# Hopf envelopes of finite-dimensional bialgebras

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*Based on an ongoing project with A. Ardizzoni and C. Menini*

Fix a base field $\Bbbk$ and let $B$ be a $\Bbbk$-bialgebra.

## Definition (Manin, 1988)

The Hopf envelope of $B$, aka the free Hopf algebra generated by $B$, is a Hopf algebra $\mathrm{H}(B)$ together with a bialgebra morphism $\eta_B \colon B \to \mathrm{H}(B)$ such that for every Hopf algebra $H$ and any bialgebra map $f \colon B \to H$, there exists a unique Hopf algebra map $\hat{f} \colon \mathrm{H}(B) \to H$ such that the following commutes

$$\begin{tikzcd} & \mathrm{H}(B) \arrow[dd, "\hat{f}"] \\ B \arrow[ur, "\eta_B"] \arrow[dr, "f"'] & \\ & H \end{tikzcd}$$

## The existence of the Hopf envelope

- Define a sequence of bialgebras $\{B_n \mid n \in \mathbb{N}\}$ by

$$B_0 := B, \qquad B_{n+1} := B_n^{\mathrm{op,cop}}$$

- Let $\mathcal{B}$ be the bialgebra coproduct of $\{B_n \mid n \in \mathbb{N}\}$ with injections $j_n\colon B_n \to \mathcal{B}$
- $\exists!$ bialgebra morphism $\mathcal{S}\colon \mathcal{B} \to \mathcal{B}^{\mathrm{op,cop}}$ s.t. the following diagrams commute

$$\begin{tikzcd}
B_n \arrow[r, "j_n"] \arrow[d, "\mathrm{id}"'] & \mathcal{B} \arrow[d, "\mathcal{S}"] \\
B_{n+1}^{\mathrm{op,cop}} \arrow[r, "j_{n+1}"'] & \mathcal{B}^{\mathrm{op,cop}}
\end{tikzcd}$$

- The two-sided ideal $\mathcal{I}$ of $\mathcal{B}$ generated by

$$\{(\mathcal{S} * \mathrm{id} - u \circ \varepsilon)(b_n), (\mathrm{id} * \mathcal{S} - u \circ \varepsilon)(b_n) \mid b_n \in j_n(B_n), n \in \mathbb{N}\}$$

  is a bi-ideal
- The quotient $\mathcal{B}/\mathcal{I}$ with the composition $B \xrightarrow{\pi \circ j_0} \mathcal{B}/\mathcal{I}$ is the Hopf envelope of $B$

Theorem (Ardizzoni, Menini, S., 2024)

*Let $B$ be a finite-dimensional bialgebra. Then* $\mathrm{H}(B) = \dfrac{B}{KB}$ *where $K$ is the kernel of the canonical morphism*

$$i_B \colon B \longrightarrow \frac{B \otimes B}{(B \otimes B)B^+}, \qquad b \longmapsto \overline{b \otimes 1}.$$

**Key ingredients:**

- the surjectivity of $i_B$
- the notion of (one-sided) $n$-Hopf algebras

The free Hopf module functor $(-)\otimes B\colon \mathfrak{M}\to\mathfrak{M}_B^B$ forms part of an adjoint triple

$$(-)\otimes_B \Bbbk \quad\dashv\quad (-)\otimes B \quad\dashv\quad (-)^{\mathrm{co}B}$$

and there is a natural transformation $\sigma_M\colon M^{\mathrm{co}B}\to M\otimes_B\Bbbk,\ m\mapsto m\otimes_B 1_\Bbbk$.

$$B\oslash B \;:=\; \frac{B_\bullet\otimes B_\bullet}{(B_\bullet\otimes B_\bullet)\,B^+}\;\simeq\;(B_\bullet\otimes B_\bullet^\bullet)\otimes_B\Bbbk,$$

coalgebra in ${}_{B\otimes B^{\mathrm{cop}}}\mathfrak{M}$ with respect to $(a\otimes b)\cdot(x\oslash y)=ax\oslash by$,

$$\Delta(x\oslash y)=(x_1\oslash y_2)\otimes(x_2\oslash y_1)\qquad\text{and}\qquad \varepsilon(x\oslash y)=\varepsilon(x)\varepsilon(y).$$

### Theorem

*The following are equivalent:*

*(1) $(-)\otimes B\colon\mathfrak{M}\to\mathfrak{M}_B^B$ is Frobenius.*

*(2) The canonical natural transformation $\sigma\colon(-)^{\mathrm{co}B}\to(-)\otimes_B\Bbbk$ is invertible.*

*(3) The map $i_B\colon B\to B\oslash B,\ b\mapsto b\oslash 1_B,$ is invertible.*

*(4) $B$ is right Hopf algebra with right antipode $S^r$ which is an anti-bialgebra map.*

## Proposition

*The following assertions are equivalent for a bialgebra $B$.*

(1) *The map $i_B : B \to B \oslash B$, $x \mapsto x \oslash 1$, is surjective.*

(2) *There is $S \in \mathrm{End}_{\Bbbk}(B)$ such that $1 \oslash y = S(y) \oslash 1$, for every $y \in B$.*

(3) *There is $S \in \mathrm{End}_{\Bbbk}(B)$ such that $x \oslash y = xS(y) \oslash 1$, for every $x, y \in B$.*

(4) *For every $y \in B$ there is $y' \in B$ such that $1 \oslash y = y' \oslash 1$.*

(5) *For every $x, y \in B$ there is $y' \in B$ such that $x \oslash y = xy' \oslash 1$.*

## Example

Let $M$ be a regular monoid: $\forall\, x \in M$, $\exists\, x^\dagger \in M$ such that $x \cdot x^\dagger \cdot x = x$.
$i_{\Bbbk M}$ is surjective since in $\Bbbk M \oslash \Bbbk M$ we have

$$1 \oslash x \;=\; x^\dagger \cdot x \oslash x \cdot x^\dagger \cdot x \;=\; x^\dagger \cdot x \oslash x \;=\; x^\dagger \oslash 1.$$

E.g., let $f : H \to G$ be a group homomorphism and let $M := H \sqcup G$ with

$$h \cdot h' = hh', \quad h \cdot g = f(h)g, \quad g \cdot h = gf(h), \quad g \cdot g' = gg', \quad x^\dagger = x^{-1}.$$

## Definition

A left $n$-Hopf algebra is a bialgebra $B$ with a minimal $n \in \mathbb{N}$ for which there is an $S \in \mathrm{End}_{\Bbbk}(B)$, called a left $n$-antipode, such that $S * \mathrm{id}_B^{*n+1} = \mathrm{id}_B^{*n}$.

Similarly one defines a right $n$-Hopf algebra and a right $n$-antipode.

A left and right $n$-Hopf algebra is an $n$-Hopf algebra. If the same $S$ is both a left $n$-antipode and a right $n$-antipode, then we call it a two-sided $n$-antipode.

## Example

For $n \in \mathbb{N}$, consider the monoid $M = \langle x \mid x^{n+1} = x^n \rangle$. The monoid algebra $\Bbbk M$ is a $n$-Hopf algebra: $\mathrm{id}_{\Bbbk M}^{*n+1} = \mathrm{id}_{\Bbbk M}^{*n}$ so that $\mathrm{id}_{\Bbbk M}$ and $u\varepsilon$ are two-sided $n$-antipodes.

## Example

Let $M$ be a commutative regular monoid: $\forall\, x \in M$, $\exists\, x^\dagger$ such that $x^\dagger \cdot x^2 = x$. If we perform a choice of $x^\dagger$ for every $x \in M$, then $\Bbbk M$ is a 1-Hopf algebra with 1-antipode $S$ uniquely determined by $S(x) := x^\dagger$ for all $x \in M$.

## Theorem

*Let $B$ be a bialgebra such that* $\mathrm{id}$ *is algebraic over* $\Bbbk$ *in the convolution algebra* $\mathrm{End}_{\Bbbk}(B)$ *(e.g., $B$ is finite-dimensional). Then $B$ has a two-sided n-antipode $S$ for some $n \in \mathbb{N}$ which moreover satisfies $S * \mathrm{id}_B = \mathrm{id}_B * S$.*

## Proposition

*Let $B$ be a left n-Hopf algebra with left n-antipode $S$ (e.g., $B$ is finite-dimensional). Then $S(y) \oslash 1 = 1 \oslash y$ for all $y \in B$ and $i_B \colon B \to B \oslash B$ is surjective.*

## Proposition

*Let $B$ be a bialgebra such that $i_B$ is surjective (e.g., $B$ is finite-dimensional). Then, the map $\eta_B \colon B \to \mathrm{H}(B)$ is surjective.*

## Proposition

*(1)* $B/\ker(i_B)B$ *is a bialgebra and* $q_B\colon B \to B/\ker(i_B)B$ *is a bialgebra map.*

*(2) For any bialgebra map* $f\colon B \to C$ *into a bialgebra* $C$ *with* $i_C$ *injective, there is a (necessarily unique) bialgebra map* $\hat{f}\colon B/\ker(i_B)B \to C$ *such that* $\hat{f} \circ q_B = f$*.*

*(3) If* $i_B$ *is surjective, then* $q_B$ *is right convolution invertible.*

## Theorem

*Let* $B$ *be a finite-dimensional bialgebra. Then* $\mathrm{H}(B) = B/\ker(i_B)B$*.*

**Proof:** Set $K := \ker(i_B)$. Since $B$ is finite-dimensional, $i_B$ is surjective.

Thus, the canonical projection $q_B\colon B \to B/KB$ is right convolution invertible.

Since $B/KB$ is a quotient of $B$, it is finite-dimensional and so it is a left $n$-Hopf algebra for some $n \in \mathbb{N}$ and hence, in fact, a Hopf algebra.

$B/KB$ has the desired universal property, as any Hopf algebra $H$ has $i_H$ injective. $\square$

$$B := \Bbbk\left\langle x, y \mid yx = -xy, x^3 = x, y^2 = 0 \right\rangle$$

which is 6-dimensional with basis $\{1, x, x^2, y, xy, x^2y\}$. Its coalgebra structure is uniquely determined by $\Delta(x) = x \otimes x$ and $\Delta(y) = x \otimes y + y \otimes 1$.

## Proposition

*$B$ has a two-sided invertible 1-antipode $S : B \to B$ which is an anti-algebra map and it is defined by setting $S(x) = x$ and $S(y) = (1 - x^2 - x)\, y$.*

## Proposition

*$B \oslash B \simeq H_4$ and, in fact, $H_4 = \mathrm{H}(B)$.*

$$B := \Bbbk\left\langle x \mid x^{m+n+1} = x^n \right\rangle$$

for $m, n \in \mathbb{N}$, monoid algebra with $\Delta(x) = x \otimes x$.

### Proposition

*$B$ has a two-sided $n$-antipode $S = \mathrm{id}^{*m}$.*

### Remark

In general, $S$ is neither injective nor surjective.

When $m = 0$, we get $x^{n+1} = x^n$ and $S = u \circ \varepsilon$.

When $m = 2$ and $n = 2$, we get $x^5 = x^2$ and $S = \mathrm{id}^{*2}$: $S(x^4) = x^8 = x^2 = S(x)$.

### Proposition

$B \oslash B \ \simeq\ \Bbbk\langle x \mid x^{m+1} = 1\rangle \ =\ \mathrm{H}(B)$.

The same result holds for a left Artinian bialgebra $B$.

### Proposition

*If $B$ is a left Artinian bialgebra, then $i_B$ is surjective.*

### Corollary

*Let $B$ be a left Artinian bialgebra. Then, $\eta_B$ is surjective and $\mathrm{H}(B)$ is finite-dimensional.*

### Proposition

*Let $f : B \to B'$ be a bialgebra map which is right convolution invertible. If $f$ is surjective and $B'$ is left Artinian, then $B'$ is a Hopf algebra.*

### Theorem

*Let $B$ be a left Artinian bialgebra. Then $\mathrm{H}(B) = B/\ker(i_B)B$.*

However, we do not know if an Artinian bialgebra is necessarily finite-dimensional.

**Thank you**