural transformation

$$\mathcal{C}(L_{L_Z Y} X, V) \xrightarrow{\sim} \mathcal{C}(X, [L_Z Y, V]) \xrightarrow{p_{X, [Y, [Z, V]]}} \mathcal{C}(X, [Y, [Z, V]]) \xrightarrow{\sim} \mathcal{C}(L_Z L_Y X, V) . \quad (2.4)$$

Such natural transformations are in bijection with $\mathcal{C}(L_Z L_Y X, L_{L_Z Y} X)$ by the Yoneda lemma, and this gives a *skew associator* $\alpha_{X, Y, Z}: L_Z L_Y X \rightarrow L_{L_Z Y} X$ as the morphism corresponding to (2.4).

The two *skew unitors* are obtained from the adjunction as follows. Firstly, $\lambda_X: L_X \mathbf{1} \rightarrow X$ corresponds simply to $j_X: \mathbf{1} \rightarrow [X, X]$. For the other, consider the natural transformation

$$\mathcal{C}(L_{\mathbf{1}} X, Y) \xrightarrow{\sim} \mathcal{C}(X, [\mathbf{1}, Y]) \xrightarrow{i_{Y}^{\circ-}} \mathcal{C}(X, Y) , \quad (2.5)$$

which by the Yoneda lemma gives $\rho_X: X \rightarrow L_{\mathbf{1}} X$.

Defining \otimes by setting $- \otimes X := L_X$ concludes the construction of the skew-monoidal structure $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$, and we repeat here the following theorem.

Theorem 2.12 ([UVZ, Theorems 2.10 and 3.8]). *Let \mathcal{C} be a category with a distinguished object $\mathbf{1}$, and functors $- \otimes -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$. Assume that there are adjunctions $- \otimes X \dashv [X, -]$, natural in $X \in \mathcal{C}$. Then left skew-monoidal structures (α, λ, ρ) on $(\mathcal{C}, \otimes, \mathbf{1})$ are in bijection with left skew-closed structures (Γ, j, i) on $(\mathcal{C}, [-, -], \mathbf{1})$.*

Moreover, the left skew-monoidal structure is left/right/associative normal if and only if the left skew-closed structure is left/right/associative normal. The left skew-monoidal structure is associative normal if and only if p from (2.3) is a natural isomorphism.

Remark 2.13. (1) Remark that p from (2.3) being a natural isomorphism can be understood as L_X being a left inverse for $[X, -]$, as functors between \mathcal{C} -enriched categories. (2) For the sake of clearness, let us provide explicit formulae for the monoidal constraints obtained from the closed ones under the correspondence of Theorem 2.12. The associativity constraint $\alpha_{X,Y,Z}$ in $\mathcal{C}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ is given by

$$\begin{aligned} \alpha_{X,Y,Z} &= \text{ev}_{X \otimes (Y \otimes Z)}^Z \circ \left(\text{ev}_{[Z, X \otimes (Y \otimes Z)]}^Y \otimes Z \right) \circ \left(([\text{coev}_Y^Z, [Z, X \otimes (Y \otimes Z)]] \otimes Y) \otimes Z \right) \\ &\quad \circ \left((\Gamma_{Y \otimes Z, X \otimes (Y \otimes Z)}^Z \otimes Y) \otimes Z \right) \circ \left((\text{coev}_X^{Y \otimes Z} \otimes Y) \otimes Z \right) \end{aligned}$$

and it is the unique morphism such that

$$\begin{aligned} [Y, [Z, \alpha_{X,Y,Z}]] \circ [Y, \text{coev}_{X \otimes Y}^Z] \circ \text{coev}_X^Y & \\ = [\text{coev}_Y^Z, [Z, X \otimes (Y \otimes Z)]] \circ \Gamma_{Y \otimes Z, X \otimes (Y \otimes Z)}^Z \circ \text{coev}_X^{Y \otimes Z}; & \quad (2.6) \end{aligned}$$

the left-unit constraint $\lambda_X \in \mathcal{C}(\mathbf{1} \otimes X, X)$ is given by $\lambda_X = \text{ev}_X^X \circ (j_X \otimes X)$ and it is the unique morphism such that

$$[X, \lambda_X] \circ \text{coev}_{\mathbf{1}}^X = j_X;$$

the right-unit constraint $\rho_X \in \mathcal{C}(X, X \otimes \mathbf{1})$ is given by $\rho_X = i_{X \otimes \mathbf{1}} \circ \text{coev}_X^{\mathbf{1}}$. ∇

Definition 2.14. A (left) skew-closed and (left) skew-monoidal category is called (*left*) *skew-closed skew-monoidal* if and only if the skew-closed and the skew-monoidal structure are dual to one another in the sense of Theorem 2.12. If either, and hence both, structure satisfies all three normality conditions, this situation is termed a *right closed monoidal category* in literature (remark however that the adjective *right* here has a different meaning than the *left* before: it indicates that the left adjoint to $[X, -]$ is given by tensoring on the *right*). A monoidal category is *closed* if all functors $X \otimes -$ and $- \otimes X$ have right adjoints (leading to two, *a priori* distinct, closed structures).

Example 2.15. The category **Set** is closed monoidal, the monoidal product being given by the categorical product. The closed structure (both left and right) is the one in Example 2.3. \triangle

Example 2.16. Let R be a ring, and consider the category ${}_R\mathcal{M}_R$ of R -bimodules. This is a monoidal category under \otimes_R , and it is in fact closed monoidal. Indeed, recall that for bimodules over rings R, S, T , we have the tensor-hom adjunctions

$${}_S\mathcal{M}_T(N, {}_R\mathcal{M}(M, P)) \cong {}_R\mathcal{M}_T(M \otimes_S N, P) \cong {}_R\mathcal{M}_S(M, \mathcal{M}_T(N, P)), \quad (2.7)$$

where the bimodule structures on the hom spaces are of course $(sft)(m) = f(ms)t$ and $(rfs)(n) = rf(sn)$. Specializing to $S = T = R$, we obtain the left and right closed monoidal structures associated to the adjunctions

$$M \otimes_R - \dashv {}_R\mathcal{M}(M, -) \quad \text{and} \quad - \otimes_R M \dashv \mathcal{M}_R(M, -)$$

of endofunctors on ${}_R\mathcal{M}_R$. Note that the two internal homs are in general not isomorphic. For later use, let us record that the evaluations and coevaluations for the left closed monoidal structure are

$$\begin{aligned} \text{ev}_N^M: M \otimes_R {}_R\mathcal{M}(M, N) &\rightarrow N, & m \otimes_R f &\mapsto f(m), & \text{and} \\ \text{coev}_N^M: N &\rightarrow {}_R\mathcal{M}(M, M \otimes_R N), & n &\mapsto (m \mapsto m \otimes_R n). \end{aligned}$$

The corresponding right versions are completely analogous.

If $R = k$ is commutative, then the category of e.g. left modules ${}_k\mathcal{M}$ is closed monoidal as well. This is facilitated by first embedding ${}_k\mathcal{M}$ into ${}_k\mathcal{M}_k$ (as usual, a right action on $M \in {}_k\mathcal{M}$ is defined by $mr = rm$), then performing the relevant operations there, and finally forgetting back to ${}_k\mathcal{M}$. Note that in this case, $M \otimes N \cong N \otimes M$ canonically, and hence the left internal hom and the right internal hom agree. We obtain the closed structure from Example 2.4. The same can be done for right k -modules. \triangle

3. TOWARDS GABI-ALGEBRAS

In this section, \mathcal{C} is at least a skew-closed category. Let $T \in \text{End}(\mathcal{C})$ be a monad on \mathcal{C} .

3.1. Lifting to Eilenberg-Moore categories. We wish to lift the skew-closed structure of \mathcal{C} to the Eilenberg-Moore category \mathcal{C}^T . Thus we need to find a functor $[-, -]^T: (\mathcal{C}^T)^{\text{op}} \times \mathcal{C}^T \rightarrow \mathcal{C}^T$, and natural transformations i^T, j^T, Γ^T , forming a skew-closed structure on \mathcal{C}^T and such that the forgetful functor $U_T: \mathcal{C}^T \rightarrow \mathcal{C}$ is strictly closed. Then, $i^T = i, j^T = j, \Gamma^T = \Gamma$. Thus, right normality of \mathcal{C} is immediately transferred to \mathcal{C}^T , while this does not necessarily have to be the case for the other two normality conditions.

We invoke the theorem for mixed liftings, Theorem A.2, to obtain the following.

Lemma 3.1 (Corollary A.4). *Let \mathcal{C} be a category, $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ a functor, and (T, m, u) be a monad on \mathcal{C} . The liftings of $[-, -]$ to \mathcal{C}^T are in bijective correspondence with natural transformations $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ satisfying*

$$s_{X,Y} \circ u_{[TX,Y]} = [u_X, u_Y], \quad (3.1)$$

$$s_{X,Y} \circ m_{[TX,Y]} = [X, m_Y] \circ s_{X,TY} \circ T s_{TX,Y} \circ T^2[m_X, Y] \quad (3.2)$$

for all $X, Y \in \mathcal{C}$.

We call the conditions (3.1) and (3.2) on s *unitality* and *multiplicativity*, respectively.

Let us make explicit the bijection alluded to in Lemma 3.1. Let first $[-, -]^T$ be a lifting of $[-, -]$. For all $(M, \mu_M), (N, \mu_N) \in \mathcal{C}^T$, we then in particular have that the object $[M, N]$ is equipped with a some action of T that we denote by $\mu_M \star \mu_N: T[M, N] \rightarrow [M, N]$. We can thus define a natural transformation as above via

$$T[TX, Y] \xrightarrow{T[u_X, u_Y]} T[TX, TY] \xrightarrow{\mu_X \star \mu_Y} [TX, TY] \xrightarrow{[u_X, Y]} [X, TY],$$

where $X, Y \in \mathcal{C}$. Conversely, given a natural transformation $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$, we can define an action $\mu \star \sigma$ via

$$T[M, N] \xrightarrow{T[\mu_M, N]} T[TM, N] \xrightarrow{s_{M,N}} [M, TN] \xrightarrow{[M, \mu_N]} [M, N],$$

where now of course $(M, \mu_M), (N, \mu_N) \in \mathcal{C}^T$.

Example 3.2. Let T be a Hopf monad on the left closed monoidal category \mathcal{C} (in the sense of [BLV, §3.1]). Then T admits a *left antipode*, which is a natural transformation $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ satisfying some conditions only valid in the closed monoidal setting [BLV, §3.3]. However, by [BLV, Proposition 3.8], the antipode is an example of a natural transformation as in Lemma 3.1, which therefore may be seen as a generalization of the antipode to the closed but non-monoidal setting. \triangle

Remark 3.3. The interested reader may check that there is a bijective correspondence between natural transformations $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ for X, Y in \mathcal{C} and natural transformations $\gamma_Y^M: T[M, Y] \rightarrow [M, TY]$ for Y in \mathcal{C} and (M, μ_M) in \mathcal{C}^T . Indeed, given s we can consider

$$\gamma(s)_Y^M: T[M, Y] \xrightarrow{T[\mu_M, Y]} T[TM, Y] \xrightarrow{s_{M,Y}} [M, TY].$$

In the opposite direction, given γ we can consider

$$s(\gamma)_{X,Y}: T[TX, Y] \xrightarrow{\gamma_Y^{TX}} [TX, TY] \xrightarrow{[u_X, TY]} [X, TY].$$

These two constructions are well-defined and each others inverses. Furthermore, s satisfies (3.1) if and only if γ satisfies

$$\gamma_Y^M \circ u_{[M,Y]} = [M, u_Y] \quad (3.3)$$

and s satisfies (3.2) if and only if γ satisfies

$$[M, m_Y] \circ \gamma_{TY}^M \circ T(\gamma_Y^M) = \gamma_Y^M \circ m_{[M,Y]}. \quad (3.4)$$

The T -algebra structure on $[M, N]$ by means of γ is given by

$$T[M, N] \xrightarrow{\gamma_N^M} [M, TN] \xrightarrow{[M, \mu_N]} [M, N]$$

for all $(M, \mu_M), (N, \mu_N)$ in \mathcal{C}^T .

The fact that for any (M, μ_M) there exists γ_Y^M natural in Y and satisfying (3.3) and (3.4) is equivalent to require that there exists a functor $[(M, \mu_M), -]^T: \mathcal{C}^T \rightarrow \mathcal{C}^T$ such that $\omega[(M, \mu_M), -]^T = [M, \omega(-)]$. Naturality of γ in M is equivalent to the fact that the assignment $(\mathcal{C}^T)^{\text{op}} \rightarrow \text{Funct}(\mathcal{C}^T, \mathcal{C}^T), (M, \mu_M) \mapsto [(M, \mu_M), -]^T$, is functorial. See [LS2, §2.1], or [Bö2, Theorems 2.27 and 2.30] for a more elementary approach. ∇

In order to obtain a lifting of the entire skew-closed structure, one now simply needs to require two things: the existence of some T -algebra structure $T\mathbf{1} \rightarrow \mathbf{1}$ on $\mathbf{1}$ and that the structural morphisms i, j, Γ of \mathcal{C} are intertwiners of the T -algebra structures granted by the lifting $[-, -]^T$ of the bifunctor $[-, -]$. The following theorem collects necessary and sufficient conditions for the skew-closed structure to lift.

Theorem 3.4. *Let T be a monad on the left skew-closed category \mathcal{C} . Then \mathcal{C}^T is left skew-closed such that the forgetful functor to \mathcal{C} is strictly closed if and only if:*

- (1) *there is a morphism $\mu_{\mathbf{1}}: T\mathbf{1} \rightarrow \mathbf{1}$ such that $(\mathbf{1}, \mu_{\mathbf{1}}) \in \mathcal{C}^T$,*
- (2) *T lifts $[-, -]$ via a natural transformation $s: T[T-, \sim] \Rightarrow [-, T\sim]$ as in Lemma 3.1,*

and they satisfy

$$Ti_X = i_{TX} \circ s_{1,X} \circ T[\mu_1, X], \quad \forall X \in \mathcal{C}, \quad (3.5)$$

$$j_M \circ \mu_1 = [M, \mu_M \circ m_M] \circ s_{M, TM} \circ Tj_{TM}, \quad \forall (M, \mu_M) \in \mathcal{C}^T, \quad (3.6)$$

and for all $X, Y \in \mathcal{C}$ and $(P, \mu_P) \in \mathcal{C}^T$

$$\Gamma_{X, TY}^P \circ s_{X, Y} = [[\mu_P, X], s_{P, Y}] \circ s_{[TP, X], [TP, Y]} \circ T[s_{P, X}, [\mu_P, Y]] \circ T\Gamma_{TX, Y}^P. \quad (3.7)$$

Proof. Let us prove the necessity part. Suppose that i_M is a morphism of T -algebras for any (M, μ_M) in \mathcal{C}^T . Then, given any X in \mathcal{C} , in the following diagram

$$\begin{array}{ccccc}
 T[T\mathbf{1}, X] & \xleftarrow{T[\mu_1, X]} & T[\mathbf{1}, X] & \xrightarrow{Ti_X} & TX \\
 \downarrow T[T\mathbf{1}, u_X] & & \downarrow T[\mathbf{1}, u_X] & & \downarrow Tu_X \\
 T[T\mathbf{1}, TX] & \xleftarrow{T[\mu_1, TX]} & T[\mathbf{1}, TX] & \xrightarrow{Ti_{TX}} & T^2X \\
 \downarrow s_{1, TX} & & \downarrow \mu_{[\mathbf{1}, TX]} & & \downarrow \mu_{TX} \\
 [\mathbf{1}, T^2X] & \xrightarrow{[\mathbf{1}, \mu_{TX}]} & [\mathbf{1}, TX] & \xrightarrow{i_{TX}} & TX
 \end{array}
 \quad \begin{array}{l}
 (a) \quad (b) \\
 (c) \quad (d)
 \end{array}
 \quad \begin{array}{l}
 \text{id}_{TX} \\
 \text{curved arrow from } TX \text{ to } T^2X
 \end{array}$$

(a) commutes by bifactoriality of $[-, -]$, (b) commutes by naturality of i , (c) commutes by definition of $\mu_{[\mathbf{1}, M]}$ and (d) commutes by hypothesis. The right-most relation follows from the fact that $\mu_{TX} = m_X$. Since, moreover, s is natural in both entries, $s_{1, TX} \circ T[T\mathbf{1}, u_X] = [\mathbf{1}, Tu_X] \circ s_{1, X}$, whence

$$Ti_X = i_{TX} \circ [\mathbf{1}, \mu_{TX} \circ Tu_X] \circ s_{1, X} \circ T[\mu_1, X] = i_{TX} \circ s_{1, X} \circ T[\mu_1, X]$$

Now, suppose that j_M is a morphism of T -algebras for every (M, μ_M) in \mathcal{C}^T . This entails that the following diagram commutes for every (M, μ_M) in \mathcal{C}^T

$$\begin{array}{ccccc}
 & & T[TM, TM] & \xrightarrow{s_{M, TM}} & [M, T^2M] \\
 & & \uparrow Tj_{TM} & & \uparrow [M, T\mu_M] \\
 T\mathbf{1} & \xrightarrow{Tj_M} & T[M, M] & \xrightarrow{T[\mu_M, M]} & T[TM, M] \\
 \downarrow \mu_1 & & \downarrow \mu_{[M, M]} & & \downarrow s_{M, M} \\
 \mathbf{1} & \xrightarrow{j_M} & [M, M] & \xleftarrow{[M, \mu_M]} & [M, TM]
 \end{array}
 \quad \begin{array}{l}
 (a) \quad (b) \\
 (c) \quad (d)
 \end{array}$$

since (a) commutes by dinaturality of j , (b) commutes by naturality of s , (c) commutes by hypothesis and (d) commutes by definition of $\mu_{[M, M]}$. That is,

$$j_M \circ \mu_1 = [M, \mu_M \circ T\mu_M] \circ s_{M, TM} \circ Tj_{TM} = [M, \mu_M \circ m_M] \circ s_{M, TM} \circ Tj_{TM}.$$

Finally, suppose that $\Gamma_{M,N}^P$ is a morphism of T -algebras for all (M, μ_M) , (N, μ_N) , (P, μ_P) in \mathcal{C}^T . This entails that the following diagram commutes for all (M, μ_M) , (N, μ_N) , (P, μ_P) in \mathcal{C}^T

$$\begin{array}{ccccc}
T[TM, N] & \xrightarrow{T\Gamma_{TM,N}^P} & T[[P, TM], [P, N]] & \xrightarrow{T[s_{P,M}, [P, N]]} & T[T[TP, M], [P, N]] \\
\uparrow T[\mu_{M,N}] & (a) & \uparrow T[[P, \mu_M], [P, N]] & (b) & \downarrow T[T[\mu_P, M], [P, N]] \\
T[M, N] & \xrightarrow{T\Gamma_{M,N}^P} & T[[P, M], [P, N]] & \xrightarrow{T[\mu_{[P,M]}, [P, N]]} & T[T[P, M], [P, N]] \\
\downarrow \mu_{[M,N]} & (c) & \downarrow \mu_{[[P,M], [P,N]]} & (d) & \downarrow s_{[P,M], [P,N]} \\
[M, N] & \xrightarrow{\Gamma_{M,N}^P} & [[P, M], [P, N]] & \xleftarrow{[[P,M], \mu_{[P,N]]}} & [[P, M], T[P, N]] \\
\uparrow [M, \mu_N] & (e) & \uparrow [[P, M], [P, \mu_N]] & (f) & \downarrow [[P, M], T[\mu_P, N]] \\
[M, TN] & \xrightarrow{\Gamma_{M, TN}^P} & [[P, M], [P, TN]] & \xleftarrow{[[P,M], s_{P,N}]} & [[P, M], T[TP, N]] \\
& & & & \downarrow T[T[TP, M], [\mu_P, N]] \\
& & & & T[T[TP, M], [TP, N]] \\
& & & & \downarrow s_{[TP, M], [TP, N]} \\
& & & & [[TP, M], T[TP, N]] \\
& & & & \downarrow [[\mu_P, M], T[TP, N]] \\
& & & & [[P, M], T[TP, N]]
\end{array}$$

since (a) and (e) commute by naturality of Γ , the left-most bended diagram, (b), (d) and (f) commute by definition of $\mu_{[M,N]}$, (c) commutes by hypothesis, and (g) commutes by naturality of s . As a consequence,

$$\begin{aligned}
& \Gamma_{M, TY}^P \circ s_{M, Y} \circ T[\mu_M, Y] \\
&= [[P, M], [P, \mu_{TY} \circ Tu_Y]] \circ \Gamma_{M, TY}^P \circ s_{M, Y} \circ T[\mu_M, Y] \\
&\stackrel{(*)}{=} [[P, M], [P, \mu_{TY}]] \circ \Gamma_{M, T^2Y}^P \circ s_{M, TY} \circ T[\mu_M, TY] \circ T[M, u_Y] \\
&\stackrel{(*)}{=} [[P, M], [P, \mu_{TY}]] \circ [[\mu_P, M], s_{P, TY}] \circ s_{[TP, M], [TP, TY]} \circ T[s_{P, M}, [\mu_P, TY]] \circ \\
&\quad \circ T\Gamma_{TM, TY}^P \circ T[\mu_M, TY] \circ T[M, u_Y] \\
&\stackrel{(*)}{=} [[P, M], [P, \mu_{TY} \circ Tu_Y]] \circ [[\mu_P, M], s_{P, Y}] \circ s_{[TP, M], [TP, Y]} \circ T[s_{P, M}, [\mu_P, Y]] \circ \\
&\quad \circ T\Gamma_{TM, Y}^P \circ T[\mu_M, Y] \\
&= [[\mu_P, M], s_{P, Y}] \circ s_{[TP, M], [TP, Y]} \circ T[s_{P, M}, [\mu_P, Y]] \circ T\Gamma_{TM, Y}^P \circ T[\mu_M, Y]
\end{aligned}$$

where (\star) follow from the naturality of Γ and s and $(*)$ is the commutativity of the diagram above with $N = TY$. Therefore

$$\begin{aligned}
& \Gamma_{X, TY}^P \circ s_{X, Y} \\
&= \Gamma_{X, TY}^P \circ s_{X, Y} \circ T[\mu_{TX} \circ Tu_X, Y] \\
&\stackrel{(*)}{=} [[P, u_X], [P, TY]] \circ \Gamma_{TX, TY}^P \circ s_{TX, Y} \circ T[\mu_{TX}, Y] \\
&\stackrel{(*)}{=} [[P, u_X], [P, TY]] \circ [[\mu_P, TX], s_{P, Y}] \circ s_{[TP, TX], [TP, Y]} \circ \\
&\quad \circ T[s_{P, TX}, [\mu_P, Y]] \circ T\Gamma_{T^2X, Y}^P \circ T[\mu_{TX}, Y] \\
&\stackrel{(*)}{=} [[\mu_P, X], s_{P, Y}] \circ s_{[TP, X], [TP, Y]} \circ T[s_{P, X}, [\mu_P, Y]] \circ T\Gamma_{TX, Y}^P \circ T[\mu_{TX} \circ Tu_X, Y]
\end{aligned}$$

$$= [[\mu_P, X], s_{P,Y}] \circ s_{[TP,X],[TP,Y]} \circ T[s_{P,X}, [\mu_P, Y]] \circ T\Gamma_{TX,Y}^P$$

for all X, Y in \mathcal{C} , by the naturality of the maps involved (to be used in (\star)), by the equality above (to be used in $(*)$) and because $\mu_{TX} = m_X$.

To conclude, a close inspection of the diagrams above shall convince the reader that the three conditions are also sufficient. \square

Remark 3.5. Continuing Remark 3.3, a technical but otherwise straightforward check reveals that s satisfies (3.5), (3.6) and (3.7) if and only if γ satisfies

$$Ti_X = i_{TX} \circ \gamma_X^1, \quad (3.8)$$

$$j_M \circ \mu_1 = [M, \mu_M] \circ \gamma_M^M \circ Tj_M, \quad (3.9)$$

and

$$\Gamma_{TX,TY}^P \circ \gamma_Y^{TX} = [[P, TX], \gamma_Y^P] \circ \gamma_{[P,Y]}^{[P,TX]} \circ T\Gamma_{TX,Y}^P, \quad (3.10)$$

respectively, for all $(M, \mu_M), (P, \mu_P)$ in \mathcal{C}^T and all X, Y in \mathcal{C} .

It is worth noticing that in [W] (see also the recent [St3]) the author studies a related question to the one studied in this section. More precisely, in [W] a closed comonad G on a closed category \mathcal{C} is considered, and sufficient and necessary conditions are given for G such that the Eilenberg-Moore category \mathcal{C}^G is closed and the forgetful functor is strict closed. The difference with our approach is twofold. Firstly, even if a monad on a category \mathcal{C} encodes the same data as a comonad on \mathcal{C}^{op} , a closed structure on a category \mathcal{C} does not necessarily induce a closed structure on \mathcal{C}^{op} . Therefore, one cannot directly derive results in the setting of monads on closed categories from results in the setting of comonads on closed categories, or vice versa. Nevertheless, it should be possible to use a similar approach as the one from [W] (or the one from this paper) in a dual setting, in order to be able to compare both. A more fundamental difference however is that, in contrast to [W], we do not assume the monad to be closed. The main ingredient in the characterization of [W], called the Wood fusion morphism in [St3], is a morphism that exactly looks as an inverse to the transformation γ^M from Remark 3.3. One could wonder if the compatibility conditions that are required on our γ^M as listed in this remark are related to, or even can be deduced from, those for a Wood fusion morphism associated to a closed monad (dualizing the results from [W]). ∇

3.2. The skew-closed skew-monoidal setting. Let now \mathcal{C} be, in addition, left skew-closed skew-monoidal. To help spell out the intertwining condition in the examples, we employ the skew-closed skew-monoidal structure and we denote by L_X the left adjoint of $[X, -]$. The adjunction establishes a bijection between natural transformations $T[TX, -] \Rightarrow [X, T-]$ and natural transformations $L_X T- \Rightarrow T L_{TX} -$. Explicitly, the latter sends a natural transformation $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ to $t_{Y,X}: L_X TY \rightarrow T L_{TX} Y$ given by

$$L_X TY \xrightarrow{L_X T \text{coev}_Y^{TX}} L_X T[TX, L_{TX} Y] \xrightarrow{L_X s_{X, L_{TX} Y}} L_X [X, T L_{TX} Y] \xrightarrow{\text{ev}_{T L_{TX} Y}^X} T L_{TX} Y. \quad (3.11)$$

This yields a natural transformation $L_-(T(\sim)) \Rightarrow T(L_{T-}(\sim))$ between bifunctors, which we shall call the *mate* of s . For the sake of completeness, if we are given $t_{Y,X}: L_X TY \rightarrow T L_{TX} Y$

then the natural transformation $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ is given by

$$T[TX, Y] \xrightarrow{\text{coev}_{T[TX, Y]}^X} [X, L_X T[TX, Y]] \xrightarrow{[X, t_{TX, Y}, X]} [X, TL_{TX}[TX, Y]] \xrightarrow{[X, T \text{ev}_Y^{TX}]} [X, TY].$$

Remark 3.6. For every object X in \mathcal{C} ,

$$t_{-,X}: \begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ L_{TX} \downarrow & \not\cong & \downarrow L_X \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array} \quad \text{is indeed the mate of} \quad s_{X,-}: \begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ \uparrow [TX, -] & \not\cong & \uparrow [X, -] \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array}$$

in the categorical sense; one would just need to adapt the definition of a mate as in [L, §6.1, page 186] to fit the setting of [Mc, Chapter IV, §7, page 98]. Thus, our “parametrized” version of the terminology is justified. ∇

Notice that $t_{X,Y} \in \mathcal{C}(L_Y TX, T(L_{TY} X))$ is the unique morphism such that

$$[Y, t_{X,Y}] \circ \text{coev}_{TX}^Y = s_{Y, L_{TY} X} \circ T(\text{coev}_X^{TY}). \quad (3.12)$$

Now, recall from Remark 2.13(2) the definitions of α, λ, ρ in terms of i, j, Γ . The properties of s translate to the mate in the following way.

Proposition 3.7. *Let $s_{X,Y}: T[TX, Y] \rightarrow [X, TY]$ be natural, and let t be its mate. Then*

(1) *s is unital, i.e. satisfies (3.1), if and only if t satisfies*

$$t_{X,Y} \circ L_Y u_X = u_{L_{TY} X} \circ L_{u_Y} X \quad (3.13)$$

for all X, Y in \mathcal{C} ;

(2) *s is multiplicative, i.e. satisfies (3.2), if and only if t satisfies*

$$t_{X,Y} \circ L_Y m_X = T(L_{m_Y} X) \circ m_{L_{T^2 Y} X} \circ T t_{X, TY} \circ t_{TX, Y} \quad (3.14)$$

for all X, Y in \mathcal{C} ;

(3) *s satisfies (3.5) if and only if t satisfies*

$$TL_{\mu_1} X \circ t_{X, 1} \circ \rho_{TX} = T\rho_X \quad (3.15)$$

for all X in \mathcal{C} ;

(4) *s satisfies (3.6) if and only if t satisfies*

$$\lambda_M \circ L_M \mu_1 = \mu_M \circ m_M \circ T(\lambda_{TM}) \circ t_{1, M} \quad (3.16)$$

for all (M, μ_M) in \mathcal{C}^T ;

(5) *s satisfies (3.7) if and only if t satisfies*

$$t_{X, L_M Y} \circ \alpha_{TX, Y, M} = T\left(L_{T(L_{\mu_M^{(2)} Y})} X\right) \circ T(L_{t_{Y, TM}} X) \circ T(\alpha_{X, TY, TM}) \circ t_{L_{TY} X, M} \circ L_M t_{X, Y} \quad (3.17)$$

for all X, Y in \mathcal{C} and all (M, μ_M) in \mathcal{C}^T , where $\mu_M^{(2)}$ denotes $\mu_M \circ m_M = \mu_M \circ T\mu_M$.

Proof. The proof is a technical but otherwise straightforward exercise. The dinaturality (see Lemma 2.11) of coev^X and ev^X is to be used.

Nevertheless, since Claim (5) is not elementary, let us sketch the underlying argument for the benefit of the reader. First of all, in view of the bijection

$$\mathcal{C}\left(L_P L_{[P,X]}(T[TX, Y]), TY\right) \cong \mathcal{C}\left(T[TX, Y], [[P, X], [P, TY]]\right)$$

there exists a unique morphism F on the left such that

$$[[P, X], [P, F]] \circ [[P, X], \text{coev}_{L_{[P,X]}T[TX,Y]}^P] \circ \text{coev}_{T[TX,Y]}^{[P,X]} = \Gamma_{X,TY}^P \circ s_{X,Y}$$

and we claim that

$$F = T(\text{ev}_Y^{TX}) \circ T\left(L_{T(\text{ev}_X^P)}[TX, Y]\right) \circ t_{[TX,Y], L_P[P,X]} \circ \alpha_{T[TX,Y], [P,X], P}.$$

We check this by direct computation:

$$\begin{aligned} & [[P, X], [P, T(\text{ev}_Y^{TX})]] \circ [[P, X], [P, T(L_{T(\text{ev}_X^P)}[TX, Y])]] \circ [[P, X], [P, t_{[TX,Y], L_P[P,X]}]] \circ \\ & \circ [[P, X], [P, \alpha_{T[TX,Y], [P,X], P}]] \circ [[P, X], \text{coev}_{L_{[P,X]}T[TX,Y]}^P] \circ \text{coev}_{T[TX,Y]}^{[P,X]} = \\ & \stackrel{(2.6)}{=} [[P, X], [P, T(\text{ev}_Y^{TX})]] \circ [[P, X], [P, T(L_{T(\text{ev}_X^P)}[TX, Y])]] \circ [[P, X], [P, t_{[TX,Y], L_P[P,X]}]] \circ \\ & \circ [\text{coev}_{[P,X]}^P, [P, L_{L_P([P,X])}T[TX, Y]]] \circ \Gamma_{L_P([P,X]), L_{L_P([P,X])}T[TX,Y]}^P \circ \text{coev}_{T[TX,Y]}^{L_P([P,X])} \\ & \stackrel{(*)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ [L_P([P, X]), T(\text{ev}_Y^{TX})] \circ \\ & \circ [L_P([P, X]), T(L_{T(\text{ev}_X^P)}[TX, Y])] \circ [L_P[P, X], t_{[TX,Y], L_P[P,X]}] \circ \text{coev}_{T[TX,Y]}^{L_P([P,X])} \\ & \stackrel{(3.12)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ [L_P([P, X]), T(\text{ev}_Y^{TX})] \circ \\ & \circ [L_P([P, X]), T(L_{T(\text{ev}_X^P)}[TX, Y])] \circ s_{L_P([P,X]), L_{T(L_P([P,X])}[TX,Y])} \circ T\left(\text{coev}_{[TX,Y]}^{T(L_P([P,X]))}\right) \\ & \stackrel{(*)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ s_{L_P([P,X]), Y} \circ T[T(L_P([P, X])), \text{ev}_Y^{TX}] \circ \\ & \circ T[T(L_P([P, X])), L_{T(\text{ev}_X^P)}[TX, Y]] \circ T\left(\text{coev}_{[TX,Y]}^{T(L_P([P,X]))}\right) \\ & \stackrel{(*)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ s_{L_P([P,X]), Y} \circ T[T(L_P([P, X])), \text{ev}_Y^{T(L_P[P,X])}] \circ \\ & \circ T[T(L_P([P, X])), L_{T(L_P[P,X])}[T(\text{ev}_X^P), Y]] \circ T\left(\text{coev}_{[TX,Y]}^{T(L_P([P,X]))}\right) \\ & \stackrel{(*)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ s_{L_P([P,X]), Y} \circ T[T(L_P([P, X])), \text{ev}_Y^{T(L_P[P,X])}] \circ \\ & \circ T\left(\text{coev}_{[T(L_P[P,X]), Y]}^{T(L_P([P,X]))}\right) \circ T[T(\text{ev}_X^P), Y] \\ & = [\text{coev}_{[P,X]}^P, [P, TY]] \circ \Gamma_{L_P([P,X]), TY}^P \circ s_{L_P([P,X]), Y} \circ T[T(\text{ev}_X^P), Y] \\ & \stackrel{(*)}{=} [\text{coev}_{[P,X]}^P, [P, TY]] \circ [[P, \text{ev}_X^P], [P, TY]] \circ \Gamma_{X, TY}^P \circ s_{X,Y} = \Gamma_{X, TY}^P \circ s_{X,Y} \end{aligned}$$

where $(*)$ follow by naturality of the morphisms involved and (\star) by dinaturality of ev . Similarly, one can check that the unique morphism G such that

$$\begin{aligned} & [[P, X], [P, G]] \circ [[P, X], \text{coev}_{L_{[P,X]}T[TX,Y]}^P] \circ \text{coev}_{T[TX,Y]}^{[P,X]} \\ & = [[\mu_P, X], s_{P,Y}] \circ s_{[TP,X], [TP,Y]} \circ T[s_{P,X}, [\mu_P, Y]] \circ T\Gamma_{TX,Y}^P \end{aligned}$$

is

$$G = T(\text{ev}_Y^{TX}) \circ T\left(L_{T(\text{ev}_X^P)}[TX, Y]\right) \circ T\left(L_{T\left(L_{\mu_P^{(2)}}[P, X]\right)}[TX, Y]\right) \circ T\left(L_{t_{[P, X], TP}}[TX, Y]\right) \circ \\ \circ T\left(\alpha_{[TX, Y], T[P, X], TP}\right) \circ t_{L_{T[P, X]}[TX, Y], P} \circ L_P\left(t_{[TX, Y], [P, X]}\right).$$

Therefore, since s satisfies (3.7), we have $F = G$. That is,

$$T(\text{ev}_Y^{TX}) \circ T\left(L_{T(\text{ev}_X^P)}[TX, Y]\right) \circ t_{[TX, Y], L_P[P, X]} \circ \alpha_{T[TX, Y], [P, X], P} \\ = T(\text{ev}_Y^{TX}) \circ T\left(L_{T(\text{ev}_X^P)}[TX, Y]\right) \circ T\left(L_{T\left(L_{\mu_P^{(2)}}[P, X]\right)}[TX, Y]\right) \circ T\left(L_{t_{[P, X], TP}}[TX, Y]\right) \circ \\ \circ T\left(\alpha_{[TX, Y], T[P, X], TP}\right) \circ t_{L_{T[P, X]}[TX, Y], P} \circ L_P\left(t_{[TX, Y], [P, X]}\right) \quad (3.18)$$

for every X, Y in \mathcal{C} , (P, μ_P) in \mathcal{C}^T . Now,

$$t_{X, L_M Y} \circ \alpha_{TX, Y, M} \\ \stackrel{(*)}{=} T\left(\text{ev}_{L_{TL_M Y} X}^{TL_M Y}\right) \circ t_{[TL_M Y, L_{TL_M Y} X], L_M Y} \circ \alpha_{T[TL_M Y, L_{TL_M Y} X], Y, M} \circ L_M L_Y T \text{coev}_X^{TL_M Y} \\ \stackrel{(*)}{=} T\left(\text{ev}_{L_{TL_M Y} X}^{TL_M Y}\right) \circ T\left(L_{T(\text{ev}_{L_M Y}^M)}[TL_M Y, L_{TL_M Y} X]\right) \circ t_{[TL_M Y, L_{TL_M Y} X], L_M[M, L_M Y]} \circ \\ \circ \alpha_{T[TL_M Y, L_{TL_M Y} X], [M, L_M Y], M} \circ L_M L_{\text{coev}_Y^M} T[T(L_M Y), L_{TL_M Y} X] \circ L_M L_Y T \text{coev}_X^{TL_M Y} \\ \stackrel{(3.18)}{=} T\left(\text{ev}_{L_{TL_M Y} X}^{TL_M Y}\right) \circ T\left(L_{T(\text{ev}_{L_M Y}^M)}[TL_M Y, L_{TL_M Y} X]\right) \circ \\ \circ T\left(L_{T\left(L_{\mu_M^{(2)}}[M, L_M Y]\right)}[TL_M Y, L_{TL_M Y} X]\right) \circ T\left(L_{t_{[M, L_M Y], TM}}[TL_M Y, L_{TL_M Y} X]\right) \circ \\ \circ T\left(\alpha_{[TL_M Y, L_{TL_M Y} X], T[M, L_M Y], TM}\right) \circ t_{L_{T[M, L_M Y]}[TL_M Y, L_{TL_M Y} X], M} \circ \\ \circ L_M\left(t_{[TL_M Y, L_{TL_M Y} X], [M, L_M Y]}\right) \circ L_M L_{\text{coev}_Y^M} T[T(L_M Y), L_{TL_M Y} X] \circ L_M L_Y T \text{coev}_X^{TL_M Y} \\ \stackrel{(*)}{=} T\left(\text{ev}_{L_{TL_M Y} X}^{TL_M Y}\right) \circ T\left(L_{T\left(L_{\mu_M^{(2)}} Y\right)}[TL_M Y, L_{TL_M Y} X]\right) \circ T\left(L_{t_{Y, TM}}[TL_M Y, L_{TL_M Y} X]\right) \circ \\ \circ T\left(\alpha_{[TL_M Y, L_{TL_M Y} X], TY, TM}\right) \circ t_{L_{TY}[TL_M Y, L_{TL_M Y} X], M} \circ L_M\left(t_{[TL_M Y, L_{TL_M Y} X], Y}\right) \circ L_M L_Y T \text{coev}_X^{TL_M Y} \\ \stackrel{(*)}{=} T\left(L_{T\left(L_{\mu_M^{(2)}} Y\right)} X\right) \circ T\left(L_{t_{Y, TM}} X\right) \circ T\left(\alpha_{X, TY, TM}\right) \circ t_{L_{TY} X, M} \circ L_M(t_{X, Y}),$$

where $(*)$ follow by naturality and the triangles of the adjunction $L_X \dashv [X, -]$ (i.e., the snake identities for ev and coev). We leave to the interested reader to check that also the other implication holds. \square

Remark 3.8. Let us take advantage of the convention $X \otimes Y := L_Y(X)$ as we did in § 2.3. Then $t_{X, Y}: TX \otimes Y \rightarrow T(X \otimes TY)$. With this notation, (3.13) becomes

$$t_{X, Y} \circ (u_X \otimes Y) = u_{X \otimes TY} \circ (X \otimes u_Y) \quad (3.19)$$

and (3.14) becomes

$$t_{X, Y} \circ (m_X \otimes Y) = T(X \otimes m_Y) \circ m_{X \otimes T^2 Y} \circ T t_{X, TY} \circ t_{TX, Y}. \quad (3.20)$$

Analogously, (3.15), (3.16) and (3.17) become

$$T(X \otimes \mu_1) \circ t_{X,1} \circ \rho_{TX} = T\rho_X, \quad (3.21)$$

$$\lambda_M \circ (\mu_1 \otimes M) = \mu_M \circ m_M \circ T(\lambda_{TM}) \circ t_{1,M}, \quad (3.22)$$

$$\begin{aligned} t_{X,Y \otimes M} \circ \alpha_{TX,Y,M} &= T(X \otimes T(Y \otimes \mu_M^{(2)})) \circ T(X \otimes t_{Y,TM}) \circ T(\alpha_{X,TY,TM}) \\ &\circ t_{X \otimes TY,M} \circ (t_{X,Y} \otimes M) \end{aligned} \quad (3.23)$$

respectively, for all X, Y in \mathcal{C} and (M, μ_M) in \mathcal{C}^T and where $\mu_M^{(2)}$ denotes $\mu_M \circ m_M = \mu_M \circ T\mu_M$. ∇

Let us now write the action of T on $[-, -]$ in terms of t , for later use. To this end, recall that given T -algebras (M, μ_M) and (N, μ_N) , the object $[M, N]$ becomes a T -algebra with action $\mu_M \star \mu_N = [M, \mu_N] \circ s_{M,N} \circ T[\mu_M, N]$. In terms of t , one finds that the T -algebra structure is the composition

$$\begin{array}{ccccc} T[M, N] & \xrightarrow{\text{coev}_{T[M,N]}^M} & [M, L_M T[M, N]] & \xrightarrow{[M, t_{[M,N],M}]} & [M, TL_{TM}[M, N]] \\ \mu_{[M,N]} \downarrow & & & & \downarrow [M, TL_{\mu_M}[M, N]] \\ [M, N] & \xleftarrow{[M, \mu_N]} & [M, TN] & \xleftarrow{[M, T \text{ev}_N^M]} & [M, TL_M[M, N]] \end{array} \quad (3.24)$$

Remark 3.9. Continuing Remark 3.5, the mate τ of γ (and in this case τ^M is exactly the mate of γ^M in the sense of [L, §6.1, page 186], for every M in \mathcal{C}^T) is the natural transformation $\tau_X^M: L_M T(X) \rightarrow TL_M(X)$ given by

$$L_M T(X) \xrightarrow{L_M T \text{coev}_X^M} L_M T[M, L_M(X)] \xrightarrow{L_M \gamma_{L_M(X)}^M} L_M[M, TL_M(X)] \xrightarrow{\text{ev}_{TL_M(X)}^M} TL_M(X).$$

The natural transformation γ satisfies (3.3) and (3.4) if and only if τ satisfies

$$\tau_X^M \circ (u_X \otimes M) = u_{X \otimes M} \quad \text{and} \quad \tau_X^M \circ (m_X \otimes M) = m_{X \otimes M} \circ T\tau_X^M \circ \tau_{TX}^M,$$

respectively. In terms of τ , the T -algebra structure on $[M, N]$ looks like

$$\begin{array}{ccc} T[M, N] & \xrightarrow{\text{coev}_{T[M,N]}^M} & [M, T[M, N] \otimes M] \\ \mu_{[M,N]} \downarrow & & \downarrow [M, \tau_{[M,N]}^M] \\ [M, N] & \xleftarrow{[M, \mu_N]} & [M, T([M, N] \otimes M)] \\ & & \downarrow [M, T(\text{ev}_N^M)] \\ & & [M, TN] \end{array}$$

(by replacing τ with its definition, we find the action via γ). It follows that γ satisfies (3.8) and (3.9) if and only if τ satisfies

$$\begin{aligned} T\rho_X &= \tau_X^1 \circ \rho_{TX}, \\ \lambda_M \circ (\mu_1 \otimes M) &= \mu_M \circ T\lambda_M \circ \tau_M^1, \end{aligned}$$

respectively. Furthermore, γ satisfies (3.10) if and only if τ satisfies

$$\tau_X^{T(Y \otimes M)} \circ (TX \otimes \tau_Y^M) \circ \alpha_{TX, TY, M} = T(X \otimes \tau_Y^M) \circ T(\alpha_{X, TY, M}) \circ \tau_{X \otimes TY}^M \circ (\tau_X^{TY} \otimes M)$$

for all X, Y in \mathcal{C} and (M, μ_M) in \mathcal{C}^T . ∇

Example 3.10. Let T be a left Hopf monad on the closed monoidal category \mathcal{C} , with left fusion operator $H_{A,B}^l: T(A \otimes TB) \rightarrow TA \otimes TB$. Choose $L_X = - \otimes X$. Then the natural transformation defined as the composition

$$t_{A,B}: L_B TA = TA \otimes B \xrightarrow{TA \otimes u_B} TA \otimes TB \xrightarrow{(H^l)_{A,B}^{-1}} T(A \otimes TB) = TL_{TB} A$$

satisfies unitality (3.13) and multiplicativity (3.14) as in Proposition 3.7. This can be seen from the formulas in [BLV, Propositions 3.8 and 3.9]. We can re-express $(H^l)_{A,B}^{-1}$ in terms of t as the composition

$$TA \otimes TB \xrightarrow{t_{A,TB}} T(A \otimes T^2 B) \xrightarrow{T(A \otimes m_B)} T(A \otimes TB).$$

This observation implicitly appeared in [BLV, Proposition 3.9]. \triangle

3.3. Set-theoretic solutions. We continue Example 2.3, the cartesian closed category \mathbf{Set} . As we are used to, take $L_X := - \times X$. Let M be an algebra in \mathbf{Set} , i.e. a monoid, and consider the monad $T = M \times -$. We write multiplication in M by juxtaposition, the unit is 1. The action of M on an M -set U will be denoted by $M \times U \rightarrow U, (m, u) \mapsto m.u$.

Since an element in a set S is the same thing as function $* \rightarrow S$ from the point, a natural transformation $t_{X,Y}: (M \times X) \times Y \rightarrow M \times (X \times (M \times Y))$ as in (3.11) is completely determined by its value $t_{*,*}$. Thus, such a natural transformation is completely determined by a function $\delta: M \rightarrow M \times M, m \mapsto (m_+, m_-)$, via $t_{X,Y}(m, x, y) = (m_+, x, m_-, y)$.

Unitality of t means that

$$(1_+, x, 1_-, y) = (t_{X,Y} \circ (u_X \times Y))(x, y) \stackrel{(3.19)}{=} (u_{X \times M \times Y} \circ (X \times u_Y))(x, y) = (1, x, 1, y),$$

while multiplicativity means

$$\begin{aligned} ((mn)_+, x, (mn)_-, y) &= (t_{X,Y} \circ (m_X \times Y))(m, n, x, y) \\ &\stackrel{(3.20)}{=} ((M \times X \times m_Y) \circ m_{X \times M \times M \times Y} \circ (M \times t_{X, M \times Y}) \circ t_{M \times X, Y})(m, n, x, y) \\ &= (m_+ n_+, x, n_- m_-, y) \end{aligned}$$

for all sets X, Y , all $x \in X, y \in Y$ and $m, n \in M$. Thus we get a lift of the inner hom functor of \mathbf{Set} if and only if $\delta: M \rightarrow M \times M^{\text{op}}$ is a monoid map. Here M^{op} is the opposite monoid, i.e. the set M with multiplication $m \cdot n = nm$ and unit element 1.

Given two M -sets U and V , from (3.24) one computes the action of M on $[U, V] = \mathbf{Set}(U, V)$ to be

$$(m.f)(u) = m_+.f(m_-.u)$$

for all $m \in M, u \in U, f \in \mathbf{Set}(U, V)$. The unique function $\varepsilon: M \rightarrow *$ to the terminal object serves as an augmentation: it is automatically an algebra map.

To lift the full *skew-closed* structure of \mathbf{Set} to ${}_M\mathbf{Set} = \mathbf{Set}^{M \times -}$, we need the transformations i, j, Γ specified in Example 2.3 to be M -module morphisms. To this aim, t satisfies (3.21) (up to identifying $X \times *$ with X) if and only if

$$\begin{aligned} (m_+, x) &= \left((M \times X \times \varepsilon) \circ t_{X,*} \circ \rho_{M \times X} \right) (m, x) \\ &= (M \times \rho_X)(m, x) = (m, x) \end{aligned}$$

for every set X , all $x \in X$ and all $m \in M$, which is equivalent to $\delta(m) = (m, m_-)$. Furthermore, t satisfies (3.22) (up to identifying $* \times X$ with X) if and only if

$$u = \left(\lambda_U \circ (\varepsilon \times U) \right) (m, u) = \left(\mu_U \circ m_U \circ (M \times \lambda_{M \times U}) \circ t_{*,U} \right) (m, u) = m_+ m_- . u$$

for each M -set U and for every $u \in U$. This yields the necessary and sufficient condition $m_+ m_- = 1$ for all $m \in M$.

Before discussing Γ , we note that this already forces M to be a group, possibly subject to extra conditions imposed by Γ . Indeed, $m m_- = m_+ m_- = 1$ says that every element $m \in M$ has a right inverse. In particular, for a given m , we have that m_- has m_{--} as right inverse. Since M is associative, the simple computation

$$m = m(m_- m_{--}) = (m m_-) m_{--} = m_{--}$$

shows that m is a two-sided inverse of m_- , and so every element of M is invertible.

For Γ , we have that t satisfies (3.23) if and only if

$$\begin{aligned} (m_+, x, m_-, y, u) &= \left(t_{X, Y \times U} \circ \alpha_{M \times X, Y, U} \right) (m, x, y, u) \\ &\stackrel{(3.23)}{=} \left(\left(M \times X \times M \times Y \times \mu_U^{(2)} \right) \circ (M \times X \times t_{Y, M \times U}) \circ (M \times \alpha_{X, TY, TU}) \right. \\ &\quad \left. \circ t_{X \times M \times Y, U} \circ (t_{X, Y} \times U) \right) (m, x, y, u) \\ &= (m_{++}, x, m_{-+}, y, m_{--} m_{+-} . u) \end{aligned}$$

for all sets X, Y , all M -sets (U, μ_U) , and for every $x \in X, y \in Y, u \in U$ and $m \in M$. This leads to $(m_{++}, m_{-+}, m_{--} m_{+-}) = (m_+, m_-, 1)$ for all $m \in M$.

Finally, this last condition is also satisfied for groups, i.e. if $(m_+, m_-) = (m, m^{-1})$. Indeed,

$$(m_{++}, m_{-+}, m_{--} m_{+-}) = (m, m^{-1}, (m^{-1})^{-1} m^{-1}) = (m, m^{-1}, 1) = (m_+, m_-, 1) .$$

Similar considerations can be done for the monad $- \times M$ and we summarize this example in the following Theorem.

Theorem 3.11. *Let $M \in \mathbf{Set}$ be a monoid. The following are equivalent:*

- (1) M is a group.
- (2) The category ${}_M\mathbf{Set}$ (\mathbf{Set}_M) of left (right) M -sets is left skew-closed such that the forgetful functor to \mathbf{Set} is strictly closed.

3.4. The linear case. We now look at solutions in ${}_k\mathcal{M}$ for a commutative ring k , following up on Example 2.4. Here $L_M = - \otimes M$.

For a k -algebra A , the functor $T = A \otimes -$ is a monad on ${}_k\mathcal{M}$ in a natural way. We need to exhibit a natural k -module map $t_{M,N}: (A \otimes M) \otimes N \rightarrow A \otimes (M \otimes (A \otimes N))$ with certain properties. Note that such a map is determined by $t_{k,k}$: we have

$$\begin{aligned} t_{M,N}(a \otimes m \otimes n) &= (t_{M,N} \circ (A \otimes c_M(m) \otimes c_N(n)))(a \otimes 1 \otimes 1) \\ &= ((A \otimes c_M(m) \otimes A \otimes c_N(n)) \circ t_{k,k})(a \otimes 1 \otimes 1) \end{aligned}$$

by naturality, where $c_M(m): k \rightarrow M$ is the unique k -linear map such that $c_M(m)(1_k) = m$ for all $m \in M$. We define, in a Sweedler-like notation,

$$\delta: A \rightarrow A \otimes A, \quad a \mapsto a_+ \otimes a_-,$$

(summation understood) to be the obvious map such that $t_{k,k}(a) = a_+ \otimes 1 \otimes a_- \otimes 1$.

From here on, one carries out, *mutatis mutandis*, the computations from § 3.3. That is, one declines the formulas from Remark 3.8 in this specific situation. We leave the details to the reader, and instead just state:

Theorem 3.12. *Let A be an algebra over a commutative ring k . Then the closed structure of ${}_k\mathcal{M}$ lifts to a left skew-closed structure on ${}_A\mathcal{M}$ if and only if:*

- (1) (A, ε) is an augmented k -algebra,
- (2) there exists an algebra map $\delta: A \rightarrow A \otimes A^{\text{op}}$, $\delta(a) = a_+ \otimes a_-$ (summation understood), and they satisfy

$$a_+ \varepsilon(a_-) = a \tag{GA1}$$

$$a_+ a_- = \varepsilon(a) 1_A \tag{GA2}$$

$$a_{++} \otimes a_{-+} \otimes a_{--} a_{+-} = a_+ \otimes a_- \otimes 1 \tag{GA3}$$

In this case, A acts on ${}_k\mathcal{M}(M, V)$ as $(a.f)(m) = a_+ f(a_- m)$.

Remark 3.13. (1) The above Theorem details the case of left modules, i.e. we looked at algebras over the monad $A \otimes -$. This choice was arbitrary: we could have chosen the monad $- \otimes A$, whose algebras are exactly the right A -modules. One then finds that lifting the skew-closed structure of ${}_k\mathcal{M}$ to \mathcal{M}_A is equivalent to A being augmented via ε , and equipped with an algebra map $\delta': A \rightarrow A^{\text{op}} \otimes A$, $a \mapsto a_- \otimes a_+$, satisfying similar properties as above. In detail, we have

$$\varepsilon(a_-) a_+ = a, \tag{GA1'}$$

$$a_- a_+ = \varepsilon(a) 1_A, \tag{GA2'}$$

$$a_{+-} a_{--} \otimes a_{-+} \otimes a_{++} = 1 \otimes a_- \otimes a_+ \tag{GA3'}$$

for all $a \in A$. In this case, A acts on ${}_k\mathcal{M}(M, V)$ as $(f.a)(m) = f(m a_-) a_+$.

- (2) Theorem 3.12 is the result of a fruitful discussion [Böl] between Gabriella Böhm and Paolo Saracco during the CIMPA school “Quantum Symmetries” in Bogotá in 2019. ▽

4. GABI-ALGEBRAS: BASIC PROPERTIES AND EXAMPLES

For the rest of the paper, we will be concerned with the situation of Theorem 3.12.

4.1. **Definition and examples.** Let us now define the main character of the paper.

Definition 4.1. An algebra A over k satisfying the equivalent conditions of Theorem 3.12 is called a *left gabi-algebra*. If A satisfies the conditions from Remark 3.13(1), then we call it a *right gabi-algebra*. If A is a gabi-algebra such that the lifted skew-closed structure on ${}_A\mathcal{M}$ is normal, then we call A a *normal gabi-algebra*.

Remark 4.2. (1) Given the lifted skew-closed structure on ${}_A\mathcal{M}$, we can obtain the gabi-algebra structure on A explicitly in the following way. The fact that k is the unit of the closed structure makes that k must be endowed with a left A -module structure. The augmentation map ε is then given by $\varepsilon(a) = a \cdot 1_k$. Now consider the left A -modules $.A \otimes A$ and $A \otimes .A$, where the dot indicates how A acts regularly on the first and second tensorand, respectively. Since the closed structure is lifted from ${}_k\mathcal{M}$, we have that $[A \otimes .A, .A \otimes A] = {}_k\mathcal{M}(A \otimes .A, .A \otimes A)$, which contains the identity map $\text{id}: A \otimes A \rightarrow A \otimes A$. Then the comultiplication map δ is given by

$$\delta(a) = (a \cdot \text{id})(1_A \otimes 1_A).$$

(2) Of course, one can work with coalgebras and formally dualise (GA1), (GA2) and (GA3) from Theorem 3.12 to obtain what we would call a *left gabi-coalgebra*, or (GA1'), (GA2') and (GA3') from (1) above to obtain a *right gabi-coalgebra*.

As for algebras, a gabi-coalgebra structure on a given coalgebra leads to additional structure on its category of comodules. For example, let C be a right gabi-coalgebra, that is a coalgebra C endowed with two coalgebra maps, a unit $1_C: k \rightarrow C$ and a multiplication $\nabla: C \otimes C^{\text{cop}} \rightarrow C$, $c \otimes d \mapsto c \cdot d$, satisfying

$$c \cdot 1_C = c, \tag{GC1'}$$

$$c_{(1)} \cdot c_{(2)} = \varepsilon(c)1_C, \tag{GC2'}$$

$$\varepsilon(e)c \cdot d = (c \cdot e_{(2)}) \cdot (d \cdot e_{(1)}). \tag{GC3'}$$

Consider two right comodules N and P . Then the k -linear hom space between them can be endowed with the following structure map

$$\begin{aligned} {}_k\mathcal{M}(N, P) &\longrightarrow {}_k\mathcal{M}(N, P \otimes C) \\ \varphi &\longmapsto \left(n \mapsto \sum_i \varphi(n_{0_i})_0 \otimes \varphi(n_{0_i})_1 \cdot n_{1_i} \right) \end{aligned}$$

If N is moreover finite-dimensional, then we have a natural k -linear isomorphism

$${}_k\mathcal{M}(N, P \otimes C) \cong {}_k\mathcal{M}(N, P) \otimes C$$

and the above map, combined with this isomorphism, endows ${}_k\mathcal{M}(N, P)$ with a right C -comodule structure. In particular, this will endow the category of all finite-dimensional right C -comodules with a skew closed structure in such a way that the forgetful functor to finite-dimensional vector spaces strictly preserves the closed structure.

By using classical Tannaka duality, the above reasoning can be reversed. First, recall that for any finite-dimensional right comodule N over an arbitrary coalgebra C , N^* is a left C -comodule with respect to

$$N^* \rightarrow C \otimes N^*, \quad f \mapsto \sum f(e_{i0})e_{i1} \otimes e^i,$$

where $\sum_i e_i \otimes e^i$ is a dual basis of N , and so it is a right C^{cop} -comodule with respect to

$$N^* \rightarrow N^* \otimes C^{\text{cop}}, \quad f \mapsto \sum e^i \otimes f(e_{i0})e_{i1}.$$

This shows that the contravariant auto-equivalence on the category of finite-dimensional vector spaces given by taking the linear dual lifts to a contravariant equivalence between the category of finite-dimensional right C -comodules and the category of finite-dimensional right C^{cop} -comodules. Suppose now that the category $\mathcal{M}_{\text{fd}}^C$ of finite-dimensional right C -comodules is skew closed in such a way that the forgetful functor to finite-dimensional vector spaces strictly preserves the closed structure. Then, by applying the above equivalence, the internal hom functor

$$[-, -] : (\mathcal{M}_{\text{fd}}^C)^{\text{op}} \times \mathcal{M}_{\text{fd}}^C \rightarrow \mathcal{M}_{\text{fd}}^C$$

can be rewritten as

$$[-, -] : \mathcal{M}_{\text{fd}}^{C^{\text{cop}}} \times \mathcal{M}_{\text{fd}}^C \rightarrow \mathcal{M}_{\text{fd}}^C$$

which translates by Tannaka duality to a coalgebra morphism

$$\nabla : C \otimes C^{\text{cop}} \rightarrow C$$

which will endow C with a right gabi-coalgebra structure. ∇

Henceforth, we focus on the left-hand side case and we call left gabi-algebras simply *gabi-algebras*.

Example 4.3 (Hopf algebras). Let H be a Hopf algebra over k . Then $\delta(h) = h_{(1)} \otimes S(h_{(2)})$ turns H into a gabi-algebra. Indeed, H is augmented; $h_{(1)}\varepsilon(S(h_{(2)})) = h$ holds since S is an anti-coalgebra map and Δ is (right) counital; $h_{(1)}S(h_{(2)}) = \varepsilon(h)1$ holds since S is a (right) convolution inverse of the identity; and finally

$$\begin{aligned} h_{(1,1)} \otimes S(h_{(2)})_{(1)} \otimes S(S(h_{(2)})_{(2)})S(h_{(1,2)}) &= h_{(1,1)} \otimes S(h_{(2)})_{(1)} \otimes S(h_{(1,2)})S(h_{(2)})_{(2)} \\ &= h_{(1,1)} \otimes S(h_{(2,2)}) \otimes S(h_{(1,2)})S(h_{(2,1)}) = h_{(1)} \otimes S(h_{(4)}) \otimes S(h_{(2)})S(h_{(3)}) \\ &= h_{(1)} \otimes S(h_{(2)}) \otimes 1. \end{aligned}$$

If the antipode of H is invertible, then also the map $h \mapsto h_{(2)} \otimes S^{-1}(h_{(1)})$ turns H into a gabi-algebra. The choice of using the antipode or the inverse antipode corresponds to lifting either the (entire) right or left closed monoidal structure to the category of left modules. Evidently, $h \mapsto S(h_{(1)}) \otimes h_{(2)}$ yields a right gabi-algebra structure. \triangleleft

Example 4.4 (one-sided Hopf algebras). In Example 4.3 we did not use the fact the antipode S is a left convolution inverse of the identity. This suggests another class of examples, namely one-sided Hopf algebras.

A *right* (resp., *left*) *Hopf algebra* is a bialgebra B in which the identity has a right (resp., left) convolution inverse. A right Hopf algebra whose right antipode is anti-multiplicative and anti-comultiplicative carries the structure of a left gabi-algebra with respect to ε and

$\delta(b) := b_{(1)} \otimes S(b_{(2)})$ for all $b \in B$, i.e., it lifts the closed structure of ${}_k\mathcal{M}$ to its category of left modules (symmetrically for the left Hopf algebra case). It is easy to see that a one-sided convolution inverse of the identity is always unital and counital (see, e.g., [Sa2, Remark 3.8]), so an anti-multiplicative and anti-comultiplicative one-sided antipode is automatically an anti-bialgebra morphism.

Examples of one-sided but not two-sided Hopf algebras exist in the literature: see [GNT, LT, NT, RT]. Namely, [GNT, Example 21] exhibits an example of a genuine left Hopf algebra whose left antipode is an anti-bialgebra morphism, constructed as the free left Hopf algebra over a coalgebra. In [NT, §3], a similar construction is used to exhibit an example of a left Hopf algebra in which no left antipode can be an anti-bialgebra morphism. In [RT, §3], a new example of a left Hopf algebra is provided by modifying the construction of $SL_q(2)$; here as well the left antipode is neither an anti-algebra nor an anti-coalgebra morphism. The latter example is extended in [LT] to provide a whole family of left Hopf algebras $\tilde{S}L_q(n)$ for all $n \geq 2$. \triangle

Inspired by Example 4.3 and Example 4.4, let us introduce the following definition.

Definition 4.5. The *antipode* of a gabi-algebra A is the map $\sigma: A \rightarrow A$ defined by $\sigma(a) = \varepsilon(a_+)a_-$ for all $a \in A$.

The antipode σ is a composition of algebra morphisms,

$$\sigma = A \xrightarrow{\delta} A \otimes A^{\text{op}} \xrightarrow{\varepsilon \otimes \text{id}} A^{\text{op}},$$

and hence itself an algebra morphism. Furthermore, it is a morphism of augmented algebras since for all $a \in A$,

$$(\varepsilon \circ \sigma)(a) = \varepsilon(a_+)\varepsilon(a_-) = \varepsilon(a).$$

Remark 4.6. One can now give trivial examples of an algebra H such that the forgetful functor $\omega: {}_H\mathcal{M} \rightarrow {}_k\mathcal{M}$ is both strictly closed and strictly monoidal, but not closed monoidal. We simply need a bialgebra structure implementing a monoidal structure, and an unrelated Hopf algebra structure implementing the closed structure as in Example 4.3. For example, $H = k[x]$, with the bialgebra structure

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1,$$

and the Hopf algebra structure

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

A more significant class of examples is offered by one-sided Hopf algebras as in Example 4.4. We will treat the latter case in more detail in § 4.5. ∇

In view of what we mentioned at the beginning of § 3.1, for a gabi-algebra A the lifted skew-closed structure on ${}_A\mathcal{M}$ is automatically right normal. The interested reader may now wonder what happens with left and associative normality. Since it will be easier to answer after discussing tricocycloids and tensor-hom adjunctions in § 4.2 and § 4.3, we postpone answering them until Proposition 5.10 and Proposition 5.11.

4.2. Gabi-algebras and tricocycloids. Let us now show that a gabi-algebra A canonically admits the structure of a so-called *lax tricocycloid* in ${}_k\mathcal{M}$. This allows us also to discuss a few conditions under which a gabi-algebra structure on an algebra A is coming from a Hopf algebra structure (see Theorem 5.3).

Definition 4.7 ([St1, §3]). Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a braided monoidal category with braiding c . A *lax tricocycloid* in \mathcal{C} is an object A together with a morphism $v: A \otimes A \rightarrow A \otimes A$ satisfying

$$(v \otimes A) \circ (A \otimes c_{A,A}) \circ (v \otimes A) = (A \otimes v) \circ (v \otimes A) \circ (A \otimes v). \quad (4.1)$$

The morphism $c_{A,A}^{-1} \circ v$ is a *fusion operator* (see [St1, Proposition 1.1]). When v is invertible, we drop the adjective ‘lax’. A (lax) tricocycloid (A, v) is *augmented* if there are morphism $\eta: \mathbf{1} \rightarrow A$ and $\varepsilon: A \rightarrow \mathbf{1}$ such that

$$(A \otimes \varepsilon) \circ v \circ (A \otimes \eta) = A, \quad (\varepsilon \otimes A) \circ v = A \otimes \varepsilon, \quad v \circ (\eta \otimes A) = A \otimes \eta, \quad \varepsilon \circ \eta = \mathbf{1}. \quad (4.2)$$

The notion of tricocycloid encompasses both bialgebras and Hopf algebras in \mathcal{C} . In fact, augmented lax tricocycloid structures (v, η, ε) on an object A are in one-to-one correspondence with bialgebra structures $(m, \Delta, \eta, \varepsilon)$ on A with the inverse braiding, and this bialgebra is a Hopf algebra if and only if v is invertible [LS1, Proposition 2.3]. The tricocycloid equation (4.1) encodes associativity, coassociativity, and the fact that Δ is a multiplicative map with respect to the inverse braiding.

Example 4.8. Let H be a Hopf algebra over k . Then $v(a \otimes b) = a_{(2)}b \otimes a_{(1)}$ and $v'(a \otimes b) = b_{(1)} \otimes S(b_{(2)})a$ define lax tricocycloid structures. In fact, $v' = v^{-1}$, and $(A, v, \mathbf{1}, \varepsilon)$ is an augmented tricocycloid. \triangle

But we do not need a Hopf algebra or a bialgebra structure in order to get a tricocycloid.

Proposition 4.9. *Let $A \in {}_k\mathcal{M}$ be a gabi-algebra and set $v(a \otimes b) := b_+ \otimes b_- a$ for all $a, b \in A$. Then (A, v) is a lax tricocycloid.*

Proof. We compute

$$\begin{aligned} ((v \otimes A) \circ (A \otimes \mathbf{tw}_{A,A}) \circ (v \otimes A))(a \otimes b \otimes c) &= ((v \otimes A) \circ (A \otimes \mathbf{tw}_{A,A}))(b_+ \otimes b_- a \otimes c) \\ &= (v \otimes A)(b_+ \otimes c \otimes b_- a) \\ &= c_+ \otimes c_- b_+ \otimes b_- a, \end{aligned}$$

where \mathbf{tw} is the usual twist, and on the other hand,

$$\begin{aligned} ((A \otimes v) \circ (v \otimes A) \circ (A \otimes v))(a \otimes b \otimes c) &= ((A \otimes v) \circ (v \otimes A))(a \otimes c_+ \otimes c_- b) \\ &= (A \otimes v)(c_{++} \otimes c_{+-} a \otimes c_- b) \\ &= c_{++} \otimes (c_- b)_+ \otimes (c_- b)_- c_{+-} a \\ &= c_{++} \otimes c_{-+} b_+ \otimes b_- c_{--} c_{+-} a \\ &\stackrel{\text{(GA3)}}{=} c_+ \otimes c_- b_+ \otimes b_- a. \end{aligned}$$

The expressions agree, and hence the claim follows. \square

4.3. Tensor-hom adjunction. As it turns out, gabi-algebras allow for a ‘tensor-hom’-like adjunction on their categories of modules.

Let A be a gabi-algebra and M be a left A -module. Consider the k -module $A \otimes M$, which we turn into an A -bimodule via

$$a.(b \otimes m).c = abc_+ \otimes c_-m .$$

We denote this bimodule by $A \odot M$.

Theorem 4.10. *For any $M \in {}_A\mathcal{M}$, there is an adjunction*

$$\begin{array}{ccc} & \xrightarrow{(A \odot M) \otimes_A -} & \\ {}_A\mathcal{M} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & {}_A\mathcal{M} \\ & \xleftarrow{{}_k\mathcal{M}(M, -)} & \end{array} .$$

The unit and counit of the adjunction are

$$\text{coev}_N^M : N \rightarrow {}_k\mathcal{M}(M, (A \odot M) \otimes_A N), \quad n \mapsto (m \mapsto (1_A \odot m) \otimes_A n)$$

and

$$\text{ev}_N^M : (A \odot M) \otimes_A {}_k\mathcal{M}(M, N) \rightarrow N, \quad (a \odot m) \otimes_A f \mapsto af(m) .$$

It is routine to check the unit-counit adjunction. Instead of giving the proof, we will briefly explain where the adjunction comes from. First of all, the free-forgetful adjunction for ${}_A\mathcal{M}$ gives an isomorphism ${}_k\mathcal{M}(M, N) \cong {}_A\mathcal{M}(A \otimes M, N)$ of k -modules, for any k -module M and A -module N . Transporting the left A -module structure on ${}_k\mathcal{M}(M, N)$ through this isomorphism, one arrives at the right module structure on $A \odot M$. Finally, one uses the standard tensor-hom adjunction for bimodules over rings, see (2.7). The chain of isomorphisms is thus

$${}_A\mathcal{M}(N, {}_k\mathcal{M}(M, P)) \cong {}_A\mathcal{M}(N, {}_A\mathcal{M}(A \odot M, P)) \cong {}_A\mathcal{M}((A \odot M) \otimes_A N, P) .$$

Remark 4.11. Theorem 4.10 actually holds more generally: this adjunction exists whenever A is an algebra admitting an algebra map $\delta : A \rightarrow A \otimes A^{\text{op}}$. ∇

4.4. Skew-monoidal structure. Any skew-closed category in which the inner hom functor admits a left adjoint possesses a skew-monoidal structure (see § 2.2). Here we give the skew-monoidal structure of the category of modules over a gabi-algebra, associated to the tensor-hom adjunction from Theorem 4.10.

Consider the bifunctor

$$\boxtimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad M \boxtimes N = (A \odot N) \otimes_A M .$$

Proposition 4.12. *Let A be a gabi-algebra. The \boxtimes tensor product defined above provides a left skew-monoidal structure on ${}_A\mathcal{M}$ with unit k , unitors*

$$\begin{aligned} \lambda_N : k \boxtimes N &\rightarrow N, & (a \odot n) \otimes_A 1_k &\mapsto an, \\ \rho_M : M &\xrightarrow{\sim} M \boxtimes k, & m &\mapsto (1_A \odot 1_k) \otimes_A m, \end{aligned} \tag{4.3}$$

and associator

$$\begin{aligned} \alpha_{L,M,N}: (L \boxtimes M) \boxtimes N &\rightarrow L \boxtimes (M \boxtimes N), \\ (a \odot n) \otimes_A ((b \odot m) \otimes_A l) &\mapsto \left(ab_+ \odot \left((1_A \odot b_- n) \otimes_A m \right) \right) \otimes_A l. \end{aligned} \quad (4.4)$$

Moreover, this is a monoidal structure (i.e., the unitors and associator are isomorphisms) if and only if A is a normal gabi-algebra.

Proof. The adjunction from Theorem 4.10 is given in terms of hom-sets as

$${}_A\mathcal{M}(X \boxtimes Y, V) \xrightarrow{\sim} {}_A\mathcal{M}(X, {}_k\mathcal{M}(Y, V)), \quad f \mapsto \left(f^\# : x \mapsto (y \mapsto f((1 \odot y) \otimes_A x)) \right)$$

with inverse

$${}_A\mathcal{M}(X, {}_k\mathcal{M}(Y, V)) \xrightarrow{\sim} {}_A\mathcal{M}(X \boxtimes Y, V) \quad g \mapsto \left(g^b : (a \odot y) \otimes_A x \mapsto ag(x)(y) \right),$$

where the action of a is in V . We leave to the interested reader to make explicit each time to which adjunction $(-)^{\#}$ and $(-)^b$ refer.

From the paragraph above (2.5), we can now immediately describe the left unitor as

$$\lambda_X : k \boxtimes X = (A \odot X) \otimes_A k \rightarrow X, \quad \lambda_X((a \odot x) \otimes_A 1_k) = j_X^b((a \odot x) \otimes_A 1_k) = ax,$$

while unpacking (2.5) entails that the right unitor has to be

$$\rho_X : X \rightarrow X \boxtimes k = (A \odot k) \otimes_A X, \quad x \mapsto (1_A \odot 1_k) \otimes_A x.$$

Notice that (2.5) is an isomorphism, since ${}_k\mathcal{M}(k, X) \cong X$ as left A -modules, and $\rho_X = \text{id}_{X \boxtimes k}^{\#}$. Furthermore, ρ_X is invertible, as expected, with inverse

$$\rho_X^{-1} : X \boxtimes k \rightarrow X, \quad \rho_X^{-1}((a \odot 1_k) \otimes_A x) = \text{id}_X^b((a \odot 1_k) \otimes_A x) = ax$$

(the invertibility of ρ_X follows from Theorem 2.12 and the fact that ${}_A\mathcal{M}$ is right normal).

Now, we compute the associator. First of all, the natural transformation from (2.3) is given by $(p_{X,Y,Z}(h))(x)(y) = h(1 \odot y \otimes_A x)$, for any linear map $h : X \otimes Y \rightarrow Z$. The natural transformation (2.4) between hom functors can be described as $f \mapsto (p \circ f^\#)^{bb}$, for an A -linear morphism $f : X \boxtimes (Y \boxtimes Z) \rightarrow V$. Explicitly, and by omitting a few \boxtimes ,

$$\begin{aligned} (p \circ f^\#)^{bb}(a \odot z \otimes_A (b \odot y \otimes_A x)) &= (p_{X,Y,X(YZ)} \circ f^\#)^{bb}(a \odot z \otimes_A (b \odot y \otimes_A x)) \\ &= a(p_{X,Y,X(YZ)} \circ f^\#)^b(b \odot y \otimes_A x)(z) \\ &\stackrel{(\star)}{=} a \left[b.(p_{X,Y,X(YZ)} \circ f^\#)(x)(y) \right] (z) \\ &= ab_+(p_{X,Y,X(YZ)} \circ f^\#)(x)(y)(b_- z) \\ &= ab_+ f^\#(x)(1 \odot b_- z \otimes_A y) \\ &= ab_+ f(1 \odot (1 \odot b_- z \otimes_A y) \otimes_A x) \\ &= f(ab_+ \odot (1 \odot b_- z \otimes_A y) \otimes_A x), \end{aligned}$$

where in the step marked (\star) care has to be taken: the action of b takes place in ${}_k\mathcal{M}(Z, V)$. By using the Yoneda lemma, we finally find $\alpha_{X,Y,Z} = (p_{X,Y,X(YZ)} \circ \text{id}_{X(YZ)}^{\#})^{bb}$ and it is clear that this coincides with the associator in the statement.

The last statement is a direct consequence of Theorem 2.12 and the definition of a normal gabi-algebra. \square

Note also that

$$k \boxtimes X = (A \odot X) \otimes_A k \cong (A \odot X) \otimes_A A/A^+ \cong (A \odot X)/(A \odot X)A^+,$$

where $A^+ := \ker(\varepsilon)$ is the augmentation ideal of A .

4.5. A family of non-trivial examples: one-sided Hopf algebras. We already saw in Example 4.4 that right Hopf algebras with anti-multiplicative and anti-comultiplicative one-sided antipode are examples of gabi-algebras, whence their categories of modules are left skew-closed. It is interesting to remark explicitly that these provide an example of left skew-closed categories which are neither left normal nor associative normal. Let (B, Δ, ε) be a k -bialgebra admitting an anti-multiplicative and anti-comultiplicative right antipode S which is not a left antipode. Then B is a left gabi-algebra with respect to ε and $\delta(b) := b_{(1)} \otimes S(b_{(2)})$ for all $b \in B$, whence the category ${}_B\mathcal{M}$ is a left skew-closed category with respect to $[M, N] := {}_k\mathcal{M}(M, N)$ with action $(b \cdot f)(m) := b_{(1)}f(S(b_{(2)})m)$ for all $b \in B$, $m \in M$, $f \in {}_k\mathcal{M}(M, N)$ and for all M, N in ${}_B\mathcal{M}$. Now, consider the k -linear map

$$\beta: B \otimes B \rightarrow B \otimes B, \quad a \otimes b \mapsto a_{(1)} \otimes S(a_{(2)})b, \quad (4.5)$$

and endow the domain with the diagonal B -module structure $a \cdot (b \otimes c) := a_{(1)}b \otimes a_{(2)}c$ and the codomain with the left regular B -module structure $a \cdot (b \otimes c) := ab \otimes c$ for all $a, b, c \in B$. It is clear that β is not B -linear, otherwise the relation

$$a_{(1)} \otimes S(a_{(2)})a_{(3)} = \beta(a \cdot (1 \otimes 1)) = a \cdot \beta(1 \otimes 1) = a \otimes 1$$

would entail that S is also a left antipode, contradicting our choice. Nevertheless,

$$\begin{aligned} a_{(1)} \cdot \beta(S(a_{(2)}) \cdot (b \otimes c)) &= a_{(1)} \cdot \beta(S(a_{(3)})b \otimes S(a_{(2)})c) \\ &= a_{(1)} \cdot \left(S(a_{(3)})_{(1)}b_{(1)} \otimes S(S(a_{(3)})_{(2)}b_{(2)})S(a_{(2)})c \right) \\ &= a_{(1)}S(a_{(4)})b_{(1)} \otimes S(a_{(2)}S(a_{(3)})b_{(2)})c = \varepsilon(a)\beta(b \otimes c) \end{aligned}$$

implies that the assignment $1_k \mapsto \beta$ is an element in ${}_B\mathcal{M}(k, [B \otimes B, B \otimes B])$ which does not come from an element in ${}_B\mathcal{M}(B \otimes B, B \otimes B)$ via the morphism in (N1). Therefore, the skew-closed structure is not left normal. It is not associative normal either, or Proposition 5.10 would imply that β is invertible, which is not the case.

As a final observation remember that, since ${}_B\mathcal{M}$ is not left normal as a skew-closed category, Theorem 2.12 entails that the unitor λ_M from (2.2) cannot be a natural isomorphism. It is interesting to see explicitly why this is the case in our concrete example. To this aim, notice that even if β from (4.5) is not B -linear, it induces a left B -linear map

$$\hat{\beta}: \frac{B \odot (.B \otimes .B)}{(B \odot (.B \otimes .B))B^+} \rightarrow \frac{B \odot (.B \otimes B)}{(B \odot (.B \otimes B))B^+}, \quad \overline{a \otimes (b \otimes c)} \mapsto \overline{a \otimes (b_{(1)} \otimes S(b_{(2)})c)},$$

because for all $x \in B^+$ we have

$$\overline{ax_{(1)} \otimes (S(x_{(3)})_{(1)}b_{(1)} \otimes S(S(x_{(3)})_{(2)}b_{(2)})S(x_{(2)})c} = \overline{ax_{(1)} \otimes (S(x_{(2)})b_{(1)} \otimes S(b_{(2)})c}$$

$$= \varepsilon(x) \overline{a \otimes (b_{(1)} \otimes S(b_{(2)})c)} = 0.$$

If, in this framework,

$$\lambda_{B \otimes B}: \frac{B \odot (.B \otimes .B)}{(B \odot (.B \otimes .B))B^+} \rightarrow .B \otimes .B, \quad \overline{a \otimes (b \otimes c)} \mapsto a_{(1)}b \otimes a_{(2)}c,$$

is invertible, with inverse necessarily given by

$$\lambda_{B \otimes B}^{-1}: .B \otimes .B \rightarrow \frac{B \odot (.B \otimes .B)}{(B \odot (.B \otimes .B))B^+}, \quad a \otimes b \mapsto \overline{a_{(1)} \otimes (1 \otimes S(a_{(2)})b)},$$

then we have the equality

$$\overline{a \otimes (1 \otimes 1)} = \overline{a_{(1)} \otimes (1 \otimes S(a_{(2)})a_{(3)})}$$

in $B \odot (.B \otimes .B)/(B \odot (.B \otimes .B))B^+$. By applying $\hat{\beta}$, the latter is an equality in $B \odot (.B \otimes B)/(B \odot (.B \otimes B))B^+$ as well, but now we may apply $\varepsilon \otimes \varepsilon \otimes \text{id}$ to it (it is well-defined) to conclude that $\varepsilon(a) = S(a_{(1)})a_{(2)}$. A contradiction.

Modules over one-sided Hopf algebras also offer an example of a monoidal and left skew-closed category whose structures are not compatible, that is, which does not form a left skew-closed skew-monoidal category in the sense of Definition 2.14. Indeed, if B is as above, then $({}_B\mathcal{M}, \otimes, k)$ is a monoidal category and $({}_B\mathcal{M}, {}_k\mathcal{M}(-, -), k)$ is a left skew-closed category, but the functor $- \otimes M$ is not left adjoint to the functor ${}_k\mathcal{M}(M, -)$: while the unit

$$\text{coev}_N^M: N \rightarrow {}_k\mathcal{M}(M, N \otimes M), \quad n \mapsto (m \mapsto n \otimes m)$$

is left B -linear, the unit

$$\text{ev}_N^M: {}_k\mathcal{M}(M, N) \otimes M \rightarrow N, \quad f \otimes m \mapsto f(m)$$

is not, unless S is a two-sided antipode.

5. WHEN ARE GABI-ALGEBRAS HOPF ALGEBRAS?

We saw (cf. Example 4.3) that Hopf algebras are examples of gabi-algebras, but also that not any gabi-algebra is Hopf (see Example 4.4). This section is devoted to answer the question when the gabi and Hopf notions coincide.

5.1. Finite-dimensional double gabi-algebras are Hopf. We first establish an easy criterion for tricocycloids to give Hopf algebras.

Lemma 5.1. *Let (A, v) be a lax tricocycloid in a symmetric monoidal category.*

- (1) *If v is invertible, then (A, v^{-1}) is a lax tricocycloid as well.*
- (2) *Assume that there are morphisms $\eta: \mathbf{1} \rightarrow A$ and $\varepsilon: A \rightarrow \mathbf{1}$ such that*

$$\varepsilon \otimes A = (A \otimes \varepsilon) \circ v, \quad \eta \otimes A = v \circ (A \otimes \eta), \quad \varepsilon \circ \eta = 1.$$

If v is invertible and $(A \otimes \varepsilon) \circ v^{-1} \circ (A \otimes \eta) = A$ holds, then $(A, v^{-1}, \eta, \varepsilon)$ is an augmented tricocycloid, and thus equips A with the structure of a Hopf algebra.

Proof. (1) follows directly from a simple computation, using that the braiding is its own inverse. (2) is easy as well: if v is invertible and satisfies the given equations, then v^{-1} satisfies

$$(\varepsilon \otimes A) \circ v^{-1} = A \otimes \varepsilon, \quad v^{-1} \circ (\eta \otimes A) = A \otimes \eta .$$

Together with $\varepsilon \circ \eta = 1$ and the assumption $(A \otimes \varepsilon) \circ v^{-1} \circ (A \otimes \eta) = A$, these are exactly the defining equations (4.2) making the tricocycloid (A, v^{-1}) augmented. \square

Example 5.2. Continuing Example 4.8, $(A, v', 1, \varepsilon)$ satisfies the assumptions of the second point of Lemma 5.1. \triangle

Define now for a gabi-algebra A the *canonical map*

$$\beta: A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto a_+ \otimes a_- b . \quad (5.1)$$

If β is invertible, we set $\Delta(a) = \beta^{-1}(a \otimes 1)$. We can now show that gabi-algebras with invertible canonical map are quite close to Hopf algebras. This will also be helpful in proving Proposition 5.11.

Theorem 5.3. *Let A be a gabi-algebra. If β is invertible and Δ is left counital, i.e. $(\varepsilon \otimes A) \circ \Delta = A$, then (A, Δ, ε) is a coalgebra. In fact, A is a Hopf algebra whose antipode is the antipode σ of A .*

Proof. Let \mathbf{tw} be the flip map in ${}_k\mathcal{M}$, i.e. the canonical symmetric braiding. By Proposition 4.9, $(A, v = \beta \circ \mathbf{tw})$ is a lax tricocycloid. Clearly, v is invertible if and only if β is. We also have $\varepsilon \circ \eta = 1$, and one computes

$$((A \otimes \varepsilon) \circ v)(a \otimes b) = b_+ \varepsilon(b_- a) \stackrel{\text{(GA1)}}{=} (\varepsilon \otimes A)(a \otimes b) ,$$

and

$$(v \circ (A \otimes \eta))(a) = 1_+ \otimes 1_- a = 1 \otimes 1a = (\eta \otimes A)(a) .$$

Finally, we have

$$(A \otimes \varepsilon) \circ v^{-1} \circ (A \otimes \eta) = A \iff (\varepsilon \otimes A) \circ \beta^{-1} \circ (A \otimes \eta) = A ,$$

and thus all the assumptions for the second point of Lemma 5.1 are satisfied. This shows that we indeed get a Hopf algebra. By the general theory of augmented lax tricocycloids with invertible structure map, the antipode is given by

$$\begin{aligned} S(a) &= ((\varepsilon \otimes A) \circ v \circ (\eta \otimes A))(a) = ((\varepsilon \otimes A) \circ \beta \circ (A \otimes \eta))(a) \\ &= ((\varepsilon \otimes A) \circ \beta)(a \otimes 1) = (\varepsilon \otimes A)(a_+ \otimes a_- 1) = \varepsilon(a_+) a_- = \sigma(a) , \end{aligned}$$

as claimed. \square

The previous result allows us to reduce a first class of gabi-algebras to Hopf algebras.

Corollary 5.4. *Let k be a field and A be a finite-dimensional gabi-algebra over k with invertible antipode σ . Define $\Delta(a) = a_+ \otimes \sigma^{-1}(a_-)$. Then $(A, m, 1, \Delta, \varepsilon, \sigma)$ is a Hopf algebra.*

Proof. In A , we have the identity

$$a \otimes 1 \stackrel{\text{(GA3)}}{=} a_{++} \otimes \varepsilon(a_{-+})a_{--}a_{+-} = a_{++} \otimes \sigma(a_{-})a_{+-} ,$$

which upon applying $1 \otimes \sigma^{-1}$ becomes $a \otimes 1 = a_{++} \otimes \sigma^{-1}(a_{+-})a_{-}$. Thus the composition

$$a \otimes b \xrightarrow{\beta} a_{+} \otimes a_{-}b \mapsto a_{++} \otimes \sigma^{-1}(a_{+-})a_{-}b$$

is the identity, i.e. the canonical map β has a left inverse. By finite-dimensionality of A , β is invertible. As in the paragraph preceding Theorem 5.3, we set $\Delta(a) = \beta^{-1}(a \otimes 1)$, i.e. $\Delta(a) = a_{+} \otimes \sigma^{-1}(a_{-})$. If Δ is left counital, then by Theorem 5.3 we are done. So we compute

$$((\varepsilon \otimes A) \circ \Delta)(a) = \varepsilon(a_{+})\sigma^{-1}(a_{-}) = \sigma^{-1}(\sigma(a)) = a ,$$

exactly as needed. \square

Our next aim is to state a sufficient condition for the antipode of a gabi-algebra to be invertible. First, note that for any *closed monoidal* category \mathcal{C} with left and right internal homs $[-, -]^l$ and $[-, -]^r$, respectively, one has natural isomorphisms

$$\mathcal{C}(X, [Y, Z]^r) \cong \mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, [X, Z]^l) .$$

One can now generalize this to the closed (non-monoidal) setting by simply demanding two closed structures with the same unit on the category such that an isomorphism between the two outer hom-sets hold. Let us apply this to the gabi-algebra setting.

Lemma 5.5. *Let (A, ε) be an augmented algebra endowed with two gabi-algebra structures denoted by $\delta(a) = a_{+} \otimes a_{-}$ and $\delta'(a) = a_{+'} \otimes a_{-}'$. Denote the associated closed structures on ${}_A\mathcal{M}$ by $[-, -]$ and $[-, -]'$ respectively. Then, the following are equivalent:*

(1) *The k -linear isomorphism*

$$\phi: {}_k\mathcal{M}(X, {}_k\mathcal{M}(Y, Z)) \rightarrow {}_k\mathcal{M}(Y, {}_k\mathcal{M}(X, Z)), \quad \phi(f)(y)(x) = f(x)(y)$$

whose inverse is given by

$$\phi^{-1}: {}_k\mathcal{M}(Y, {}_k\mathcal{M}(X, Z)) \rightarrow {}_k\mathcal{M}(X, {}_k\mathcal{M}(Y, Z)), \quad \phi^{-1}(g)(x)(y) = g(y)(x)$$

restricts and corestricts to an isomorphism

$${}_A\mathcal{M}(X, [Y, Z]) \cong {}_A\mathcal{M}(Y, [X, Z]')$$

for all X, Y, Z in ${}_A\mathcal{M}$.

(2) *$a_{+'}a_{-}' \otimes a_{-}' = 1 \otimes a$ and $a_{+}a_{-+'} \otimes a_{-}' = 1 \otimes a$ hold for all $a \in A$.*

If one of these equivalent conditions holds, then we call A a double gabi-algebra.

Proof. We will first show that for all f in ${}_A\mathcal{M}(X, [Y, Z])$, $\phi(f)$ is A -linear if and only if $a_{+'}a_{-}' \otimes a_{-}' = 1 \otimes a$ for all $a \in A$. Now, observe that the left A -linearity of $\phi(f)$ means that

$$\begin{aligned} (a \cdot \phi(f)(y))(x) &= a_{+'}\phi(f)(y)(a_{-}'x) = a_{+'}f(a_{-}'x)(y) = a_{+'}(a_{-}' \cdot f(x))(y) \\ &= a_{+'}a_{-}'f(x)(a_{-}'y) \end{aligned}$$

equals

$$\phi(f)(ay)(x) = f(x)(ay)$$

for all $x \in X$, $y \in Y$, $a \in A$. Clearly, if $a_{+'}a_{-'} \otimes a_{--} = 1 \otimes a$ holds for all $a \in A$, then this is the case. Conversely, assume that (1) holds. Let $X = Y = {}_A A$ and $Z = {}_A A \otimes A$ and consider $f \in {}_A \mathcal{M}(A, [A, A \otimes A])$ uniquely determined by $f(1)(a) := 1 \otimes a$ for all $a \in A$. Then we have

$$\begin{aligned} 1 \otimes a &= f(1)(a) = \phi(f)(a)(1) = (a \cdot \phi(f)(1))(1) = a_{+'} \phi(f)(1)(a_{-'}) = a_{+'} f(a_{-'})(1) \\ &= a_{+'} (a_{-'} \cdot f(1))(1) = a_{+'} a_{-'} f(1)(a_{--}) = a_{+'} a_{-'} \otimes a_{--} \end{aligned}$$

for all $a \in A$, as desired. A straightforward adaptation of this argument shows that ϕ^{-1} applied on ${}_A \mathcal{M}(Y, [X, Z]')$ takes values in ${}_A \mathcal{M}(X, [Y, Z])$ if and only if $a_{+}a_{-+} \otimes a_{--'} = 1 \otimes a$ hold for all $a \in A$. \square

Lemma 5.6. *Let A be a double gabi-algebra, then the antipode σ is invertible.*

Proof. In this case, one has

$$\sigma(\sigma'(a)) = \varepsilon(a_{+'}a_{-'} a_{--}) = a \quad \text{and} \quad \sigma'(\sigma(a)) = \varepsilon(a_{+}a_{-+} a_{--'}) = a,$$

so that the antipode is invertible. \square

Combining Lemma 5.6 with Corollary 5.4 we immediately obtain the following.

Corollary 5.7. *A finite-dimensional double gabi-algebra is a Hopf algebra.*

5.2. Commutative gabi-algebras are Hopf algebras. Throughout this section, A shall always be a gabi-algebra over the commutative ring k .

Proposition 5.8. *If the gabi-algebra A is commutative, then it is a Hopf algebra with comultiplication $\Delta(a) := a_{(1)} \otimes a_{(2)} := a_{+} \otimes \sigma(a_{-})$ and antipode σ .*

Proof. Since A is commutative, on the one hand we have that

$$a_{++} \otimes a_{+-} \sigma(a_{-}) = a_{++} \otimes \varepsilon(a_{-+}) a_{+-} a_{--} = a_{++} \varepsilon(a_{-+}) \otimes a_{--} a_{+-} \stackrel{\text{(GA3)}}{=} a \otimes 1,$$

and on the other the computation

$$\sigma^2(a) = \varepsilon(a_{+}) \sigma(a_{-}) \stackrel{\text{(GA2)}}{=} a_{++} a_{+-} \varepsilon(a_{-+}) a_{--} = a_{++} \varepsilon(a_{-+}) a_{--} a_{+-} \stackrel{\text{(GA3)}}{=} a$$

entails that also

$$a_{++} \otimes \sigma(a_{+-}) a_{-} = a_{++} \otimes \sigma(\sigma(a_{-}) a_{+-}) = a \otimes 1$$

for all $a \in A$, whence the canonical map β from (5.1) is invertible with inverse $\beta^{-1}(a \otimes b) := a_{+} \otimes \sigma(a_{-}) b$. If we set $\Delta(a) = a_{+} \otimes \sigma(a_{-})$, then it is left counital because

$$(\varepsilon \otimes A)(\Delta(a)) = \varepsilon(a_{+}) \sigma(a_{-}) = \sigma^2(a) = a$$

and so the statement follows from Theorem 5.3. \square

Corollary 5.9. *A commutative algebra lifts the closed structure of ${}_k \mathcal{M}$ if and only if it lifts the closed monoidal structure.*

5.3. Gabi-algebras and normality. We are now in the position of establishing which additional properties on A correspond to associative and left normality of the skew-closed structure, as promised in § 4.1.

Proposition 5.10. *Let A be a gabi-algebra over k . Then the skew-closed structure on ${}_A\mathcal{M}$ is associative normal if and only if the canonical map β from (5.1) is invertible.*

Proof. Recall from Theorem 2.12 that the closed structure on ${}_A\mathcal{M}$ is associative normal if and only if every component of the associative constraint from (4.4), that is

$$\begin{aligned} \alpha_{L,M,N}: (L \boxtimes M) \boxtimes N &\rightarrow L \boxtimes (M \boxtimes N), \\ (a \odot n) \otimes_A ((b \odot m) \otimes_A l) &\mapsto (ab_+ \odot ((1_A \odot b_- n) \otimes_A m)) \otimes_A l, \end{aligned}$$

is an isomorphism. Therefore, we are going to show that $\alpha_{L,M,N}$ is an isomorphism for every L, M, N in ${}_A\mathcal{M}$ if and only if β admits an inverse, that we write $\beta^{-1}(a \otimes b) = a_{(1)} \otimes a_{(2)} b$ for all $a, b \in A$ by a slight abuse of notation that will be soon justified (see Proposition 5.11). In such a case,

$$\begin{aligned} a_{(1)_+} \otimes a_{(1)_-} a_{(2)} &= a \otimes 1 = a_{+(1)} \otimes a_{+(2)} a_- \\ \text{and } a_+ \otimes a_{-(1)} \otimes a_{-(2)} &= a_{++} \otimes a_- \otimes a_{+-} \end{aligned} \tag{5.2}$$

for all $a \in A$ (the latter follows by applying $\text{id} \otimes \beta$ to both sides and comparing the results). Observe that since $\beta(a \otimes 1) = \delta(a) = a_+ \otimes a_-$ is unital and multiplicative, we have that

$$\begin{aligned} \beta(a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}) &= (a_{(1)} b_{(1)})_+ \otimes (a_{(1)} b_{(1)})_- a_{(2)} b_{(2)} = a_{(1)_+} b_{(1)_+} \otimes b_{(1)_-} a_{(1)_-} a_{(2)} b_{(2)} \\ &= ab_{(1)_+} \otimes b_{(1)_-} b_{(2)} = ab \otimes 1 = \beta\beta^{-1}(ab \otimes 1), \end{aligned}$$

whence $A \rightarrow A \otimes A, a \mapsto \beta^{-1}(a \otimes 1)$ is a morphism of algebras.

To begin with, let us slightly simplify (4.4). Note that

$$(L \boxtimes M) \boxtimes N = (A \odot N) \otimes_A (L \boxtimes M) = (A \odot N) \otimes_A ((A \odot M) \otimes_A L) \cong ((A \odot N) \odot M) \otimes_A L$$

via $(a \odot n) \otimes_A ((b \odot m) \otimes_A l) \mapsto ((ab_+ \odot b_- n) \odot m) \otimes_A l$ in one direction, for all $a, b \in A, m \in M, n \in N, l \in L$, and via $((a \odot n) \odot m) \otimes_A l \mapsto (a \odot n) \otimes_A ((1_A \odot m) \otimes_A l)$ in the other. Up to the latter isomorphism,

$$\begin{aligned} \alpha_{L,M,N}: ((A \odot N) \odot M) \otimes_A L &\rightarrow (A \odot ((A \odot N) \otimes_A M)) \otimes_A L, \\ ((a \odot n) \odot m) \otimes_A l &\mapsto (a \odot ((1_A \odot n) \otimes_A m)) \otimes_A l, \end{aligned}$$

Suppose now that β is invertible and consider

$$\begin{aligned} (A \odot ((A \odot N) \otimes_A M)) \otimes_A L &\rightarrow ((A \odot N) \odot M) \otimes_A L, \\ (a \odot ((b \odot n) \otimes_A m)) \otimes_A l &\mapsto ((a \odot b_{(2)} n) \odot b_{(1)} m) \otimes_A l, \end{aligned}$$

which is well-defined because of (5.2). By a direct check, it turns out to be a two-sided inverse of $\alpha_{L,M,N}$, so one implication is proved.

For the other implication, suppose that α is a natural isomorphism and consider the isomorphism

$$(A \odot A) \odot A \cong \left((A \odot A) \odot A \right) \otimes_A A \xrightarrow{\alpha_{A,A,A}} \left(A \odot \left((A \odot A) \otimes_A A \right) \right) \otimes_A A \cong A \odot (A \otimes A),$$

$$(a \odot b) \odot c \mapsto a \odot (c_+ \otimes c_- b),$$

where $A \otimes A$ is a left A -module with regular left A -action on the left-most tensorand. If we look at it as a k -linear morphism

$$\alpha': (A \otimes A) \otimes A \rightarrow A \otimes (A \otimes A), \quad (a \otimes b) \otimes c \mapsto a \otimes (c_+ \otimes c_- b),$$

its inverse is uniquely determined by

$$(x' \otimes x'') \otimes x''' := \alpha'^{-1}(1 \otimes (x \otimes 1))$$

for all $x \in A$. The latter satisfies

$$1 \otimes x \otimes 1 = x' \otimes x'''_+ \otimes x'''_- x'' \quad \text{and} \quad 1 \otimes 1 \otimes y = y_+' \otimes y_+'' y_- \otimes y_+'''$$

for all $x, y \in A$. In particular,

$$x \otimes 1 = x'''_+ \otimes x'''_- \varepsilon(x') x''$$

entails that $a \otimes b \mapsto a''' \otimes \varepsilon(a') a'' b$ is a section of β , while

$$y \otimes 1 = y_+''' \otimes \varepsilon(y_+') y_+'' y_-$$

entails that it is a retraction of β as well, hence it is its inverse. \square

Proposition 5.11. *Let A be a gabi-algebra whose canonical morphism β is invertible. Then the closed structure on ${}_A \mathcal{M}$ is left normal, too, if and only if A is a Hopf algebra with comultiplication $\beta^{-1}(a \otimes 1)$, counit ε and antipode $\sigma(a) = \varepsilon(a_+) a_-$ for all $a \in A$.*

Proof. Clearly, if the gabi-algebra structure on A comes from a Hopf algebra structure, then the claim in the statement holds. Thus, let us focus on the other implication. Recall from Theorem 2.12 that the closed structure on ${}_A \mathcal{M}$ is left normal if and only if every component

$$\lambda_M: (A \odot M)/(A \odot M)A^+ \rightarrow M, \quad \overline{a \odot m} \mapsto am,$$

of the left unit constraint from (4.3) is an isomorphism. Since, by hypothesis, β is invertible with inverse $\beta^{-1}(a \otimes b) = a_{(1)} \otimes a_{(2)} b$ for all $a, b \in A$, we know that

$$\overline{a \odot m} = \overline{a_{(1)+} \odot a_{(1)-} a_{(2)} m} = \overline{1_A \odot \varepsilon(a_{(1)}) a_{(2)} m}$$

for all $a \in A, m \in M$. In particular, if λ_A is invertible, then

$$\overline{1_A \odot ab} = \overline{a \odot b} = \overline{1_A \odot \varepsilon(a_{(1)}) a_{(2)} b}$$

and so $ab = \varepsilon(a_{(1)}) a_{(2)} b$ for all $a, b \in A$. The conclusion now follows from Theorem 5.3. \square

We are now ready to formulate the main theorem of this paper, the proof of which now follows directly from the result above.

Theorem 5.12. *Let A be an algebra. Then there is a bijective correspondence between normal gabi-algebra structures on A and Hopf algebra structures on A .*

In other words, ${}_A\mathcal{M}$ is a normal closed category with strictly closed forgetful functor ${}_A\mathcal{M} \rightarrow {}_k\mathcal{M}$ if and only if ${}_A\mathcal{M}$ is closed monoidal and the forgetful functor ${}_A\mathcal{M} \rightarrow {}_k\mathcal{M}$ is strictly closed monoidal.

Remark 5.13. One could wonder under which conditions Theorem 5.12 can be extended to the more general setting of algebras in a general closed monoidal category (possibly different from ${}_k\mathcal{M}$). To this end, let us make the following observation, which allows us to recover the above result in such a more general setting. Consider two closed monoidal categories \mathcal{C} and \mathcal{D} , and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an (essentially) surjective-on-objects, strict closed functor. Then we have the following natural isomorphisms:

$$\begin{aligned} [F(X \otimes Y), FZ] &= F[X \otimes Y, Z] \cong F[X, [Y, Z]] \\ &= [FX, F[Y, Z]] = [FX, [FY, FZ]] \cong [FX \otimes FY, FZ], \end{aligned}$$

where the equalities follow from the fact that the functor F is strict closed, and the isomorphisms follow from the closed monoidal structures of \mathcal{C} and \mathcal{D} . Since F is surjective on objects and by invoking enriched Yoneda (see, e.g., [K, §1.9]), we obtain a natural isomorphism $F(X \otimes Y) \cong FX \otimes FY$, so F is also strong monoidal.

Then, consider a normal gabi-algebra A in a closed braided monoidal category \mathcal{D} , let \mathcal{C} be the category of A -modules and $F : {}_A\mathcal{M} \rightarrow \mathcal{D}$ the forgetful functor. We showed that modules are skew closed monoidal and normality of A means that ${}_A\mathcal{M}$ is closed monoidal (see Proposition 4.12). Moreover F is strict closed by construction and essentially surjective on objects since thanks to the augmentation map ε , we can endow any object of \mathcal{D} with a trivial A -module structure. Then by the above F is also strong monoidal and therefore usual Tannaka-Krein duality implies that A is a Hopf algebra in \mathcal{D} . ∇

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APPENDIX A. MIXED LIFTINGS

Consider categories \mathcal{D} , \mathcal{E} , and \mathcal{C} . On them, consider a comonad (V, Δ, ε) and monads (W, m^W, u^W) and (T, m^T, u^T) , respectively. Now let $F: \mathcal{D} \times \mathcal{E} \rightarrow \mathcal{C}$ be a functor. The question is: when does this lift to a functor $\mathcal{D}^V \times \mathcal{E}^W \rightarrow \mathcal{C}^T$?

To answer it, let us introduce the following notion.

Definition A.1. A natural transformation of the form $\nu_{X,Y}: TF(VX, Y) \rightarrow F(X, WY)$ is said to satisfy the *lifting property of F with respect to (V, W, T)* if the the following hold:

- *mixed (co)unitality:*

$$\begin{array}{ccc} F(VX, Y) & \xrightarrow{u_{F(VX, Y)}^T} & TF(VX, Y) \\ & \searrow F(\varepsilon_X, u_Y^W) & \downarrow \nu_{X, Y} \\ & & F(X, WY) \end{array}$$

and

- *mixed (co)multiplicativity:*

$$\begin{array}{ccc} T^2F(VX, Y) & \xrightarrow{m_{F(VX, Y)}^T} & TF(VX, Y) \\ T^2F(\Delta_X, Y) \downarrow & & \downarrow \nu_{X, Y} \\ T^2F(V^2X, Y) & \xrightarrow{T\nu_{VX, Y}} TF(VX, WY) \xrightarrow{\nu_{X, WY}} F(X, W^2Y) \xrightarrow{F(X, m_Y^W)} & F(X, WY) \end{array} .$$

Similar to the usual lifting theorem for monads, we can use these to classify liftings in our current situation.

Theorem A.2. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}, V, W, T$, and F be as above. The liftings of F are in one-to-one correspondence with natural transformations satisfying the lifting property of F w.r.t. (V, W, T) .*

More precisely, if ν satisfies the lifting property, and $((X, x), (Y, y)) \in \mathcal{D}^V \times \mathcal{E}^W$, then $F(X, Y)$ is equipped with the T -action

$$\triangleright_{x, y}^\nu = TF(X, Y) \xrightarrow{TF(x, Y)} TF(VX, Y) \xrightarrow{\nu_{X, Y}} F(X, WY) \xrightarrow{F(X, y)} F(X, Y) .$$

The action on morphisms is determined by F . Conversely, if the functor lifts, i.e. every pair of (co)actions x, y lifts to an action $\triangleright_{x, y}: TF(X, Y) \rightarrow F(X, Y)$, then the natural transformation $\nu_{X, Y}^\triangleright$ satisfying the lifting property is obtained as the composition

$$TF(VX, Y) \xrightarrow{TF(VX, u_Y^W)} TF(VX, WY) \xrightarrow{\triangleright_{\Delta_X, m_Y^W}} F(VX, WY) \xrightarrow{F(\varepsilon_X, WY)} F(X, WY) .$$

Before proving Theorem A.2, let us record the following useful lemma.

Lemma A.3. *Let everything be as in Theorem A.2, and assume that F lifts. Also, let (X, x) and (Y, y) be a V -coalgebra and a W -algebra, respectively. Then*

- (1) $F(\varepsilon_X, Y) \in \mathcal{C}^T \left((F(VX, Y), \triangleright_{\Delta_X, y}), (F(X, Y), \triangleright_{x, y}) \right)$
- (2) $F(X, u_Y^W) \in \mathcal{C}^T \left((F(X, Y), \triangleright_{x, y}), (F(X, WY), \triangleright_{x, m_Y^W}) \right)$

(3) *The natural transformation built from the lifting satisfies*

$$\nu_{X,Y}^\triangleright = F(\varepsilon_X, u_Y^W) \circ \triangleright_{\Delta_X, y} = F(X, u_Y^W) \circ \triangleright_{x, y} \circ TF(\varepsilon^X, Y) = \triangleright_{x, m_Y^W} \circ TF(\varepsilon_X, u_Y^W)$$

Proof. (1) follows immediately, since F lifts by assumption and $\varepsilon_X: (VX, \Delta_X) \rightarrow (X, x)$ as V -comodules. (2) is completely analogous. Finally, (3) follows from appropriate applications of (1) and (2). \square

Proof of Theorem A.2. We first establish the bijection. Let ν be given. Then

$$\begin{aligned} \nu_{X,Y}^\triangleright &= F(\varepsilon_X, WY) \circ \triangleright_{\Delta_X, m_Y^W}^\nu \circ TF(VX, u_Y^W) \\ &= F(\varepsilon_X, WY) \circ F(VX, m_Y^W) \circ \nu_{VX, WY} \circ TF(\Delta_X, WY) \circ TF(VX, u_Y^W) \\ &= F(\varepsilon_X, WY) \circ F(VX, m_Y^W) \circ \nu_{VX, WY} \circ TF(V^2X, u_Y^W) \circ TF(\Delta_X, Y) \\ &= F(\varepsilon_X, WY) \circ F(VX, m_Y^W) \circ TF(VX, W u_Y^W) \circ \nu_{VX, Y} \circ TF(\Delta_X, Y) \\ &= F(\varepsilon_X, WY) \circ \nu_{VX, Y} \circ TF(\Delta_X, Y) \\ &= \nu_{X, Y} \circ TF(V\varepsilon_X, Y) \circ TF(\Delta_X, Y) \\ &= \nu_{X, Y} , \end{aligned}$$

where we have used nothing but functoriality and the (co)unitality axioms.

On the other hand, given \triangleright , we have

$$\begin{aligned} \triangleright_{x, y}^\nu &= F(X, y) \circ \nu_{X, Y}^\triangleright \circ TF(x, Y) \\ &= F(X, y) \circ F(\varepsilon_X, WY) \circ \triangleright_{\Delta_X, m_Y^W} \circ TF(VX, u_Y^W) \circ TF(x, Y) \\ &\stackrel{(i)}{=} F(X, y) \circ F(\varepsilon_X, WY) \circ F(VX, u_Y^W) \circ \triangleright_{\Delta_X, y} \circ TF(x, Y) \\ &= F(\varepsilon_X, Y) \circ \triangleright_{\Delta_X, y} \circ TF(x, Y) \\ &\stackrel{(ii)}{=} \triangleright_{x, y} \circ TF(\varepsilon_X, Y) \circ TF(x, Y) \\ &= \triangleright_{x, y} . \end{aligned}$$

Both (i) and (ii) use Lemma A.3. The other steps are again naturality and the (co)unitality axioms.

Thus the bijection is established, and we are left with showing that everything is well-defined. Let first ν be given. We check unitality of \triangleright^ν :

$$\begin{aligned} \triangleright_{x, x}^\nu \circ u_{F(X, Y)}^T &= F(X, y) \circ \nu_{X, Y} \circ TF(x, Y) \circ u_{F(X, Y)}^T \\ &= F(X, y) \circ \nu_{X, Y} \circ u_{F(VX, Y)}^T \circ F(x, Y) \\ &\stackrel{(*)}{=} F(X, y) \circ F(\varepsilon_X, u_Y^W) \circ F(x, Y) \\ &= F(X, Y) , \end{aligned}$$

where the only non-trivial step (*) uses the mixed (co)unitality of ν . To see the associativity of Δ^ν , we compute

$$\begin{aligned} \triangleright_{x, y}^\nu \circ m_{F(X, Y)}^T &= F(X, y) \circ \nu_{X, Y} \circ TF(x, Y) \circ m_{F(X, Y)}^T \\ &= F(X, y) \circ \nu_{X, Y} \circ m_{F(VX, Y)}^T \circ T^2F(x, Y) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(i)}{=} F(X, y) \circ F(X, m_Y^W) \circ \nu_{X, WY} \circ T\nu_{VX, Y} \circ T^2F(\Delta_X, Y) \circ T^2F(x, Y) \\
&\stackrel{(ii)}{=} F(X, y) \circ F(X, Wy) \circ \nu_{X, WY} \circ T\nu_{VX, Y} \circ T^2F(Vx, Y) \circ T^2F(x, Y) \\
&\stackrel{(iii)}{=} F(X, y) \circ F(X, Wy) \circ \nu_{X, WY} \circ TF(x, WY) \circ T\nu_{X, Y} \circ T^2F(x, Y) \\
&\stackrel{(iv)}{=} F(X, y) \circ \nu_{X, Y} \circ TF(VX, y) \circ TF(x, WY) \circ T\nu_{X, Y} \circ T^2F(x, Y) \\
&= F(X, y) \circ \nu_{X, Y} \circ TF(x, Y) \circ T(F(X, y) \circ \nu_{X, Y} \circ TF(x, Y)) \\
&= \triangleright_{x, y}^\nu \circ T\triangleright_{x, y}^\nu,
\end{aligned}$$

as desired. Here, (i) uses mixed (co)associativity, (ii) uses (co)associativity of (co)monad (co)actions, and (iii) and (iv) use naturality of ν .

Finally we need to check that for $(f, g): ((X, x), (Y, y)) \rightarrow ((X', x'), (Y', y'))$, the morphism $F(f, g)$ indeed intertwines the T -actions. We have

$$\begin{aligned}
F(f, g) \circ \triangleright_{x, y}^\nu &= F(f, g) \circ F(X, y) \circ \nu_{X, Y} \circ TF(x, Y) \\
&= F(X', y') \circ F(f, Wg) \circ \nu_{X, Y} \circ TF(x, Y) \\
&= F(X', y') \circ \nu_{X', Y'} \circ TF(Xf, g) \circ TF(x, Y) \\
&= F(X', y') \circ \nu_{X', Y'} \circ TF(x', Y') \circ TF(f, g) \\
&= \triangleright_{x', y'}^\nu \circ TF(f, g),
\end{aligned}$$

exactly as needed.

So now assume that F lifts. In particular, the actions \triangleright are given. We check the mixed (co)unitality of ν^\triangleright :

$$\begin{aligned}
\nu_{X, Y}^\triangleright \circ u_{F(VX, Y)}^T &= F(\varepsilon_X, WY) \circ \triangleright_{\Delta_X, m_Y^W} \circ TF(VX, u_Y^W) \circ u_{F(VX, Y)}^T \\
&= F(\varepsilon_X, WY) \circ \triangleright_{\Delta_X, m_Y^W} \circ u_{F(VX, WY)}^T \circ F(VX, u_Y^W) \\
&= F(\varepsilon_X, u_Y^W),
\end{aligned}$$

using unitality of T -actions. For the mixed (co)multiplicativity, we compute

$$\begin{aligned}
&F(X, m_Y^W) \circ \nu_{X, WY}^\triangleright \circ T\nu_{VX, Y}^\triangleright \circ T^2F(\Delta_X, Y) \\
&\stackrel{(\star)}{=} F(X, m_Y^W) \circ F(X, u_{WY}^W) \circ \triangleright_{x, m_Y^W} \circ TF(\varepsilon_X, WY) \circ T\nu_{VX, Y}^\triangleright \circ T^2F(\Delta_X, Y) \\
&\stackrel{(\star)}{=} \triangleright_{x, m_Y^W} \circ TF(\varepsilon_X, WY) \circ TF(VX, u_Y^W) \circ T\triangleright_{\Delta_X, y} \circ T^2F(\varepsilon_{VX}, Y) \circ T^2F(\Delta_X, Y) \\
&= \triangleright_{x, m_Y^W} \circ TF(\varepsilon_X, WY) \circ TF(VX, u_Y^W) \circ T\triangleright_{\Delta_X, y} \\
&\stackrel{(\star)}{=} F(\varepsilon_X, u_Y^W) \circ \triangleright_{\Delta_X, y} \circ T\triangleright_{\Delta_X, y} \\
&= F(\varepsilon_X, u_Y^W) \circ \triangleright_{\Delta_X, y} \circ m_{F(VX, Y)}^T \\
&\stackrel{(\star)}{=} \nu_{X, Y}^\triangleright \circ m_{F(VX, Y)}^T.
\end{aligned}$$

Here, Lemma A.3 was used in all steps marked (\star) .

Finally, we show that ν^\triangleright is natural. Let $(f, g): (X, Y) \rightarrow (X', Y')$ in $\mathcal{D} \times \mathcal{E}$. Then

$$\nu_{X', Y'}^\triangleright \circ TF(Vf, g) = F(\varepsilon_{X'}, WY') \circ \triangleright_{\Delta_{X'}, m_{Y'}^W} \circ TF(VX', u_{Y'}^W) \circ TF(Vf, g)$$

$$\begin{aligned}
&= F(\varepsilon_{X'}, WY') \circ \triangleright_{\Delta_{X'}, m_{Y'}^W} \circ TF(Vf, Wg) \circ TF(VX, u_Y^W) \\
&\stackrel{(*)}{=} F(\varepsilon_{X'}, WY') \circ F(Vf, Wg) \circ \triangleright_{\Delta_X, m_Y^W} \circ TF(VX, u_Y^W) \\
&= F(f, Wg) \circ F(\varepsilon_X, WY) \circ \triangleright_{\Delta_X, m_Y^W} \circ TF(VX, u_Y^W) \\
&= F(f, Wg) \circ \nu_{X,Y}^\triangleright,
\end{aligned}$$

where in (*) we used that (Vf, Wg) is a morphism in $\mathcal{D}^V \times \mathcal{E}^W$ of (co)free (co)algebras, and thus $F(Vf, Wg)$ is a morphism in \mathcal{C}^T , intertwining the actions $\triangleright_{\Delta_X, m_Y^W}$ and $\triangleright_{\Delta_{X'}, m_{Y'}^W}$. This finishes the proof. \square

Since a monad on a category is the same as a comonad on the opposite category, we immediately get the following.

Corollary A.4. *Let (T, m, u) be a monad on a category \mathcal{C} . Then a functor $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ lifts to a functor $F^\#: (\mathcal{C}^T)^{\text{op}} \times \mathcal{C}^T \rightarrow \mathcal{C}^T$ if and only if there is a natural transformation $\nu_{X,Y}: TF(TX, Y) \rightarrow F(X, TY)$ satisfying*

$$\nu_{X,Y} \circ u_{F(TX,Y)} = F(u_X, u_Y)$$

and

$$\nu_{X,Y} \circ m_{F(TX,Y)} = F(X, m_Y) \circ \nu_{X, TY} \circ T\nu_{TX,Y} \circ T^2F(m_X, Y).$$

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