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Everybody knows what a normal gabi-algebra is

Hopf Algebras and Monoidal Categories, Ferrara 2024

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Let $\Bbbk$ be a field (commutative ring), $\mathfrak{M}$ the category of $\Bbbk$-modules.

$$(A, m, u) \quad \Bbbk\text{-algebra} \quad \leftrightsquigarrow \quad {}_A\mathfrak{M} \xrightarrow{\omega} \mathfrak{M} \qquad A \cong \mathrm{Nat}(\omega, \omega)$$

$$(A, m, u, \Delta, \varepsilon) \quad \Bbbk\text{-bialgebra} \quad \leftrightsquigarrow \quad \left({}_A\mathfrak{M}, \otimes, \Bbbk\right) \xrightarrow{\omega} \left(\mathfrak{M}, \otimes, \Bbbk\right)$$

$$a \cdot (m \otimes n) = a_1 \cdot m \otimes a_2 \cdot n$$
$$a \cdot 1_\Bbbk = \varepsilon(a)$$

$$\Delta(a) = a \cdot (1_A \otimes 1_A)$$
$$\varepsilon(a) = a \cdot 1_\Bbbk$$

There is a bijective correspondence between liftings of the monoidal structure along ω and bialgebra structures on A.

If A is a bialgebra, ${}_A\mathfrak{M}$ is also biclosed monoidal

$$-\otimes N \;\dashv\; [N,-] \;=\; {}_A\mathrm{Hom}\left({}_\bullet A\otimes{}_\bullet N,-\right) \qquad (\text{right closed})$$

$$(\text{left closed}) \qquad M\otimes - \;\dashv\; \{M,-\} \;=\; {}_A\mathrm{Hom}\left({}_\bullet M\otimes{}_\bullet A,-\right)$$

There is a bijective correspondence between liftings of the right closed monoidal structure along ω and Hopf algebra structures on A.

$$\begin{array}{ccc} {}_A\mathrm{Hom}\left(A\otimes N,P\right) & \xrightarrow{\mathrm{coev}} & \mathrm{Hom}\left(N,{}_A\mathrm{Hom}\left(A\otimes N,P\right)\otimes N\right) \\ \vdots & & \downarrow{\scriptstyle \mathrm{Hom}(N,\mathrm{ev})} \\ {}_A\mathrm{Hom}\left(A\otimes N,P\right) & \xleftarrow[\cong]{} & \mathrm{Hom}\left(N,P\right) \end{array}$$

is induced by

$$A\otimes N\to A\otimes N, \qquad a\otimes n\mapsto a_1\otimes a_2\cdot n\ .$$

There is a bijective correspondence between liftings of the biclosed monoidal structure along ω and Hopf algebra structures with bijective antipode on A.

$$\begin{array}{ccc} {}_A\mathrm{Hom}\,(M \otimes A, P) & \xrightarrow{\mathrm{coev}} & \mathrm{Hom}\,(M, {}_A\mathrm{Hom}\,(M \otimes A, P) \otimes M) \\ \downarrow & & \downarrow \mathrm{Hom}(M,\mathrm{ev}) \\ {}_A\mathrm{Hom}\,(A \otimes M, P) & \xleftarrow[\cong]{} & \mathrm{Hom}\,(M, P) \end{array}$$

is induced by

$$A \otimes M \to M \otimes A, \qquad a \otimes m \mapsto a_1 \cdot m \otimes a_2 \ .$$

What can we say about A

if we lift the closed structure alone?

Definition

A skew-monoidal category is a category $\mathcal{C}$ together with

- a distinguished object $\mathbf{1}$
- a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a family $\rho_X\colon X \to X \otimes \mathbf{1}$ natural in X
- a family $\lambda_X\colon \mathbf{1} \otimes X \to X$ natural in X
- a family $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ natural in X, Y, Z

subject to the commutativity of 5 diagrams.

A skew-monoidal category is called

- right normal iff $\rho_X\colon X \to X \otimes \mathbf{1}$ is a natural isomorphism
- left normal iff $\lambda_X\colon \mathbf{1} \otimes X \to X$ is a natural isomorphism
- associative normal iff $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is a natural isomorphism

Definition (Street, 2013)

A skew-closed category is a category $\mathcal{C}$ together with

- a distinguished object $\mathbf{1}$
- a bifunctor $[-,-]\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$
- a family $i_X\colon [\mathbf{1}, X] \to X$ natural in X
- a family $j_X\colon \mathbf{1} \to [X, X]$ dinatural in X
- a family $\Gamma^Z_{X,Y}\colon [X, Y] \to [[Z, X], [Z, Y]]$ natural in X, Y and dinatural in Z

subject to the commutativity of 5 diagrams.

Example

Take R a $\Bbbk$-algebra and $\mathcal{C} = {}_R\mathfrak{M}$ with $[U, V] = \mathrm{Hom}\,(U, V)$ and $\mathbf{1} = R$.

- $i_U\colon\ \mathrm{Hom}\,(R, U) \to U, \quad f \mapsto f(1_R)$
- $j_U\colon R \to \mathrm{Hom}\,(U, U)\,, \quad 1_R \mapsto \mathrm{id}_U$

- Morally, one should think about $\Gamma^X_{Y,Z} : [Y, Z] \to [[X, Y], [X, Z]]$ as a "post-composition map":

$$\Gamma^U_{V,W} : \mathrm{Hom}(V, W) \;\to\; \mathrm{Hom}\Big(\mathrm{Hom}(U, V), \mathrm{Hom}(U, W)\Big)$$

$$\left(V \xrightarrow{\;f\;} W \right) \;\mapsto\; \left(\left(U \xrightarrow{\;g\;} V \right) \mapsto \left(\begin{array}{ccc} U & \xrightarrow{\quad f \circ g \quad} & W \\ & {\scriptstyle g} \searrow \quad \nearrow {\scriptstyle f} & \\ & V & \end{array} \right) \right)$$

Definition

A (strict) closed functor between closed categories $(\mathcal{C}, \mathbf{1}, [-,-])$ and $(\mathcal{C}', \mathbf{1}', [-,-]')$ is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ satisfying

$$F(\mathbf{1}) = \mathbf{1}' \qquad \text{and} \qquad F[X, Y] = [FX, FY]$$

and all behaves well with respect to coherence transformations.

A skew-closed category is called

- right normal iff $i_X \colon [\mathbf{1}, X] \to X$ is a natural isomorphism,
- left normal iff

$$\mathcal{C}(X, Y) \to \mathcal{C}(\mathbf{1}, [X, Y]), \qquad f \mapsto [f, Y] \circ j_Y$$

  is a bijection,
- associative normal iff

$$\int^W \mathcal{C}(X, [W, U]) \times \mathcal{C}(Y, [Z, W]) \;\to\; \mathcal{C}(X, [Y, [Z, U]])$$

$$(f, g) \mapsto \left\{ X \xrightarrow{f} [W, U] \xrightarrow{\Gamma^Z_{W,U}} [[Z, W], [Z, U]] \xrightarrow{[g,[Z,U]]} [Y, [Z, U]] \right\}$$

  is bijective.

A closed category is a skew-closed category satisfying all the normality conditions.

Theorem (Uustalu, Veltri, Zeilberg, 2020)

Let $\mathcal{C}$ be a category with a distinguished object $\mathbf{1}$ and bifunctors $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $[-,-]\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. Assume that there are adjunctions $- \otimes X \dashv [X,-]$, natural in X.

- *Skew-monoidal structures (α, λ, ρ) on $(\mathcal{C}, \otimes, \mathbf{1})$ are in bijection with skew-closed structures (Γ, j, i) on $(\mathcal{C}, [-,-], \mathbf{1})$.*
- *The skew-monoidal structure is left/right/associative normal iff the skew-closed structure is left/right/associative normal.*
- *The skew-monoidal structure is associative normal iff*

$$[X \otimes Y, Z] \xrightarrow{\Gamma^{Y}_{X\otimes Y, Z}} [[Y, X \otimes Y], [Y, Z]] \xrightarrow{[\mathrm{coev}, [Y,Z]]} [X, [Y, Z]]$$

is a natural isomorphism, iff the skew-closed structure is associative normal.

Theorem

Let (T, m, u) be a monad on a skew-closed category $\mathcal{C}$. Then $\mathcal{C}^T$ is skew-closed such that the forgetful functor to $\mathcal{C}$ is strictly closed iff

- *there is $\mu_\mathbf{1} : T\mathbf{1} \to \mathbf{1}$ in $\mathcal{C}$ such that $(\mathbf{1}, \mu_\mathbf{1}) \in \mathcal{C}^T$ and*
- *there is a family $s_{X,Y} : T[TX, Y] \to [X, TY]$ natural in X, Y*

which satisfy, for all X, Y in $\mathcal{C}$ and (M, μ_M) in $\mathcal{C}^T$,

$$\begin{aligned}
s_{X,Y} \circ u_{[TX,Y]} &= [u_X, u_Y], \\
s_{X,Y} \circ m_{[TX,Y]} &= [X, m_Y] \circ s_{X,TY} \circ Ts_{TX,Y} \circ T^2[m_X, Y], \\
Ti_X &= i_{TX} \circ s_{\mathbf{1},X} \circ T[\mu_\mathbf{1}, X], \\
j_M \circ \mu_\mathbf{1} &= [M, \mu_M \circ m_M] \circ s_{M,TM} \circ Tj_{TM}, \\
\Gamma^M_{X,TY} \circ s_{X,Y} &= \big[[\mu_M, X], s_{M,Y}\big] \circ s_{[TM,X],[TM,Y]} \circ T\big[s_{M,X}, [\mu_M, Y]\big] \circ T\Gamma^M_{TX,Y}.
\end{aligned}$$

Proposition

Let (A, m, u) be a monoid in a closed monoidal category $\mathcal{C}$. Then ${}_A\mathcal{C}$ is skew-closed such that the forgetful functor is strictly closed iff

- *there is $\varepsilon\colon A \to \mathbf{1}$ in $\mathcal{C}$ such that $(\mathbf{1}, \varepsilon \otimes \mathbf{1})$ in ${}_A\mathcal{C}$ and*
- *there is a family $t_{X,Y}\colon A \otimes X \otimes Y \to A \otimes X \otimes A \otimes Y$ natural in X, Y*

which satisfy, for all X, Y in $\mathcal{C}$ and (M, μ_M) in ${}_A\mathcal{C}$

$$t_{X,Y} \circ (u \otimes X \otimes Y) = u \otimes X \otimes u \otimes Y,$$

$$t_{X,Y} \circ (m \otimes X \otimes Y) = (m \otimes X \otimes m \otimes Y) \circ (A \otimes t_{X,A\otimes Y}) \circ t_{A\otimes X,Y},$$

$$(A \otimes X \otimes \varepsilon) \circ t_{X,\mathbf{1}} = A \otimes X,$$

$$\varepsilon \otimes M = (m \otimes M) \circ t_{\mathbf{1},M},$$

$$t_{X,Y\otimes M} = (A \otimes X \otimes A \otimes Y \otimes \mu_M^{(2)}) \circ (A \otimes X \otimes t_{Y,A\otimes M}) \circ t_{X\otimes A\otimes Y,M} \circ (t_{X,Y} \otimes M)\,.$$

Theorem

Let A be a $\Bbbk$-algebra. Then the skew-closed structure of $\mathfrak{M}$ lifts to a skew-closed structure on ${}_A\mathfrak{M}$ iff

- (A, ε) *is an augmented* $\Bbbk$*-algebra and*
- $\exists$ *an algebra map* $\delta \colon A \to A \otimes A^{\mathrm{op}}$, $\delta(a) = a_+ \otimes a_-$, *such that for all* $a \in A$

$$a_+\varepsilon(a_-) = a, \qquad a_+a_- = \varepsilon(a)1_A,$$

$$a_{++} \otimes a_{-+} \otimes a_{--}a_{+-} = a_+ \otimes a_- \otimes 1.$$

In this case, A *acts on* $\mathrm{Hom}\,(M, N)$ *as*

$$(a.f)(m) = a_+ f(a_- m).$$

Definition

Such an algebra will be called a gabi-algebra.

Example (Hopf algebras)

Any Hopf algebra H is a gabi-algebra with $\delta(h) = h_1 \otimes S(h_2)$. The category ${}_H\mathfrak{M}$ is a closed monoidal category with closed monoidal forgetful functor.

Example (One-sided Hopf algebras – Green, Nichols, Taft, 1980)

A right Hopf algebra is a bialgebra B in which id_B has a right convolution inverse.

A right Hopf algebra whose right antipode is an anti-bialgebra map carries the structure of a gabi-algebra with respect to ε and $\delta(b) := b_1 \otimes S(b_2)$ for all $b \in B$.

This skew-closed structure is not left normal, since the $\Bbbk$-linear map

$$\beta \colon {}_\bullet B \otimes {}_\bullet B \to {}_\bullet B \otimes B, \qquad a \otimes b \mapsto a_1 \otimes S(a_2)b,$$

induces an element $1_\Bbbk \mapsto \beta$ in ${}_B\mathrm{Hom}\left(\Bbbk, \mathrm{Hom}\left(B \otimes B, B \otimes B\right)\right)$ which does not come from an element in ${}_B\mathrm{Hom}\left(B \otimes B, B \otimes B\right)$.

Gabi-algebras and skew monoidal structures

Let A be a gabi-algebra and M be a left A-module. Consider the $\Bbbk$-module $A \otimes M$, which we turn into an A-bimodule via

$$a.(b \otimes m).c = abc_+ \otimes c_- m \ .$$

We denote this bimodule by $A \odot M$.

Proposition

For any $M \in {}_A\mathfrak{M}$, there is an adjunction

$$\xymatrix{ {}_A\mathfrak{M} \ar@/^/[r]^{(A \odot M) \otimes_A -} \ar@{}[r]|{\perp} & {}_A\mathfrak{M} \ar@/^/[l]^{\mathrm{Hom}(M,-)} }$$

The unit and counit of the adjunction are

$$N \to \mathrm{Hom}\left(M, (A \odot M) \otimes_A N\right), \qquad n \mapsto \left\{m \mapsto (1_A \odot m) \otimes_A n\right\},$$

$$(A \odot M) \otimes_A \mathrm{Hom}\left(M, N\right) \to N, \qquad (a \odot m) \otimes_A f \mapsto af(m) \ .$$

Recall that any skew-closed category in which $[-,-]$ admits a left adjoint possesses a skew-monoidal structure. Consider the bifunctor

$$\boxtimes\colon\ {}_A\mathfrak{M}\times{}_A\mathfrak{M}\to{}_A\mathfrak{M},\qquad M\boxtimes N=(A\odot N)\otimes_A M\ .$$

Proposition

$\boxtimes$ defines a skew-monoidal structure on ${}_A\mathfrak{M}$, with unit $\Bbbk$, unitors

$$\lambda_N\colon \Bbbk\boxtimes N\to N,\qquad (a\odot n)\otimes_A 1_\Bbbk\mapsto an,$$

$$\rho_M\colon M\xrightarrow{\sim} M\boxtimes\Bbbk,\qquad m\mapsto(1_A\odot 1_\Bbbk)\otimes_A m,$$

and associator

$$\alpha_{L,M,N}\colon (L\boxtimes M)\boxtimes N\to L\boxtimes(M\boxtimes N),$$

$$(a\odot n)\otimes_A((b\odot m)\otimes_A l)\mapsto\Big(ab_+\odot((1_A\odot b_-n)\otimes_A m)\Big)\otimes_A l\ .$$

Given a gabi-algebra (A, δ, ε) define its canonical morphism to be

$$\beta\colon A \otimes A \to A \otimes A, \qquad a \otimes b \mapsto a_+ \otimes a_- b,$$

and its antipode to be $\sigma(a) := \varepsilon(a_+)a_-$.

Proposition

If β is invertible and $\Delta(a) := \beta^{-1}(a \otimes 1)$ is left counital, then (A, Δ, ε) is a Hopf algebra with antipode σ.

Corollary (Commutative is Hopf)

Any commutative gabi-algebra A is a Hopf algebra with comultiplication $\Delta(a) := a_+ \otimes \sigma(a_-)$ and antipode σ.

Proposition

Let A be a finite-dimensional gabi-algebra with invertible antipode σ. Define $\Delta(a) := a_+ \otimes \sigma^{-1}(a_-)$. Then $(A, \Delta, \varepsilon, \sigma)$ is a Hopf algebra.

Proposition

Let A be a gabi-algebra over $\Bbbk$. Then the skew closed structure on ${}_A\mathfrak{M}$ is associative normal if and only if the canonical map β is invertible.

Proposition

Let A be a gabi-algebra whose β is invertible. Then the skew closed structure on ${}_A\mathfrak{M}$ is also left normal if and only if A is a Hopf algebra with comultiplication $\beta^{-1}(a \otimes 1)$, counit ε and antipode $\sigma(a) = \varepsilon(a_+)a_-$ for all $a \in A$.

Our conclusive slogan is:

Everybody knows what a normal gabi-algebra is.

Theorem

Let A be an algebra. Then ${}_A\mathfrak{M}$ is a closed category with strictly closed forgetful functor ${}_A\mathfrak{M} \to \mathfrak{M}$ if and only if A is a Hopf algebra.

Many thanks . . .

HOPF25

22-26 April, 2025 – Brussels

Hopf algebras, quantum groups, monoidal categories and related structures

Expression of interest (and info):

hopfalgb.ulb.be/Hopf2025

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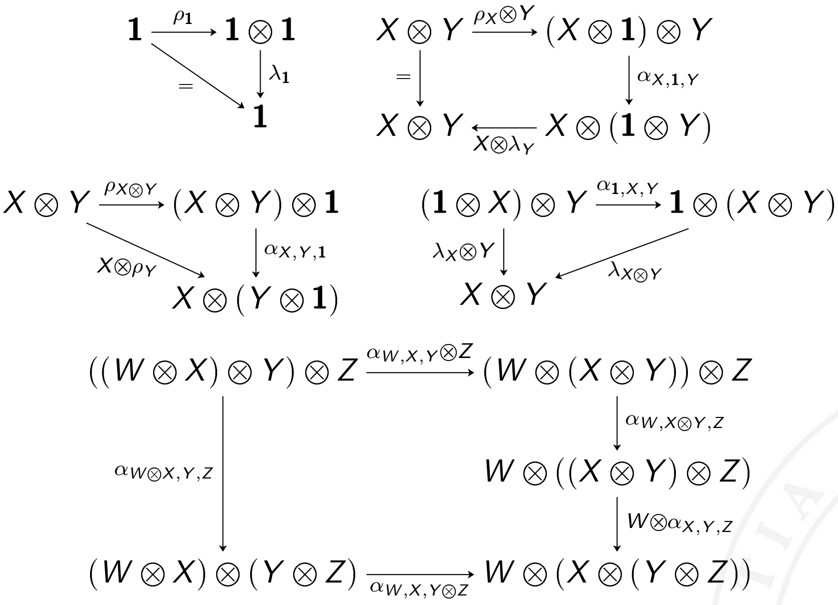
$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{j_{\mathbf{1}}} & [\mathbf{1},\mathbf{1}] \\
& \searrow^{=} & \downarrow{\scriptstyle i_{\mathbf{1}}} \\
& & \mathbf{1}
\end{array}
\qquad
\begin{array}{ccc}
[X,Y] & \xrightarrow{\Gamma^X_{X,Y}} & [[X,X],[X,Y]] \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle [j_X,[X,Y]]} \\
[X,Y] & \xleftarrow[i_{[X,Y]}]{} & [\mathbf{1},[X,Y]]
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{j_Y} & [Y,Y] \\
& \searrow_{j_{[X,Y]}} & \downarrow{\scriptstyle \Gamma^X_{Y,Y}} \\
& & [[X,Y],[X,Y]]
\end{array}
\qquad
\begin{array}{ccc}
[X,Y] & \xrightarrow{\Gamma^{\mathbf{1}}_{X,Y}} & [[\mathbf{1},X],[\mathbf{1},Y]] \\
& \searrow_{[i_X,Y]} & \downarrow{\scriptstyle [[\mathbf{1},X],i_Y]} \\
& & [[\mathbf{1},X],Y]
\end{array}
$$

$$
\begin{array}{ccc}
[W,X] & \xrightarrow{\Gamma^U_{W,X}} & [[U,W],[U,X]] \\
& & \downarrow{\scriptstyle \Gamma^{[U,V]}_{[U,W],[U,X]}} \\
{\scriptstyle \Gamma^V_{W,X}}\downarrow & & [[[U,V],[U,W]],[[U,V],[U,X]]] \\
& & \downarrow{\scriptstyle [\Gamma^U_{V,W},[[U,V],[U,X]]]} \\
[[V,W],[V,X]] & \xrightarrow[{[[V,W],\Gamma^U_{V,X}]}]{} & [[V,W],[[U,V],[U,X]]]
\end{array}
$$


$\mathbf{1} \xrightarrow{\rho_{\mathbf{1}}} \mathbf{1} \otimes \mathbf{1}$
$\lambda_{\mathbf{1}}$
$=$
$\mathbf{1}$
$X \otimes Y \xrightarrow{\rho_{X \otimes Y}} (X \otimes \mathbf{1}) \otimes Y$
$=$
$\alpha_{X,\mathbf{1},Y}$
$X \otimes Y \xleftarrow{X \otimes \lambda_Y} X \otimes (\mathbf{1} \otimes Y)$
$X \otimes Y \xrightarrow{\rho_{X \otimes Y}} (X \otimes Y) \otimes \mathbf{1}$
$\alpha_{X,Y,\mathbf{1}}$
$X \otimes \rho_Y$
$X \otimes (Y \otimes \mathbf{1})$
$(\mathbf{1} \otimes X) \otimes Y \xrightarrow{\alpha_{\mathbf{1},X,Y}} \mathbf{1} \otimes (X \otimes Y)$
$\lambda_{X \otimes Y}$
$\lambda_{X \otimes Y}$
$X \otimes Y$
$((W \otimes X) \otimes Y) \otimes Z \xrightarrow{\alpha_{W,X,Y} \otimes Z} (W \otimes (X \otimes Y)) \otimes Z$
$\alpha_{W,X \otimes Y,Z}$
$\alpha_{W \otimes X,Y,Z}$
$W \otimes ((X \otimes Y) \otimes Z)$
$W \otimes \alpha_{X,Y,Z}$
$(W \otimes X) \otimes (Y \otimes Z) \xrightarrow{\alpha_{W,X,Y \otimes Z}} W \otimes (X \otimes (Y \otimes Z))$