From groups to groupoids

A path to internal symmetries

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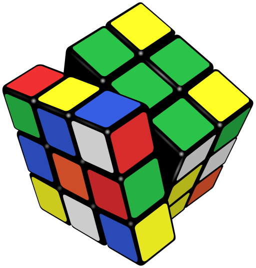


Rubik's cube[1]

Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to MacDonald's proudly announcing that they've sold more than 120 hamburgers.

– J. A. Paulos, Innumeracy

In fact, there are 43.252.003.274.489.856.000 possible patterns.

[1]By *This image was created by me, Booyabazooka*, CC BY-SA 3.0, commons.wikimedia.org/w/index.php?curid=4771790

Definition

A group is a set G (of transformations) together with a composition law

$$\circ\ :\ G \times G \to G, \qquad (g,h) \mapsto g \circ h \qquad\qquad \text{satisfying}$$

associativity : $(g \circ h) \circ k = g \circ (h \circ k)$ for all $g, h, k \in G$

unitality : $\exists\, e \in G$ such that $e \circ g = g = g \circ e$ for all $g \in G$

inverse : $\forall\, g \in G,\ \ \exists\, g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$

Definition

An action of a group G on a set Y is a function

$$G \times Y \to Y, \qquad (g,y) \mapsto g(y) \qquad\qquad \text{which is}$$

associative : $(g \circ h)(y) = g(h(y))$ for all $g, h \in G,\ y \in X$

unital : $e(y) = y$ for all $y \in Y$

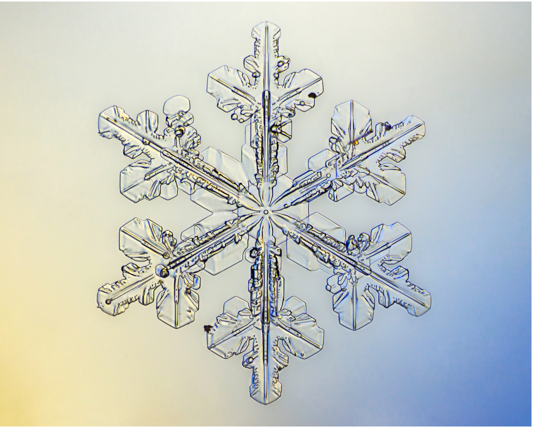
Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection (H. Weyl, 1952)

Vitruvius, 1st Century BC:

"Symmetry is proportioned correspondence of the elements of the work itself, a response, in any given part, of the separate parts to the appearance of the entire figure as a whole.

Hermann Weyl, 1952:

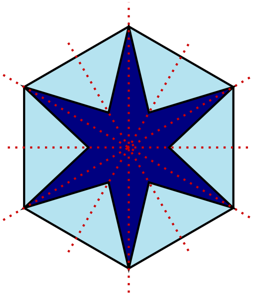
"Given a spatial configuration $\mathscr{F}$, those automorphisms of space which leave $\mathscr{F}$ unchanged form a group, and this group describes exactly the symmetry possessed by $\mathscr{F}$".

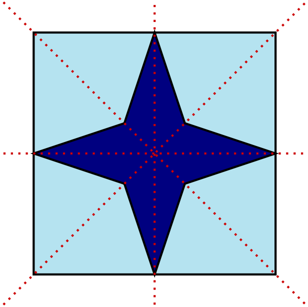


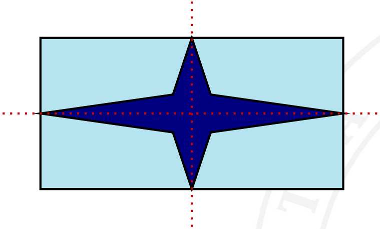
Snowflake[2]

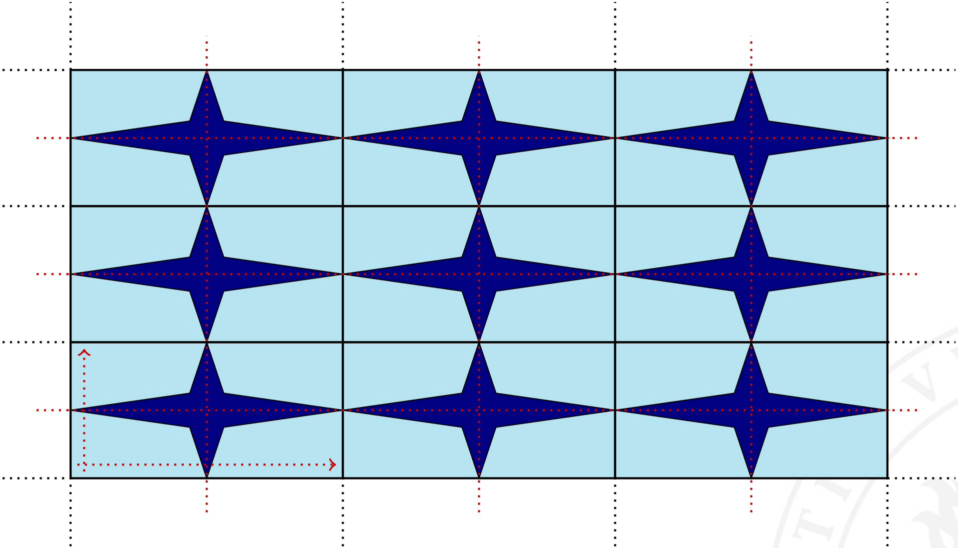
[2] By *Janek Lass - Own work*, CC BY 4.0, commons.wikimedia.org/w/index.php?curid=145499964

The classical examples. . .

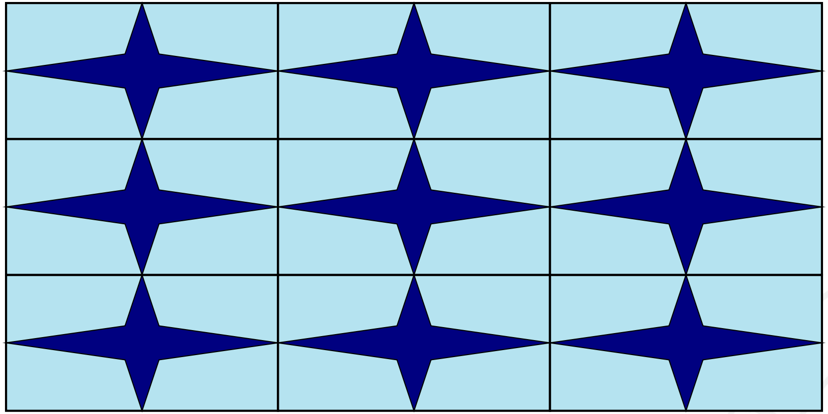








…the tiling is limited in space ? For instance, a real floor…



Definition (Exel, 1994)

A partial action of a group G on a set X is the datum of

- a collection $\{X_g \mid g \in G\}$ of subsets of X (the domains)
- a collection $\{\alpha_g : X_g \to X_{g^{-1}} \mid g \in G\}$ of bijections

such that

(1) $X_e = X$ and α_e is the identity morphism

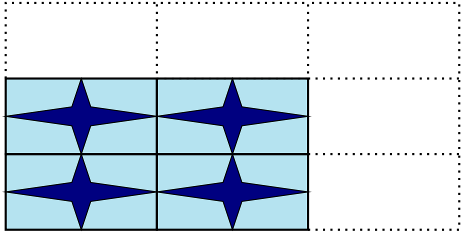
(2) $\alpha_g^{-1}(X_h \cap X_{g^{-1}}) = X_g \cap X_{hg}$ (compatibility of domains)

(3) $\alpha_h(\alpha_g(x)) = \alpha_{hg}(x)$ for each $x \in \alpha_g^{-1}(X_h \cap X_{g^{-1}})$

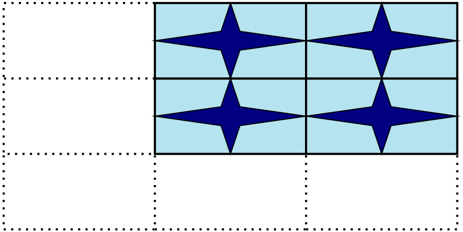
Example

The group Γ of symmetries of the planar tiling acts partially on the finite tiling X. For example, if $\tau \in \Gamma$ is the translation by the vector $(1,2)$, then

$X_\tau =$

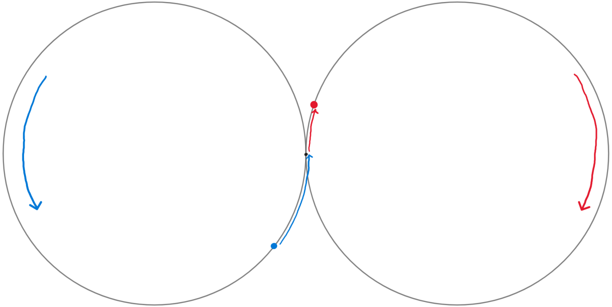


$X_{\tau^{-1}} =$



Example

$G = S^1 \times S^1$ acts partially on



15 puzzle[3]

[3] By *Arcinides - Own work*, CC BY-SA 4.0, commons.wikimedia.org/w/index.php?curid=140073742

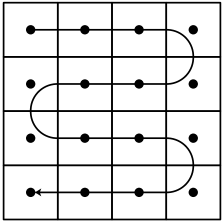
1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Loyd's 15 puzzle

Question : Is there a sequence of legal moves that allows to rearrange Loyd's puzzle into the "natural arrangment" ?

Price : $\sim$ \$34.000

(1) There are 16 possible locations for the empty square $\rightsquigarrow$ 16 possible "states" $s_1, \ldots, s_{16}$ according to



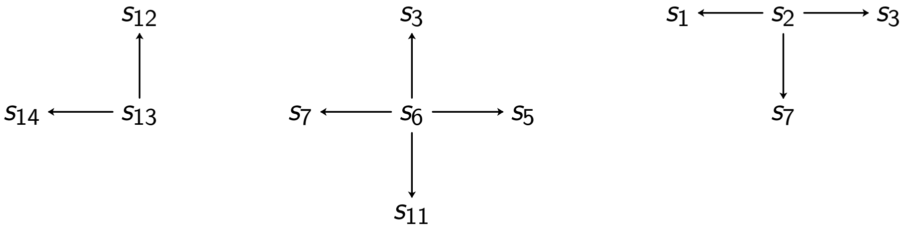
E.g. $s_7 =$

*	*	*	*
*		*	*
*	*	*	*
*	*	*	*

$s_{16} =$

*	*	*	*
*	*	*	*
*	*	*	*
	*	*	*

(2) A legal move allows us to pass from one state to another, e.g.



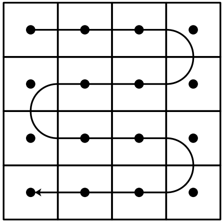
However :

- There are many ways to pass from one state to another,

$$\text{e.g.} \qquad s_1 \rightarrow s_8 \qquad \text{and} \qquad s_1 \rightarrow s_2 \rightarrow s_7 \rightarrow s_8$$

- A legal move also changes the "inner structure" of the puzzle

(3) Given a state, let us number the cases from 1 to 16 according to



again. Each legal move corresponds then to a permutation :

c_1	c_2	c_3	c_4
c_8	c_7	c_6	c_5
c_9	c_{10}	c_{11}	c_{12}
c_{16}	c_{15}	c_{14}	c_{13}

$\longrightarrow$

c_1	c_2	c_3	c_4
c_8	c_{10}	c_6	c_5
c_9	c_7	c_{11}	c_{12}
c_{16}	c_{15}	c_{14}	c_{13}

$\leftrightsquigarrow \quad (7, 10)$

- We can represent a sequence of legal moves as

$$s_i \xrightarrow{(j,\alpha,i)} s_j$$

where α is the corresponding permutation.

- $s_1 \xrightarrow{\left(8,\,(1{,}8),\,1\right)} s_8$ while

$$s_1 \xrightarrow{\left(2,\,(1{,}2),\,1\right)} s_2 \xrightarrow{\left(7,\,(2{,}7),\,2\right)} s_7 \xrightarrow{\left(8,\,(7{,}8),\,7\right)} s_8 = s_1 \xrightarrow{\left(8,\,(1{,}8{,}7{,}2),\,1\right)} s_8$$

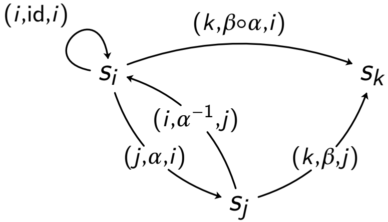
- $s_1 \xrightarrow{\left(1,\,\mathrm{id},\,1\right)} s_1$

$$s_1 \xrightarrow{\left(1,\,(2{,}8{,}7),\,1\right)} s_1$$

$$s_1 \xrightarrow{\left(1,\,(2{,}8{,}7{,}6{,}3),\,1\right)} s_1$$

We set

$$\mathscr{L}_4 := \left\{(j,\alpha,i) \mid i,j = 1,\ldots,16,\ \alpha \in \mathfrak{S}_{16} \text{ corr. to a sequence of legal moves}\right\}$$



Definition

A groupoid is a pair $(\mathscr{G}_0, \mathscr{G}_1)$ ("objects", "arrows") together with

$$\mathscr{G}_0 \overset{\tau}{\underset{\sigma}{\leftleftarrows}} \xrightarrow{1} \mathscr{G}_1 \circlearrowright^{i} \xleftarrow{\circ} \mathscr{G}_1 \times_{\mathscr{G}_0} \mathscr{G}_1 = \{(g,f) \mid \sigma(g) = \tau(f)\}$$

such that

$$h\circ(g\circ f) = (h\circ g)\circ f, \qquad 1_{\tau(f)}\circ f = f = f\circ 1_{\sigma(f)}, \qquad \tau(g\circ f) = \tau(g),$$

$$\sigma(g\circ f) = \sigma(f), \qquad \sigma(1_p) = p = \tau(1_p), \qquad i^2 = \mathrm{id}$$

$$\sigma(f^{-1}) = \tau(f), \qquad \tau(f^{-1}) = \sigma(f), \qquad f\circ f^{-1} = 1_{\tau(f)}, \qquad f^{-1}\circ f = 1_{\sigma(f)}$$

for all $f, g, h \in \mathscr{G}_1$ suitably compatible and for all $p \in \mathscr{G}_0$.

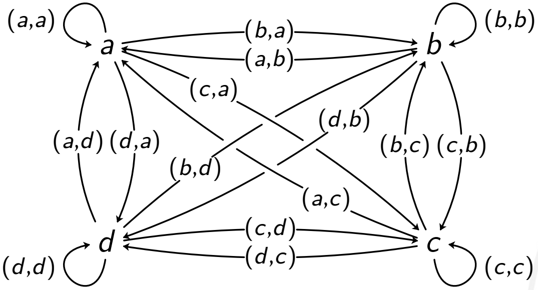
Example

(a) $\mathscr{L}_4$ is a groupoid.

(b) For any set X, $(X, X \times X)$ is a groupoid, called the groupoid of pairs, with

$$1_x = (x,x), \qquad \sigma(y,x) = x, \qquad \tau(y,x) = y,$$
$$(z,w)\circ(y,x) = (z,x) \Leftrightarrow w = y, \qquad (y,x)^{-1} = (x,y).$$

E.g.



Definition

Let $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ be a groupoid and let X be a set. An action of $\mathscr{G}$ on X is the datum of a function $\mu\colon X \to \mathscr{G}_0$, called the moment map, and of a function

$$\big\{(f,x) \in \mathscr{G}_1 \times X \mid \sigma(f) = \mu(x)\big\} \;=\; \mathscr{G}_1 \times_{\mathscr{G}_0} X \;\to\; X, \qquad (f,x) \mapsto f(x),$$

the action, such that

(1) $\mu(f(x)) \;=\; \tau(f)$

(2) $1_{\mu(x)}(x) \;=\; x$

(3) $g(f(x)) \;=\; (g \circ f)(x)$

Example

Describe a 15 puzzle as a pair (s, c) where $s \in \{s_1, \ldots, s_{16}\}$ is its state and $c \in \mathfrak{S}_{16}$ is the configuration of the tiles (including the empty case).

$$X := \Big\{(s, c) \mid (s, c) \text{ is a 15 puzzle}\Big\}.$$

$\mathscr{L}_4$ acts on X via $\mu(s, c) := s$ and $(j, \sigma, i)(s_i, c) := (s_j, \sigma \circ c)$.

$(8, (1, 2, 7, 8), 7)$ applied to

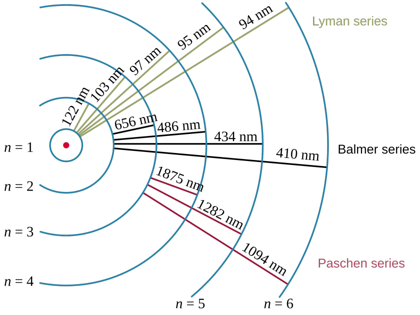
13	10	11	6
5		4	8
1	12	14	9
3	15	2	7

=

5	13	11	6
	10	4	8
1	12	14	9
3	15	2	7

An example from quantum mechanics

- A hydrogen atom consists of an electron orbiting its nucleus.
- The electromagnetic force between the electron and the nuclear proton leads to a set of quantum states for the electron, each with its own energy.



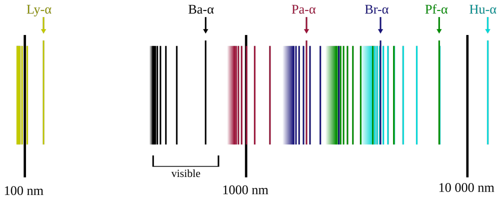
Hydrogen electron transitions and their resulting wavelengths (energy levels are not to scale).[4]

[4] Original: *Szdori*, Derivative work: *OrangeDog*, CC BY 2.5, commons.wikimedia.org/w/index.php?curid=6273602

An example from quantum mechanics

- Spectral emission occurs when the electron jumps from a higher energy state to a lower one.

- The emission spectrum is the spectrum of frequencies emitted due to electrons making a transition from a high energy state to a lower one.



The spectral series of hydrogen (on a logarithmic scale)[5] and the four visible hydrogen emission spectrum lines in the Balmer series[6]

[5] By OrangeDog, CC BY-SA 3.0, commons.wikimedia.org/w/index.php?curid=6278485

[6] By Merikanto, Adrignola, CC0, commons.wikimedia.org/w/index.php?curid=16417920

An example from quantum mechanics

- Rydberg, 1890ca : from experimental discovery one deduces that
  - there exists a set I of frequencies such that the spectrum is the set of differences $\nu_{ij} = \nu_j - \nu_i$ of arbitrary pairs of elements of I.

    One can combine two frequencies ν_{ij} and ν_{jk} to obtain a third,

$$\nu_{ik} = \nu_{ij} + \nu_{jk}.$$

- Ritz-Rydberg combination principle : the sum of certain frequencies of the spectrum is again a frequency of the spectrum.

- These experimental results could not be explained in the framework of the theoretical physics of the 19th century:

  it predicted that the set of frequencies forms a subgroup of $\mathbb{R}$, but it does not.

An example from quantum mechanics

Heisenberg's questioning of classical mechanics:

- In the classical model, the algebra of observable physical quantities can be directly read from the group Γ of emitted frequencies :

  it is the convolution algebra of this group of frequencies.

- In reality one is not dealing with a group of frequencies but rather, due to the Ritz-Rydberg combination principle, with a groupoid

$$\Delta = \{(i,j) : i,j \in I\}$$

  having the composition rule

$$(i,j) \circ (j,k) = (i,k),$$

  i.e. a groupoid of pairs !

Let G be a finite group. The space $\mathbb{C}^G$ of complex-valued functions on G is an algebra with respect to the convolution product :

$$1(g) = \delta_{e,g} \qquad \text{and} \qquad (\varphi \cdot \psi)(g) = \sum_{h,k \mid hk=g} \varphi(h)\psi(k).$$

If $\mathscr{G}$ is a finite groupoid, then the space $\mathbb{C}^{\mathscr{G}_1}$ of complex-valued functions on $\mathscr{G}_1$ is still an algebra with respect to the convolution product :

$$1(g) = \sum_{p \in \mathscr{G}_0} \delta_{1_p,g} \qquad \text{and} \qquad (\varphi \cdot \psi)(g) = \sum_{h,k \mid h \circ k=g} \varphi(h)\psi(k).$$

Example

Let X be a finite set and $(X, X \times X)$ its groupoid of pairs. Then

$$(\varphi \cdot \psi)(i,k) \;=\; \sum_{j \in X} \varphi(i,j)\psi(j,k)$$

and $\mathbb{C}^{X \times X} \;\cong\; \mathrm{M}_{|X|}(\mathbb{C})$ via $\varphi \mapsto \big(\varphi(i,j)\big)_{i,j \in X}$. The convolution algebra of the groupoid Δ is none other than the algebra of matrices !

Heisenberg replaced classical mechanics, in which the observable quantities commute pairwise, by matrix mechanics, in which observable quantities no longer commute.

Definition

Given an action $G \times Y \to Y$ of the group G on the set Y, we have a groupoid $\mathscr{G}_{G,Y} = (Y, G \times Y)$ where

$$\begin{array}{rclcrclcrcl} 1_y & = & (e,y), & & \sigma(g,y) & = & y, & & \tau(g,y) & = & g(y), \\ (g,y')\circ(h,y) & = & (gh,y) & \Leftrightarrow & y' = h(y), & & & & (g,y)^{-1} & = & (g^{-1},g(y)). \end{array}$$

This is called the action (or transformation) groupoid.

If $X \subseteq Y$, we have the restriction $\mathscr{G}_{G,X} = (X, G \bullet X)$ of the action groupoid, where

$$G \bullet X = \left\{ (g,x) \in G \times X \mid g(x) \in X \right\}.$$

Example

If $Y =$ infinite tiling, Γ its group of symmetries, and $X =$ finite tiling, then $\mathscr{G}_{\Gamma,X}$ describes the symmetries of the finite tiling (and it is not an action groupoid!).

Definition

Let G be a group and $\{X_g, \alpha_g \mid g \in G\}$ a partial action of G on a set X. Then

$$G \ltimes_\alpha X \; := \; \left\{ (g,x) \in G \times X \;\middle|\; x \in X_g \right\}$$

together with

$$1_x \; = \; (e,x), \qquad \sigma(g,x) \; = \; x, \qquad \tau(g,x) \; = \; \alpha_g(x),$$
$$(g,x') \circ (h,x) = (gh,x) \;\Leftrightarrow\; \alpha_h(x) = x', \qquad (g,x)^{-1} \; = \; (g^{-1}, \alpha_g(x)),$$

is a groupoid, called the partial action groupoid. We have a groupoid action

$$(G \ltimes_\alpha X) \; \times_X \; X \; \to \; X, \qquad ((g,x),x) \; \mapsto \; \alpha_g(x).$$

Theorem (Abadie, 2003)

For every partial action of a group G on a set X there exists a globalization :

- *a (universal) set Y with an action of G,*
- *an injection $\epsilon\colon X \rightarrowtail Y$*

such that the partial action on X is provided by restriction.

Fact

If Y is the globalization of $\{X_g, \alpha_g \mid g \in G\}$, then the restriction $(X, G \bullet X)$ of the action groupoid $\mathcal{G}_{G,Y}$ to X coincides with the partial action groupoid $(X, G \ltimes_\alpha X)$.

Let G be a group.

$$\mathscr{P}_e(G) := \left\{A \subseteq G \mid e \in A\right\}$$

The datum of $D_g := \{A \in \mathscr{P}_e(g) \mid g^{-1} \in A\}$ and

$$\mathfrak{b}_g : D_g \to D_{g^{-1}}, \qquad A \mapsto gA$$

defines a partial action of G on $\mathscr{P}_e(G)$, often called the Bernoulli partial action.

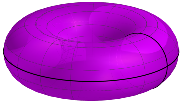
Theorem

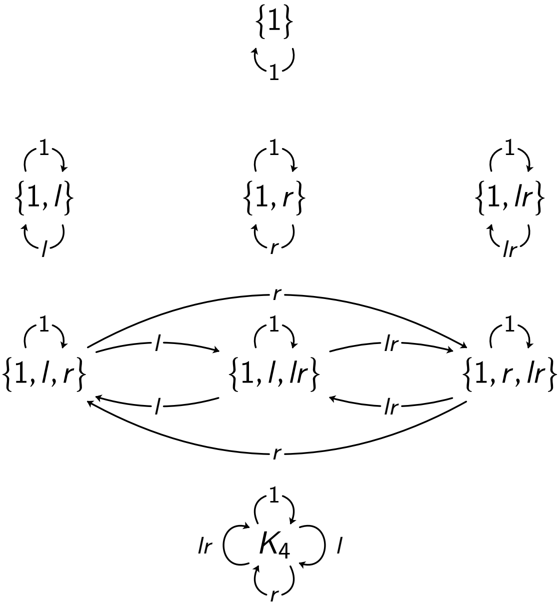
There is a bijective correspondence between

- *partial actions of G on X*
- *actions of $G \ltimes_{\mathfrak{b}} \mathscr{P}_e(G)$ on X*

Theorem (Jerez, 2024)

Every groupoid is a partial action groupoid.

- Globalizations of the partial actions :
  - Symmetries of the infinite planar tiling on the finite tiling is the infinite planar tiling itself
  - Torus on the tangent circumferences is the torus itself 
  - Any G on $\mathscr{P}_e(G)$ is $\mathscr{P}(G)$
- There is no way to rearrange Loyd's configuration into the natural arrangement, the hands-weaving answer being that to every closed path in $\mathscr{L}_4$ it corresponds an even permutation and the two arrangements have different parity



The groupoid governing the partial representation theory of the Vierergruppe $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$