ADDENDA TO "CORRESPONDENCE THEOREMS FOR HOPF ALGEBROIDS WITH APPLICATIONS TO AFFINE GROUPOIDS"

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Abstract. The proof of Theorem 3.14 contains an unsubstantiated claim. To overcome this problem, we add an hypothesis to the statement of 3.14 and we provide a new valid proof. We adjust Theorem 3.15, Corollary 3.16, Proposition 4.23, Theorem 4.26, Corollary 4.29 and Corollary 4.32 accordingly.

Let $I \subseteq \mathcal{H}$ be a left ideal coideal in a left Hopf algebroid $(A, \mathcal{H})$ and let $B \coloneqq \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}}$. In the proof of [1, Theorem 3.14] there is claimed that *the additional condition on $\gamma(B)$ we proved above ensures that ξ of (2.14) is an isomorphism in view of Lemma 3.12. In particular, $\mathcal{K}$ of Theorem 2.11 is an equivalence of categories. Therefore, the morphism $\theta_{\overline{\mathcal{H}}}$ in (3.12) from the proof of Proposition 3.8 is an isomorphism*. This argument contains a gap: the condition on $\gamma(B)$ entails that $\mathcal{K}$ is an equivalence between ${}_B\mathsf{Mod}$ and ${}^{\mathcal{H}/\mathcal{H}B^+}_{\mathcal{H}}\mathrm{HopfMod}$, but $\theta_{\overline{\mathcal{H}}}$ is a component of the counit of the adjunction between ${}_B\mathsf{Mod}$ and ${}^{\mathcal{H}/I}_{\mathcal{H}}\mathrm{HopfMod}$. Thus, the argument provided is not sufficient to conclude that $\theta_{\overline{\mathcal{H}}}$ is an isomorphism.

The following enhanced statement shall replace [1, Theorem 3.14] and its unsubstantiated proof. For the unexplained notation we refer to [1].

Theorem 0.1. *Let $(A, \mathcal{H})$ be a left Hopf algebroid such that ${}_s\mathcal{H} = {}_{A\otimes 1^o}\mathcal{H}$ is A-flat. Let I be a left ideal coideal in $\mathcal{H}$ for which $\mathcal{H} \otimes_{A^o} \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}}$ injects into $\mathcal{H} \otimes_{A^o} \mathcal{H}$ and*

$$\mathcal{H} \,\square^{\frac{\mathcal{H}}{I}}\, \mathcal{H} \underset{\mathcal{H}\otimes_A\varepsilon}{\overset{\varepsilon\otimes_A\mathcal{H}}{\rightrightarrows}} \mathcal{H} \longrightarrow \mathcal{H}/I \tag{1}$$

is a coequalizer diagram, then $I = \mathcal{H}(\mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}})^+$, that is to say, $\Phi\Psi(I) = I$.

Proof. Denote by $\beta \colon \mathcal{H}\otimes_{A^o}\mathcal{H} \to \mathcal{H}\otimes_A\mathcal{H}$, $x\otimes_{A^o}y \mapsto \sum x_1\otimes_A x_2 y$, the canonical isomorphism. Under the additional hypotheses on I, the left-most vertical morphism ζ in the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{H} \otimes_{A^o} \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} & \longrightarrow & \mathcal{H} \otimes_{A^o} \mathcal{H} & \underset{\mathcal{H}\otimes_{A^o}(\pi_I\otimes_A\mathcal{H})\Delta}{\overset{\mathcal{H}\otimes_{A^o}\pi_I t\otimes_A\mathcal{H}}{\rightrightarrows}} & \mathcal{H} \otimes_{A^o} (\mathcal{H}/I \otimes_A \mathcal{H}) \\
\zeta\downarrow & & \beta\downarrow & & \downarrow \beta\otimes_{\mathcal{H}}(\mathcal{H}/I\otimes_A\mathcal{H}) \\
\mathcal{H} \,\square^{\frac{\mathcal{H}}{I}}\, \mathcal{H} & \longrightarrow & \mathcal{H} \otimes_A \mathcal{H} & \underset{\mathcal{H}\otimes_A(\pi_I\otimes_A\mathcal{H})\Delta}{\overset{(\mathcal{H}\otimes_A\pi_I)\Delta\otimes_A\mathcal{H}}{\rightrightarrows}} & \mathcal{H} \otimes_A (\mathcal{H}/I \otimes_A \mathcal{H})
\end{array}$$

is an isomorphism, because both lines are equalizers in the category of vector spaces and β is an isomorphism. If, in addition, (1) is a coequalizer, then $\mathcal{H}/I = \mathcal{H}/\mathcal{H}(\mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}})^+$, because both lines

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in the commutative diagram

$$\begin{array}{ccccc}
\mathcal{H} \otimes_{A^{\circ}} \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} & \underset{\mathcal{H}\otimes_{A^{\circ}}\varepsilon}{\overset{\mu}{\rightrightarrows}} & \mathcal{H} & \longrightarrow & \mathcal{H}/\mathcal{H}(\mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}})^{+} \\
{\scriptstyle \zeta}\downarrow & & \| & & \vdots \\
\mathcal{H} \,\square^{\frac{\mathcal{H}}{I}}\, \mathcal{H} & \underset{\mathcal{H}\otimes_{A}\varepsilon}{\overset{\varepsilon\otimes_{A}\mathcal{H}}{\rightrightarrows}} & \mathcal{H} & \longrightarrow & \mathcal{H}/I
\end{array}$$

are coequalizers and ζ is an isomorphism (see also [2, Proposition 3.2]). □

The coequalizer condition in Theorem 0.1 is a necessary condition whose role has been observed and studied in [2]. Nevertheless, since we are not able to determine whether it follows from the other hypotheses, we need to add it to the results relying on Theorem 0.1.

For simplicity's sake, suppose that $\mathcal{H}$ is a left Hopf algebroid over A such that ${}_{s}\mathcal{H} = {}_{A\otimes 1^{\circ}}\mathcal{H}$ and $\mathcal{H}_{t} = \mathcal{H}_{1\otimes A^{\circ}}$ are flat modules. Having in mind the commutative case, this is harmless. Then, [1, Theorem 3.15 and Corollary 3.16] shall be rephrased as follows.

Theorem 0.2. *We have a well-defined inclusion-preserving bijective correspondence*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{left ideal coideals } I \textit{ in } \mathcal{H} \textit{ such}\\ \textit{that } \mathcal{H}\otimes_{\mathcal{H}^{\mathrm{co}\mathcal{H}/I}} - \textit{ is comonadic}\\ \textit{and } (1) \textit{ is a coequalizer}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{right } \mathcal{H}\textit{-comodule } A^{\circ}\textit{-subrings } B\\ \textit{of } \mathcal{H} \textit{ via } t \textit{ such that } \mathcal{H}\otimes_{B} - \textit{ is}\\ \textit{comonadic and } \gamma(B)\subseteq B\otimes_{A^{\circ}}\mathcal{H}\end{array}\right\} \\
I & \longmapsto & \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} \\
\mathcal{H}B^{+} & \longleftarrow\!\!\shortmid & B
\end{array} \tag{2}$$

Corollary 0.3. (2) *restricts to a well-defined inclusion-preserving bijective correspondence*

$$\left\{\begin{array}{c}\textit{left ideal coideals } I \textit{ in } \mathcal{H} \textit{ such}\\ \textit{that } \mathcal{H} \textit{ is right faithfully flat over}\\ \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} \textit{ and } (1) \textit{ is a coequalizer}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{right } \mathcal{H}\textit{-comodule } A^{\circ}\textit{-subrings } B\\ \textit{of } \mathcal{H} \textit{ via } t \textit{ such that } \mathcal{H}_{B} \textit{ is faithfully}\\ \textit{flat and } \gamma(B)\subseteq B\otimes_{A^{\circ}}\mathcal{H}\end{array}\right\}$$

Now, by moving to the commutative case, adding the coequalizer condition (1) to [1, Proposition 4.23, Theorem 4.26, Corollary 4.29 and Corollary 4.32] results in the following statements. The proofs can be taken verbatim from [1].

Proposition 0.4 ([1, Proposition 4.23])**.** *If $(A, \mathcal{H})$ is a commutative Hopf algebroid and ${}_{s}\mathcal{H}$ is A-flat, then we have a well-defined inclusion-preserving bijective correspondence*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{bi-ideals } I \textit{ in } \mathcal{H} \textit{ such that } \mathcal{H} \textit{ is pure}\\ \textit{over } \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} \textit{ and } (1) \textit{ is a coequalizer}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{right } \mathcal{H}\textit{-comodule } A\textit{-subalgebras}\\ B\subseteq\mathcal{H} \textit{ via } t \textit{ such that } \mathcal{H} \textit{ is pure over } B\end{array}\right\} \\
I & \longmapsto & \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} \\
\mathcal{H}B^{+} & \longleftarrow\!\!\shortmid & B
\end{array}$$

Theorem 0.5 ([1, Theorem 4.26]). *If $(A, \mathcal{H})$ is a commutative Hopf algebroid and ${}_s\mathcal{H}$ is A-flat, then we have a well-defined inclusion-preserving bijective correspondence*

$$\left\{\begin{array}{c}\text{normal Hopf ideals } I \text{ in } \mathcal{H} \text{ such that } \mathcal{H} \\ \text{is pure over } \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}} \text{ and (1) is a coequalizer}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{sub-Hopf algebroids } \mathcal{K} \subseteq \mathcal{H} \text{ such} \\ \text{that } \mathcal{H} \text{ is pure over } \mathcal{K}\end{array}\right\}$$

$$I \longmapsto \mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}}$$

$$\mathcal{H}\mathcal{K}^{+} \longleftarrow \mathcal{K}$$

Corollary 0.6 ([1, Corollary 4.29]). *Let $(A, \mathcal{H})$ be a commutative Hopf algebroid such that ${}_s\mathcal{H}$ is A-flat and let $I \subseteq \mathcal{H}$ be a normal Hopf ideal such that $\mathcal{H}$ is pure over $\mathcal{H}^I$ and* (1) *is a coequalizer. Denote by $\phi\colon \mathcal{H}^I \to \mathcal{H}$ the inclusion. Then the canonical morphism*

$$\Psi_R\colon \frac{\mathscr{G}_{\mathcal{H}}(R)}{\mathscr{G}_{\mathcal{H}/I}(R)} \to \mathrm{CAlg}_{\Bbbk}(\mathcal{H}^I, R), \qquad \mathscr{G}_{\mathcal{H}/I}(R) \bullet g \mapsto \Phi_R(g),$$

of [1, Lemma 4.22] is injective for every $R \in \mathrm{CAlg}_{\Bbbk}$. That is to say, the kernel of the morphism $\Phi_R\colon \mathrm{CAlg}_{\Bbbk}(\mathcal{H}, R) \to \mathrm{CAlg}_{\Bbbk}(\mathcal{H}^I, R)$ induced by ϕ is exactly $\mathrm{CAlg}_{\Bbbk}(\mathcal{H}/I, R)$.

Corollary 0.7 ([1, Corollary 4.32]). *Suppose that $\Bbbk$ is an algebraically closed field. Let I be a normal Hopf ideal of the commutative Hopf algebroid $(A, \mathcal{H})$ such that ${}_s\mathcal{H}$ is A-flat and $\mathcal{H}$ is pure over $\mathcal{H}^{\mathrm{co}\frac{\mathcal{H}}{I}}$ and* (1) *is a coequalizer. Then the $\Bbbk$-component $\Psi_{\Bbbk}$ of the canonical morphism from [1, Lemma 4.22] is an isomorphism. That is to say, $\mathscr{G}_{\mathcal{H}}(\Bbbk)/\mathscr{G}_{\mathcal{H}/I}(\Bbbk) \;\cong\; \mathscr{G}_{\mathcal{H}^I}(\Bbbk)$.*

It is noteworthy that [1, Example 4.27] is still providing a valid description of the bijective correspondence in the finite groupoid case: any normal Hopf ideal in the Hopf algebroid of functions on a finite groupoid satisfies the coequalizer condition (1), roughly because the corresponding subgroupoid $\mathcal{G}_1 \setminus S_1$ is the equalizer of the arrows corresponding to $\varepsilon \otimes_{\Bbbk(\mathcal{G}_0)} \Bbbk(\mathcal{G}_1), \Bbbk(\mathcal{G}_1) \otimes_{\Bbbk(\mathcal{G}_0)} \varepsilon\colon \Bbbk(\mathcal{G}_1)\square^{\Bbbk(\mathcal{G}_1\setminus S_1)}\Bbbk(\mathcal{G}_1) \to \Bbbk(\mathcal{G}_1)$.

References

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