UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI MATEMATICA GIUSEPPE PEANO

SCUOLA DI SCIENZE DELLA NATURA

Corso di Laurea Magistrale in Matematica

Tesi di Laurea Magistrale

On the Fundamental Structure Theorem for quasi-Hopf bimodules

Relatore: Dott. Alessandro Ardizzoni
Correlatore: Prof. Laiachi El Kaoutit Zerri

Candidato: Paolo Saracco

ANNO ACCADEMICO 2013 - 2014

Introduction

Let A be an algebra over the field $\Bbbk$ and denote by $\mathcal{M}$ the category of $\Bbbk$-vector spaces and by $\otimes$ the tensor product over $\Bbbk$. Assume that besides the algebra structures $m\colon A\otimes A \to A$ and $u\colon \Bbbk \to A$, A comes with two algebra morphisms: $\Delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to \Bbbk$. We call it a bialgebra if these additional operations are coassociative and counital. It can be proven (cfr. [Ka, Proposition XI.3.1]) that A is a bialgebra if and only if the category of left (resp. right) A-modules ${}_A\mathcal{M}$ (resp. $\mathcal{M}_A$) is a monoidal category, when equipped with the tensor product of $\Bbbk$-vector spaces and with the natural constraints (a, l, r). For ${}_A\mathcal{M}$ to be a monoidal category means that it looks like the category of vector spaces or the category of groups, i.e., it is a category endowed with a bifunctor $-\otimes-\colon {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{M}$ and a distinguished object $\Bbbk$ such that $\otimes$ is associative, up to a natural isomorphism a, and $\Bbbk$ is a left and right unit for $\otimes$, up to natural isomorphisms l, r. Bluntly speaking, the notion of a monoidal category is the 'categorification' of the notion of a monoid.

Furthermore, a bialgebra H is a Hopf algebra if it admits an antipode, i.e., an endomorphism of H that is the convolution inverse of the identity. Larson and Sweedler proved in 1969 that if a (finite dimensional) bialgebra H is a Hopf algebra, then a certain Structure Theorem holds for Hopf H-modules (cfr. [LS, Proposition 1, page 82]). A (right) Hopf H-module is a (right) H-module that is also a (right) H-comodule over the coassociative coalgebra H within the monoidal category of (right) H-modules $\mathcal{M}_H$. The Structure Theorem, as it appears in [LS], states that every Hopf module over a Hopf algebra is trivial, that is, for each Hopf module M there exists a vector space $M^{\mathrm{Co}H}$ (called the space of coinvariants of M) such that M has the form $M^{\mathrm{Co}H} \otimes H$. This result allowed them to show that every finite dimensional Hopf algebra H admits non-zero left integrals (actually, that for a finite dimensional bialgebra H, the existence of the antipode is equivalent to the existence of non-singular left integrals; cfr. [LS, Theorem on page 79]), from which they also proved that all finite dimensional Hopf algebras are Frobenius algebras and that, in such a case, the antipode is always bijective. In Chapter 2 we will retrieve a modern version of the Structure Theorem, that states that a bialgebra H admits an antipode if and only if the functor $-\otimes H\colon \mathcal{M} \to \mathcal{M}_H^H$ that associate to each vector space V the free Hopf module $V \otimes H_{\bullet}^{\bullet}$ is a category equivalence (cfr. also [Ab, Theorem 3.1.8], [BW, Theorem 15.5] and [Sw, Theorem 4.1.1]).

The main aim of this thesis is to look for a proper analogue of the notion of antipode for quasi-bialgebras and hence to extend the Structure Theorem to this more general framework. Let us spend a few words to highlight that quasi-bialgebras became important since 1989, because they are the basic structure on which quasi-Hopf algebras are constructed and then they are related to conformal field theory, quantum groups, the

Knizhnik-Zamolodchikov equations, the Yang-Baxter equation (cfr. for example [Dr1] and [Dr2]), apart from being interesting in themselves.

Actually, Drinfel'd definition of a quasi-bialgebra A ensures that the category of A-modules is still monoidal with tensor product given by the tensor product of $\Bbbk$-vector spaces, but with different constraints with respect to $\mathcal{M}$ ([Dr1]). However, if we try to establish an analogue of the Structure Theorem criterion for quasi-bialgebras we run into two difficulties.

The first one is that there are no Hopf modules: we will show in Chapter 3 that a quasi-bialgebra A is an associative algebra with counit and comultiplication that is coassociative only up to conjugation by an invertible element $\Phi \in A \otimes A \otimes A$, i.e.

$$\Phi(\Delta \otimes A)(\Delta(\cdot)) = (A \otimes \Delta)(\Delta(\cdot))\Phi.$$

Therefore, one does not know how to define comodules without a coalgebra structure. Fortunately, this problem was solved by Hausser and Nill in [HN], where they observe that A, with the natural left and right A-actions, is a coalgebra in the category of (A, A)-bimodules ${}_A\mathcal{M}_A$. Hence, we can still define a quasi-Hopf A-(bi)module category ${}_A\mathcal{M}_A^A$: namely, the category of (right) A-comodules over the coassociative coalgebra A within the monoidal category ${}_A\mathcal{M}_A$.

The second problem arises with the Structure Theorem itself. Consider a quasi-bialgebra A; in [Dr1, page 1424], Drinfel'd introduced the notion of quasi-Hopf algebras in order to have that the category of finite dimensional left A-modules is rigid (i.e., every object admits a right dual object and a left dual object, just as vector spaces): they are quasi-bialgebras endowed with a triple (s, α, β) composed by an antiendomorphism s and two distinguished elements α and β that satisfies certain properties. Hausser and Nill proved in [HN, Theorem 3.8] that if A is a quasi-Hopf algebra, then a certain functor ${}_A\mathcal{M} \to {}_A\mathcal{M}_A^A$ is a category equivalence, i.e., there exists a generalization of the space of coinvariants such that every quasi-Hopf bimodule M is isomorphic to ${}_\bullet M^{\mathrm{Co}A} \otimes {}_\bullet A_\bullet^\bullet$. Since quasi-Hopf algebras seem to be a good generalization of Hopf algebras (cfr. [Dr1]), it is a little bit surprising to discover that the converse of this Structure Theorem needs not to be true.

Actually, there exists an example in the dual context, due to Schauenburg, of a dual quasi-bialgebra (dual quasi-bialgebras are also referred to as coquasi-bialgebras) for which the Structure Theorem holds, but that is not a dual quasi-Hopf algebra (cfr. [Sc1] and [Sc3, Example 4.5.1]).

Ardizzoni and Pavarin studied the topic in depth in [AP1] and they came to the conclusion that a correct generalization of the antipode to dual quasi-bialgebras is what they called a *preantipode*: a $\Bbbk$-linear map $S\colon A \to A$ satisfying certain properties. Here we fit what they got to the framework of quasi-bialgebras. Even if, at a first sight, it may seem just dualizing, things are not so easy. First of all, the dual of a dual quasi-bialgebra is not a quasi-bialgebra in general (unless we are in the finite dimensional case). Secondly, we will see in Section 3.2 that, unlikely the 'dual quasi' case, we don't have a pretty definition of the space of coinvariants that helps us in defining the adjunction between ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_A^A$ by taking inspiration from the ordinary Hopf version. On the contrary, our definition of coinvariants is a strict consequence of the Structure Theorem.

In details, the subsequent work is organized as follows.

In Chapter 1 we recall some basic notions of category theory and monoidal categories. In particular, we show explicitly that the category of vector spaces over the field $\Bbbk$ is a monoidal category.

In Chapter 2 we retrieve some classical results concerning Hopf algebras and we prove the Structure Theorem for Hopf modules we referred to at the very beginning.

Chapter 3 is devoted to quasi-bialgebras and the notion of preantipode. In Section 3.2 we prove the main result: the Fundamental Structure Theorem for quasi-Hopf bimodules. It states that the adjunction $-\otimes A\colon {}_A\mathcal{M} \to {}_A\mathcal{M}_A^A$ of Hausser and Nill, that sends a left A-module into the free quasi-Hopf bimodule ${}_\bullet M \otimes {}_\bullet A_\bullet^\bullet$, is an equivalence of categories if and only if A admits a preantipode, if and only if there exists a projection map $\tau_M\colon M \to M$ for every quasi-Hopf bimodule M that satisfies certain properties. In particular, every quasi-Hopf bimodule M is of the form $N \otimes A$, where N is a suitably defined space of coinvariants of M, namely $N = \tau_M(M)$.

In Section 3.3 we introduce quasi-Hopf algebras in order to show how the classical results are now consequences of the theory we developed. The cornerstone of this section is Theorem 3.3.4, which asserts that every quasi-Hopf algebra admits a preantipode. From this result we can recover the Structure Theorem for Hopf modules (Remark 3.3.7) and Hausser and Nill version of the Structure Theorem for quasi-Hopf bimodules (Remark 3.3.8) as corollaries. Unfortunately, and unlike the dual quasi case, we are not able to exhibit an explicit example of a quasi-bialgebra with preantipode that does not admit a quasi-antipode and so we cannot say with certainty that the two concepts don't coincide, though it is very unlikely to be so.

Nevertheless, even if it will turn out that the two are equivalent, we took a step forward. Indeed, on one hand we will show how the preantipode is actually more handy than the quasi-antipode. Primarily, because it is composed by a single data: the map $S\colon A \to A$. Secondly, because it is unique (see Theorem 3.3.11) and not just unique up to an invertible element (as the quasi-antipode is). On the other hand, we will be able to choose the one that fits better our needs because, unluckily, the preantipode is just a linear map: it is not an algebra nor a coalgebra antiendomorphism (cfr. Remark 3.3.16), while the quasi-antipode is an algebra antiendomorphism by definition and it can become a coalgebra antiendomorphism via a twist (cfr. [Dr1, Proposition 2] and the preceding discussion on page 1426).

Another important result that we were able to prove is Proposition 3.3.13, that states that quasi-bialgebras with preantipode form a class of bialgebras closed under gauge twisting.

Note that all these results argue in favour of the thesis that preantipodes are a more effective candidate for generalizing antipodes than quasi-antipodes (just in the case they don't coincide, obviously).

Even if we believe that quasi-bialgebras with preantipode are a strictly larger class of quasi-bialgebras with respect to quasi-Hopf algebras, we are able to exhibit a number of cases in which the two structures are equivalent. For example: ordinary bialgebras viewed as quasi via the trivial reassociator (Proposition 3.4.1), commutative quasi-bialgebras (Corollary 3.4.2) and, last but not the least, finite dimensional quasi-bialgebras. Indeed, the very last theorem of Chapter 3 is due to Schauenburg again and it shows that, at least in the finite dimensional case, the existence of a preantipode is equivalent to the existence of a quasi-antipode.

In some of this cases we are able to recover explicitly the quasi-antipode from the preantipode, as shown at the very end of Chapter 3, for example when the distinguished element α is invertible. We will also highlight that we can do it for much of the best known examples of non-trivial quasi-Hopf algebra. Nevertheless, up to this moment, we are not able to give general guidelines to recover the quasi-antipode from the preantipode, even in the finite dimensional case. The heart of the problem lies in the fact that Schauenburg's proof invokes the Krull-Schmidt Theorem and this is a non-constructive result. Hence, as we will see, the relation between the quasi-antipode and the preantipode stays hidden behind an unknown isomorphism $\widetilde{\gamma}$ (cfr. proof of Theorem 3.4.4).

We conclude by adding an appendix dedicated to prove the Krull-Schmidt Theorem (Chapter 4).

Contents

Chapter 1

Preliminaries

In this chapter we introduce some basic definitions and properties of category theory and, in particular, we will concentrate on the category of $\Bbbk$-vector spaces that forms the framework for the subsequent work. Throughout the text, $\Bbbk$ will always denote a field.

1.1 Categories

This section consists of a collection of definition and elementary properties of categories and functors that comes from the book of Kassel, [Ka, Chapter XI], and from Mac Lane's work, [ML]. For a more exhaustive treatment we refer to [ML].

Definition 1.1.1. (Category)
A *category* $\mathcal{C}$ consists of

1. a class $\mathrm{Ob}(\mathcal{C})$ of *objects* of the category,
2. a class $\hom(\mathcal{C})$ of *morphisms* (or *arrows*) of the category,
3. two operations
$$\hom(\mathcal{C}) \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$$
called *domain* and *codomain* (or *source* and *target*) which assign to each arrow f an object called, respectively, the *domain* of f and the *codomain* of f, and
4. two additional operations, Id and $\circ$, defined by the following assignments:
$$\begin{array}{rccc} \mathrm{Id}: & \mathrm{Ob}(\mathcal{C}) & \longrightarrow & \hom(\mathcal{C}) \\ & C & \longmapsto & \mathrm{Id}_C \end{array} \quad \text{and} \quad \begin{array}{rccc} \circ: & \hom(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \hom(\mathcal{C}) & \longrightarrow & \hom(\mathcal{C}) \\ & (g,f) & \longmapsto & g \circ f \end{array}$$
called *identity* and *composition* such that
$$\mathrm{cod}(\mathrm{Id}_C) = C = \mathrm{dom}(\mathrm{Id}_C), \quad \mathrm{dom}(g \circ f) = \mathrm{dom}(f), \quad \mathrm{cod}(g \circ f) = \mathrm{cod}(g)$$
for every object C in $\mathrm{Ob}(\mathcal{C})$ and for every composable pair of arrows (g,f) in $\hom(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \hom(\mathcal{C})$, where $\hom(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \hom(\mathcal{C})$ denotes the class of couples (g,f) of *composable* morphisms in the category, i.e., such that $\mathrm{dom}(g) = \mathrm{cod}(f)$.

Furthermore, the associativity and unit axioms must be satisfied:

(ass) For any morphisms f, g, h satisfying $\mathrm{dom}(g) = \mathrm{cod}(f)$ and $\mathrm{dom}(h) = \mathrm{cod}(g)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(un) For any morphism f in $\hom(\mathcal{C})$,

$$\mathrm{Id}_{\mathrm{cod}(f)} \circ f = f \circ \mathrm{Id}_{\mathrm{dom}(f)} = f.$$

As a matter of terminology, we will call *endomorphism* a morphism from an object to itself and *isomorphism* (or simply *iso*) a morphism that admits a two-sided inverse, i.e., $f\colon C \to D$ in $\hom(\mathcal{C})$ is an isomorphism if there exists $g\colon D \to C$ in $\hom(\mathcal{C})$ such that $g \circ f = \mathrm{Id}_C$ and $f \circ g = \mathrm{Id}_D$. Moreover, if C, D are objects in $\mathrm{Ob}(\mathcal{C})$ we will indicate with $\hom_{\mathcal{C}}(C, D)$ the *set* of morphisms of the category $\mathcal{C}$ whose domain is C and whose codomain is D; it could be abbreviated also in $\mathcal{C}(C, D)$. Note that we are requesting explicitly that given two objects C, D of $\mathcal{C}$, $\hom_{\mathcal{C}}(C, D)$ is a set, i.e., $\mathcal{C}$ is actually a *set-category*.

Remark 1.1.2. To each category $\mathcal{C}$ we can associate another category, called the *opposite category* $\mathcal{C}^{\mathrm{op}}$. The class of objects of $\mathcal{C}^{\mathrm{op}}$ is the class of objects of $\mathcal{C}$, i.e., $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$, and the morphisms of $\mathcal{C}^{\mathrm{op}}$ are given by

$$\hom_{\mathcal{C}^{\mathrm{op}}}(C, D) := \hom_{\mathcal{C}}(D, C)$$

for each pair of objects C, D of $\mathcal{C}^{\mathrm{op}}$. The composition operation in the opposite category is defined by:

$$\begin{array}{rccc} \circ : & \hom(\mathcal{C}^{\mathrm{op}}) \times_{\mathrm{Ob}(\mathcal{C}^{\mathrm{op}})} \hom(\mathcal{C}^{\mathrm{op}}) & \longrightarrow & \hom(\mathcal{C}^{\mathrm{op}}) \\ & (g, f) & \longmapsto & g \circ f \end{array}$$

where $g \circ f$ is performed in $\mathcal{C}$. Bluntly speaking, the opposite category is just the category $\mathcal{C}$ with all arrows reversed. In order to avoid confusion, when we look at the morphism $f\colon D \to C$ in $\mathcal{C}$ as a morphism in $\mathcal{C}^{\mathrm{op}}$, we denote it by $f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$.

In order to simplify the subsequent treatment, we are going to concentrate on categories that are interpretations of the category axioms within set theory. This means that $\mathrm{Ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ will be sets, dom, cod, Id and $\circ$ will be functions and we will even write simply $C \in \mathcal{C}$ or $f \in \mathcal{C}$ to mean that C is an object of $\mathcal{C}$ and f is a morphism of $\mathcal{C}$ respectively (it will be always clear if we are considering an object or a morphism).

Example 1.1.3. Let us retrieve some examples of such categories:

Set: Objects: all small[1] sets; arrows: all functions between them.

Grp: Objects: all small groups; arrows: all morphisms of groups.

Ab: Objects: all small abelian groups; arrows: all morphisms of abelian groups.

[1]Assume that there exists a 'big enough' set U, that we can call the *universe*. We describe a set as *small* if it is a member of the universe (cfr. [ML, Section I.6]).

Mod-R: Objects: all small right R-modules over the ring R; arrows: all R-linear morphisms.

Vect($\Bbbk$): Objects: all small vector spaces over the field $\Bbbk$; arrows: all linear maps of vector spaces.

Top: Objects: all small topological spaces; arrows: continuous maps.

For more examples of this kind and also for some examples of categories whose objects are not sets we refer to [ML].

Definition 1.1.4. (Functors and natural transformations)
A *functor* is a morphism of categories. In details, a functor $F\colon \mathcal{C} \to \mathcal{D}$ from the category $\mathcal{C}$ to the category $\mathcal{D}$ consists of two related functions:

- a map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, called *object function*, that assigns to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$, and
- a map $F\colon \hom(\mathcal{C}) \to \hom(\mathcal{D})$, called *arrow function*, which assigns to each arrow $f\colon C \to C'$ in $\mathcal{C}$ an arrow $F(f)\colon F(C) \to F(C')$ in $\mathcal{D}$,

such that

$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f),$$

the latter whenever the composite $g \circ f$ is defined in $\mathcal{C}$.

Let F, G be two parallel functors between the category $\mathcal{C}$ and the category $\mathcal{D}$. A *natural transformation* $\eta\colon F \to G$ from F to G (also referred to as *morphism of functors*) is a family of morphisms $\eta_C\colon F(C) \to G(C)$ in $\mathcal{D}$ indexed by the objects C of $\mathcal{C}$ such that, for any morphism $f\colon C \to D$ in $\mathcal{C}$, the square:

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\eta_D"'] & G(D) \end{tikzcd}$$

commutes. We call η_C, η_D, ... the *components* of the natural transformation. Furthermore, if any component of a natural transformation is an isomorphism, then we call it a *natural isomorphism*.

Note that if $\eta\colon F \to G$ is a natural isomorphism, then also $\eta^{-1}\colon G \to F$ is.

Remark 1.1.5. The definition we gave above coincides with what is commonly known as a *covariant functor*, in the sense that it preserves the order of compositions. There also exists in literature the concept of *contravariant functors*. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *contravariant* if $F(g \circ f) = F(f) \circ F(g)$ for all composable pairs (g, f) of morphisms of $\mathcal{C}$ (i.e., it reverses compositions). We do not put too much emphasis on this idea because a contravariant functor can be seen as a simple covariant functor from the opposite category: $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Definition 1.1.6. (Isomorphism of categories, fullness and faithfulness)
An *isomorphism of categories* is a functor that is a bijection both on objects and on arrows. Equivalently, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism if and only if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ for which both composites $F \circ G$ and $G \circ F$ are identity functors.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *full* if the associated arrow function is surjective. It is *faithful* if the associated arrow function is injective.

Definition 1.1.7. (Category equivalence)
A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* (and the categories are said to be *equivalent*) when there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms

$$\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF \quad \text{and} \quad \epsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}.$$

Definition 1.1.8. (Adjoint functors)
Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. Then G is *right adjoint* to F or F is *left adjoint* to G if there exist natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF$ (called the *unit*) and $\epsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$ (called the *counit*) such that the following diagrams commute (*triangular identities*):

$$\begin{tikzcd} GFG(D) \arrow[r, "G(\epsilon_D)"] & G(D) \\ G(D) \arrow[u, "\eta_{G(D)}"] \arrow[ur, "\mathrm{Id}_{G(D)}"'] & \end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd} FGF(C) \arrow[r, "\epsilon_{F(C)}"] & F(C) \\ F(C) \arrow[u, "F(\eta_C)"] \arrow[ur, "\mathrm{Id}_{F(C)}"'] & \end{tikzcd} \tag{1.2}$$

We will write $(F, G, \eta, \epsilon)\colon \mathcal{C} \rightharpoonup \mathcal{D}$ or, simply, (F, G, η, ϵ).

If both the unit and the counit of the adjunction are natural isomorphisms, we call it an *adjoint equivalence*.

Theorem 1.1.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. (F, G) is an adjunction if and only if there exists a natural isomorphism:*

$$\varphi_{C,D}\colon\ \hom_{\mathcal{D}}(F(C), D) \to \hom_{\mathcal{C}}(C, G(D))$$

that is natural in both components.

Proof. For the 'only if' part, assume that (F, G, η, ϵ) is an adjunction and let

$$f\colon F(C) \to D$$

be a morphism in $\hom_{\mathcal{D}}(F(C), D)$. Consider $G(f)\colon GF(C) \to G(D)$ and compose it with $\eta_C\colon C \to GF(C)$:

$$\varphi_{C,D}(f) := \Big(C \xrightarrow{\eta_C} GF(C) \xrightarrow{G(f)} G(D)\Big) \tag{1.3}$$

Let us show that this φ is a natural isomorphism in both components.

- Naturality in the first component. Let $g\colon B \to C \in \mathcal{C}(B, C)$ (this is just a shortcut for $\hom_{\mathcal{C}}(B, C)$). Consider the diagram:

$$\begin{tikzcd}
\mathcal{D}(F(C), D) \arrow[r, "\varphi_{C,D}"] \arrow[d, "-\circ F(g)"'] & \mathcal{C}(C, G(D)) \arrow[d, "-\circ g"] \\
\mathcal{D}(F(B), D) \arrow[r, "\varphi_{B,D}"'] & \mathcal{C}(B, G(D))
\end{tikzcd}$$

This is commutative, since for all f in $\mathcal{D}(F(C), D)$:

$$\varphi_{B,D}(f \circ F(g)) = G(f \circ F(g)) \circ \eta_B = G(f) \circ GF(g) \circ \eta_B \overset{(*)}{=} G(f) \circ \eta_C \circ g = \varphi_{C,D}(f) \circ g,$$

where in $(*)$ we used the naturality of η.

- Naturality in the second component. Let $g\colon D \to E \in \mathcal{D}(D, E)$. Consider the diagram:

$$\begin{tikzcd}
\mathcal{D}(F(C), D) \arrow[r, "\varphi_{C,D}"] \arrow[d, "g\circ -"'] & \mathcal{C}(C, G(D)) \arrow[d, "G(g)\circ -"] \\
\mathcal{D}(F(C), E) \arrow[r, "\varphi_{C,E}"'] & \mathcal{C}(C, G(E))
\end{tikzcd}$$

This is commutative, since for all f in $\mathcal{D}(F(C), D)$:

$$\varphi_{C,E}(g \circ f) = G(g \circ f) \circ \eta_C = G(g) \circ G(f) \circ \eta_c = G(g) \circ \varphi_{C,D}(f).$$

- Define, for all C in $\mathcal{C}$, D in $\mathcal{D}$ and g in $\mathcal{C}(C, G(D))$,

$$\psi_{C,D}(g) := \left(F(C) \xrightarrow{F(g)} FG(D) \xrightarrow{\epsilon_D} D\right) \tag{1.4}$$

For all $f\colon C \to G(D)$ we have that:

$$\begin{aligned}
\varphi(\psi(f)) &= \varphi(\epsilon_D \circ F(f)) = G(\epsilon_D \circ F(f)) \circ \eta_C = \\
&= G(\epsilon_D) \circ GF(f) \circ \eta_C \overset{(*)}{=} G(\epsilon_D) \circ \eta_{G(D)} \circ f \overset{(1.1)}{=} f.
\end{aligned}$$

where in $(*)$ we used the naturality of η again. In the same way, for all $f\colon F(C) \to D$ we have that:

$$\begin{aligned}
\psi(\varphi(f)) &= \psi(G(f) \circ \eta_C) = \epsilon_D \circ F(G(f) \circ \eta_C) = \\
&= \epsilon_D \circ FG(f) \circ F(\eta_C) \overset{(**)}{=} f \circ \epsilon_{F(C)} \circ F(\eta_C) \overset{(1.2)}{=} f.
\end{aligned}$$

where in $(**)$ we used the naturality of ϵ.

Conversely, for the 'if' part, define

$$\eta_C := \varphi_{C,F(C)}(\mathrm{Id}_{F(C)}) \quad \text{and} \quad \epsilon_D := \varphi^{-1}_{G(D),D}(\mathrm{Id}_{G(D)}) \tag{1.5}$$

These are natural:

- Let $f\colon C \to B \in \mathcal{C}(C,B)$. Since φ is natural in both components, we get that:

  $$GF(f) \circ \eta_C = GF(f) \circ \varphi_{C,F(C)}(\mathrm{Id}_{F(C)}) = \varphi_{C,F(B)}(F(f) \circ \mathrm{Id}_{F(C)}) = \varphi_{C,F(B)}(F(f))$$

  by naturality on the second component, and:

  $$\varphi_{C,F(B)}(F(f)) = \varphi_{C,F(B)}(\mathrm{Id}_{F(B)} \circ F(f)) = \eta_B \circ f$$

  by naturality on the first component. This means that the following diagram commutes:

  $$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & GF(C) \arrow[d, "GF(f)"] \\ B \arrow[r, "\eta_B"'] & GF(B) \end{tikzcd}$$

- Analogously, let $g\colon D \to E \in \mathcal{D}(D,E)$. Since φ^{-1} is natural in both components, we get that:

  $$g \circ (\varphi_{G(D),D})^{-1}(\mathrm{Id}_{G(D)}) = (\varphi_{G(D),E})^{-1}(G(g))$$

  by naturality on the second component, and:

  $$(\varphi_{G(D),E})^{-1}(G(g)) = (\varphi_{G(D),E})^{-1}(\mathrm{Id}_{G(E)} \circ G(g)) = \epsilon_E \circ FG(g)$$

  by naturality on the first component. That means that the following diagram commutes:

  $$\begin{tikzcd} FG(D) \arrow[r, "\epsilon_D"] \arrow[d, "FG(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "g"] \\ FG(E) \arrow[r, "\epsilon_E"'] & E \end{tikzcd}$$

And satisfy the triangular identities:

- Let $f\colon B \to C \in \mathcal{C}$ and $g\colon D \to E \in \mathcal{D}$. By naturality of φ:

  $$\varphi_{B,D}(- \circ F(f)) = \varphi_{C,D}(-) \circ f \quad \text{and} \quad \varphi_{C,E}(g \circ -) = G(g) \circ \varphi_{C,D}(-).$$

  Hence:

  $$\begin{aligned} G(\epsilon_D) \circ \eta_{G(D)} = G(\epsilon_D) \circ \varphi_{G(D),FG(D)}(\mathrm{Id}_{FG(D)}) = \varphi_{G(D),D}(\epsilon_D \circ \mathrm{Id}_{FG(D)}) = \\ = \varphi_{G(D),D}((\varphi_{G(D),D})^{-1}(\mathrm{Id}_{G(D)})) = \mathrm{Id}_{G(D)} \end{aligned}$$

  and (1.1) follows.

- In the same way, let $f\colon B \to C \in \mathcal{C}$ and $g\colon D \to E \in \mathcal{D}$. By naturality of φ^{-1}:

  $$\begin{aligned} &(\varphi_{B,D})^{-1}(- \circ f) = (\varphi_{C,D})^{-1}(-) \circ F(f) \quad \text{and} \\ &(\varphi_{C,E})^{-1}(G(g) \circ -) = g \circ (\varphi_{C,D})^{-1}(-). \end{aligned}$$

  Therefore:

  $$\begin{aligned} \epsilon_{F(C)} \circ F(\eta_C) = (\varphi_{GF(C),C})^{-1}(\mathrm{Id}_{GF(C)}) \circ F(\eta_C) = (\varphi_{C,F(C)})^{-1}(\eta_C) = \\ = (\varphi_{C,F(C)})^{-1}(\varphi_{C,F(C)}(\mathrm{Id}_{F(C)})) = \mathrm{Id}_{F(C)} \end{aligned}$$

  and (1.2) follows as well.

$\square$

Remark 1.1.10. At the very beginning of Section 1.2 we will introduce the concept of product of categories. With this notion, it will be easy to see that the isomorphism φ of the previous theorem is just a natural isomorphism between the functors:

$$\varphi\colon \mathcal{D}(F(-),*) \to \mathcal{C}(-,G(*))$$

where

$$\mathcal{D}(*,*)\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$$

associate to each pair of objects (C,D) of $\mathcal{D}$ the set $\mathcal{D}(C,D)$ and to every pair of morphisms $(f^{\mathrm{op}},g)\colon (C,D) \to (C',D')$ the function:

$$\mathcal{D}(f,g)\colon \mathcal{D}(C,D) \longrightarrow \mathcal{D}(C',D')\colon \sigma \longmapsto g \circ \sigma \circ f.$$

Analogously $\mathcal{C}(-,-)$.

There exists another useful characterization of adjoints, that depends on the following definition.

Definition 1.1.11. (Dual to [ML, Section III.1]) (Universal arrows)
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and let $D \in \mathrm{Ob}(\mathcal{D})$ be an object. A *universal arrow* from $F(-)$ to D is a pair (E,u) consisting of an object E of $\mathcal{C}$ and an arrow $u\colon F(E) \to D$ in $\mathcal{D}(F(E),D)$ that enjoys the following universal property: for each pair (C,f) consisting of an object C of $\mathcal{C}$ and an arrow $f\colon F(C) \to D$ in $\mathcal{D}(F(C),D)$ there exists a unique arrow $\widetilde{f}\colon C \to E$ in $\mathcal{C}(C,E)$ such that the following diagram commutes:

$$\begin{tikzcd} F(E) \arrow[dr, "u"] & \\ & D \\ F(C) \arrow[uu, "F(\widetilde{f})"] \arrow[ur, "f"'] & \end{tikzcd}$$

In other words, every arrow f from $F(-)$ to D factors uniquely through the universal arrow u.

Corollary 1.1.12. (cfr. [ML, Theorem IV.1.2, page 83]) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is a left adjoint if and only if for each object $D \in \mathcal{D}$ there exists a universal arrow (E_D,u_D) from $F(-)$ to D.*

Proof. For the 'only if' part assume that there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF(-)$ and $\epsilon\colon FG(-) \to \mathrm{Id}_{\mathcal{D}}$ such that (F,G,η,ϵ) is an adjunction. Let us prove that for each $D \in \mathcal{D}$, $(G(D),\epsilon_D)$ is universal from $F(-)$ to D. In view of Theorem 1.1.9, for each pair (C,f) consisting of an object of $\mathcal{C}$ and an arrow $f\colon F(C) \to D$ there exists a unique arrow $\widetilde{f} := \varphi_{C,D}(f)\colon C \to G(D)$ defined as (cfr. (1.3))

$$\varphi_{C,D}(f) = G(f) \circ \eta_C.$$

Furthermore, the following diagram commutes

$$\begin{tikzcd}
FG(D) \arrow[r, "\epsilon_D"] & D \\
FG(F(C)) \arrow[u, "FG(f)"] \arrow[r, "\epsilon_{F(C)}"] & F(C) \arrow[u, "f"'] \\
F(C) \arrow[u, "F(\eta_C)"] \arrow[ur, "\mathrm{Id}_{F(C)}"'] &
\end{tikzcd}$$

by naturality of ϵ and (1.2). Thus:

$$\epsilon_D \circ F\left(\widetilde{f}\right) = \epsilon_D \circ FG(f) \circ F(\eta_C) = f$$

and $(G(D), \epsilon_D)$ is universal as claimed.

For the 'if' part, assume that for each $D \in \mathrm{Ob}(\mathcal{D})$ there exists a universal arrow (E_D, u_D) from $F(-)$ to D and let us construct a functor $G\colon \mathcal{D} \to \mathcal{C}$ and a natural isomorphism

$$\psi_{C,D}\colon \hom_{\mathcal{C}}(C, G(D)) \to \hom_{\mathcal{D}}(F(C), D).$$

First of all, consider the assignment

$$G\colon \mathcal{D} \longrightarrow \mathcal{C}\colon D \longmapsto E_D.$$

In order to define how G operates on morphisms, pick $g\colon D \to D'$ in $\mathcal{D}(D, D')$ and consider the following diagram:

$$\begin{tikzcd}
F(E_{D'}) \arrow[r, "u_{D'}"] & D' \\
F(E_D) \arrow[u, dashed, "F(\widehat{g})"] \arrow[ur, "g \circ u_D"] \arrow[r, "u_D"'] & D \arrow[u, "g"']
\end{tikzcd}$$

By the universal property of $(E_{D'}, u_{D'})$ there exists a unique morphism $\widehat{g}\colon E_D \to E_{D'}$ such that the above diagram commutes. Hence we can define

$$G\colon \mathcal{D}(D, D') \longrightarrow \mathcal{C}(G(D), G(D'))\colon g \longmapsto \widehat{g}$$

for all $D, D' \in \mathrm{Ob}(\mathcal{D})$. Observe that in this way we automatically get that

$$u\colon FG(-) \to \mathrm{Id}_{\mathcal{D}}$$

is natural. Let us show that this G is a functor. By functoriality of F we can note that:

- the identity Id_{E_D} makes the following diagram commutes:

$$\begin{tikzcd}
F(E_D) \arrow[r, "u_D"] & D \\
F(E_D) \arrow[u, "F(\mathrm{Id}_{E_D})"] \arrow[r, "u_D"'] \arrow[ur, phantom, "\circlearrowleft"] & D \arrow[u, "\mathrm{Id}_D"']
\end{tikzcd}$$

  and so, by uniqueness, $G(\mathrm{Id}_D) = \widehat{\mathrm{Id}_D} = \mathrm{Id}_{E_D} = \mathrm{Id}_{G(D)}$.

- for $f\colon D \to D'$ and $g\colon D' \to D''$ in $\hom(\mathcal{D})$ the following diagram commutes:

$$\begin{tikzcd}
F(E_D) \arrow[r, "u_D"] \arrow[d, "F\left(\widehat{f}\right)"] \arrow[dd, bend right, "F\left(\widehat{g\circ f}\right)"'] & D \arrow[d, "f"'] \arrow[dd, bend left, "g\circ f"] \\
F(E_{D'}) \arrow[r, "u_{D'}"] \arrow[d, "F(\widehat{g})"] & D' \arrow[d, "g"'] \\
F(E_{D''}) \arrow[r, "u_{D''}"] & D''
\end{tikzcd}$$

Since $F\left(\widehat{g}\right) \circ F\left(\widehat{f}\right) = F\left(\widehat{g} \circ \widehat{f}\right)$, we have two morphisms that satisfy the universal property: $\widehat{g \circ f}$ and $\widehat{g} \circ \widehat{f}$. Thus, $G(g \circ f) = G(g) \circ G(f)$.

Next, since (E_D, u_D) is universal from $F(-)$ to D for every $D \in \mathrm{Ob}(\mathcal{D})$, we have that for all $g\colon F(C) \to D$ there exists a unique morphism $\widetilde{g}\colon C \to G(D)$ such that

$$\begin{tikzcd}
FG(D) \arrow[r, "u_D"] & D \\
F(C) \arrow[u, "F(\widetilde{g})"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. Therefore, the assignment

$$\psi_{C,D}\colon \hom_{\mathcal{C}}(C, G(D)) \longrightarrow \hom_{\mathcal{D}}(F(C), D)\colon f \longmapsto u_D \circ F(f)$$

is a well defined isomorphism, natural in both components (the proof of the naturality is actually the same that we gave in the ‘only if’ part of Theorem 1.1.9 for $\varphi_{C,D}$, except that here we should use naturality of u). By virtue of Theorem 1.1.9 itself, we conclude. $\square$

Remark 1.1.13. In view of (1.5) in the proof of Theorem 1.1.9 we have that the unit and the counit of the adjunction that we get from the universal arrows are given explicitly by:

$$\eta_C = \psi^{-1}_{C,F(C)}\left(\mathrm{Id}_{F(C)}\right), \tag{1.6a}$$

$$\epsilon_D = \psi_{G(D),D}\left(\mathrm{Id}_{G(D)}\right) = u_D \circ F\left(\mathrm{Id}_{G(D)}\right) = u_D. \tag{1.6b}$$

Actually, by definition, η_C is the unique arrow from C to $GF(C)$ such that

$$\epsilon_{F(C)} \circ F(\eta_C) = \mathrm{Id}_{F(C)} \tag{1.7}$$

for all objects C in $\mathcal{C}$.

Observe that if (F, G, η, ϵ) is an adjoint equivalence, then the two functors that compose the adjunction are category equivalences. The apparently unexpected claim is that also the converse holds, as the subsequent result states.

Theorem 1.1.14. ([ML, Theorem IV.4.1, page 93]) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then the following assertions are equivalent:*

(1) F is an equivalence of categories,

(2) F is part of an adjoint equivalence (F, G, η, ϵ),

(3) F is fully faithful and each object $D \in \mathcal{D}$ is isomorphic to $F(C)$ for some object $C \in \mathcal{C}$. In this case we say that F is essentially surjective (on objects)*.*

Proof. We already observed that (2) implies (1). To prove that (1) implies (3), note that since there exists a natural isomorphism $\epsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$, for every object $D \in \mathcal{D}$ $D \cong F(G(D))$ and so F is essentially surjective. Moreover, naturality of η gives for each $f\colon C \to C'$ a commutative square:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd}$$

Hence, $f = \eta_{C'}^{-1} \circ GF(f) \circ \eta_C$ and so F is faithful: if $F(f) = F(g)$, then

$$f = \eta_{C'}^{-1} \circ GF(f) \circ \eta_C = \eta_{C'}^{-1} \circ GF(g) \circ \eta_C = g.$$

Analogously, naturality of ϵ ensures that also G is faithful.

To show that F is full, pick a morphism $h\colon F(C) \to F(C')$ and define

$$f := \eta_{C'}^{-1} \circ G(h) \circ \eta_C.$$

By construction, we have two morphisms from $GF(C)$ to $GF(C')$, namely $GF(f)$ and $G(h)$, such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & GF(C) \arrow[d, shift right, "G(h)"'] \arrow[d, shift left, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd}$$

Therefore, since η is a natural isomorphism, we get that $GF(f) = G(h)$ and, by faithfulness of G, $F(f) = h$.

Now, let us prove that (3) implies (2). For any object $D \in \mathcal{D}$, we know that there exists (at least) a pair (E_D, u_D) where E_D is an object in $\mathcal{C}$ and $u_D\colon F(E_D) \to D$ is an isomorphism in $\hom(\mathcal{D})$, since F is essentially surjective on objects. Moreover, if $f\colon F(C) \to D$ is any other morphism from $F(C)$ to D for some $C \in \mathrm{Ob}(\mathcal{C})$, we can consider the composition $u_D^{-1} \circ f$ that belongs to $\mathcal{D}(F(C), F(E_D))$. Since F is fully faithful, it is bijective on morphisms, and so there exists a unique morphism $\widetilde{f}\colon C \to E_D$ such that $F\left(\widetilde{f}\right) = u_D^{-1} \circ f$. This means exactly that (E_D, u_D) is universal from $F(-)$ to

D. In view of Corollary 1.1.12 and relation (1.6b) we can then construct an adjunction (F, G, η, ϵ) where

$$G\colon \mathcal{D} \longrightarrow \mathcal{C}\colon D \longmapsto E_D,$$
$$G\colon \mathcal{D}(D, D') \longrightarrow \mathcal{C}(G(D), G(D'))\colon g \longmapsto \widetilde{g \circ u_D}$$

and

$$\epsilon_D := u_D\colon FG(D) \to D \qquad (\forall\, D \in \mathrm{Ob}(\mathcal{D}))\,.$$

Furthermore, by (1.7) and since ϵ is a natural isomorphism, we have that $F(\eta_C)$ is invertible for all C, with inverse given by $\epsilon_{F(C)}$. Hence even η_C is invertible, because F is fully faithful, and so (F, G, η, ϵ) is an adjoint equivalence. $\square$

Theorem 1.1.15. ([ML, Theorem IV.8.1, page 103]) *Given two adjunctions*

$$(F, G, \eta, \epsilon)\colon \mathcal{C} \rightharpoonup \mathcal{D} \qquad \text{and} \qquad (\overline{F}, \overline{G}, \overline{\eta}, \overline{\epsilon})\colon \mathcal{D} \rightharpoonup \mathcal{E}$$

the composite functors yield an adjunction

$$(\overline{F}\,F, G\,\overline{G}, G\,\overline{\eta}\,F \circ \eta, \overline{\epsilon} \circ \overline{F}\,\epsilon\,\overline{G})\colon \mathcal{C} \rightharpoonup \mathcal{E}.$$

Proof. Define

$$\epsilon' = \left(\overline{F}\,F\,G\,\overline{G} \xrightarrow{\overline{F}\,\epsilon\,\overline{G}} \overline{F}\,\overline{G} \xrightarrow{\overline{\epsilon}} \mathrm{Id}_{\mathcal{E}}\right)$$

and

$$\eta' = \left(\mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta} GF \xrightarrow{G\,\overline{\eta}\,F} G\,\overline{G}\,\overline{F}\,F\right)$$

Let $E, E' \in \mathrm{Ob}(\mathcal{E})$, $f\colon E \to E'$ in $\hom(\mathcal{E})$ and consider the diagram:

$$\begin{tikzcd}
\overline{F}(FG(\overline{G}(E))) \arrow[r, "\overline{F}\left(\epsilon_{\overline{G}(E)}\right)"] \arrow[d, "\overline{F}(FG(\overline{G}(f)))"'] & \overline{F}(\overline{G}(E)) \arrow[r, "\overline{\epsilon}_E"] \arrow[d, "\overline{F}(\overline{G}(f))"] & E \arrow[d, "f"] \\
\overline{F}(FG(\overline{G}(E'))) \arrow[r, "\overline{F}\left(\epsilon_{\overline{G}(E')}\right)"'] & \overline{F}(\overline{G}(E')) \arrow[r, "\overline{\epsilon}_{E'}"'] & E'
\end{tikzcd}$$

The external path represents naturality of ϵ'. The left hand square commutes by naturality of ϵ applied to $\overline{G}(f)\colon \overline{G}(E) \to \overline{G}(E')$ and functoriality of $\overline{F}$. The right hand square commutes by naturality of $\overline{\epsilon}$.

Analogously, let $C, C' \in \mathrm{Ob}(\mathcal{C})$, $f\colon C \to C'$ in $\hom(\mathcal{C})$ and consider the diagram:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "f"'] & G(F(C)) \arrow[r, "G\left(\overline{\eta}_{F(C)}\right)"] \arrow[d, "G(F(f))"] & G(\overline{G}\,\overline{F}(F(C))) \arrow[d, "G(\overline{G}\,\overline{F}(F(f)))"] \\
C' \arrow[r, "\eta_{C'}"'] & G(F(C')) \arrow[r, "G\left(\overline{\eta}_{F(C')}\right)"'] & G(\overline{G}\,\overline{F}(F(C')))
\end{tikzcd}$$

The external path represents naturality of η'. The left hand square commutes by naturality of η. The right hand square commutes by naturality of $\overline{\eta}$ applied to $F(C)$ and functoriality of G. What is left is to prove that the triangular identities are satisfied.

(1.1): for $E \in \mathcal{E}$ the following diagram, that represents the first triangular identity,

$$\begin{tikzcd}
G\,\overline{G}(E) \arrow[r, "\eta_{G\,\overline{G}(E)}"] \arrow[d, "\mathrm{Id}_{G\,\overline{G}(E)}"'] & G\,F\,G\,\overline{G}(E) \arrow[dl, dashed, "G\left(\epsilon_{\overline{G}(E)}\right)"'] \arrow[d, "G\left(\overline{\eta}_{FG\overline{G}(E)}\right)"] \\
G\,\overline{G}(E) \arrow[d, "\mathrm{Id}_{G\,\overline{G}(E)}"'] \arrow[dr, dashed, "G\left(\overline{\eta}_{\overline{G}(E)}\right)"] & G\,\overline{G}\,\overline{F}\,F\,G\,\overline{G}(E) \arrow[d, "G\,\overline{G}\,\overline{F}\left(\epsilon_{\overline{G}(E)}\right)"] \\
G\,\overline{G}(E) & G\,\overline{G}\,\overline{F}\,\overline{G}(E) \arrow[l, "G\,\overline{G}(\overline{\epsilon}_E)"]
\end{tikzcd}$$

commutes. Indeed, the upper left triangle is the triangular identity (1.1) satisfied by the first adjunction applied to $\overline{G}(E)$ and so commutes. The lower left triangle commutes by functoriality of G and because it represents the same triangular identity satisfied by the second adjunction applied to E. The central square commutes by naturality of $\overline{\eta}$ applied to $\epsilon_{\overline{G}(E)} \colon FG(\overline{G}(E)) \to \overline{G}(E)$.

(1.2): for $C \in \mathcal{C}$ the following diagram, that represents the second triangular identity,

$$\begin{tikzcd}
\overline{F}\,F(C) \arrow[r, "\overline{F}\,F(\eta_C)"] \arrow[d, "\mathrm{Id}_{\overline{F}\,F(C)}"'] & \overline{F}\,F\,G\,F(C) \arrow[dl, dashed, "\overline{F}(\epsilon_{F(C)})"'] \arrow[d, "\overline{F}\,F\,G(\overline{\eta}_{F(C)})"] \\
\overline{F}\,F(C) \arrow[d, "\mathrm{Id}_{\overline{F}\,F(C)}"'] \arrow[dr, dashed, "\overline{F}(\overline{\eta}_{F(C)})"] & \overline{F}\,F\,G\,\overline{G}\,\overline{F}\,F(C) \arrow[d, "\overline{F}\left(\epsilon_{\overline{G}\,\overline{F}\,F(C)}\right)"] \\
\overline{F}\,F(C) & \overline{F}\,\overline{G}\,\overline{F}\,F(C) \arrow[l, "\overline{\epsilon}_{\overline{F}\,F(C)}"]
\end{tikzcd}$$

commutes, too. As above, the upper left triangle commutes by functoriality of $\overline{F}$ and by the triangular identity (1.2) satisfied by (F, G, η, ϵ). The lower left triangle represents (1.2) again for $(\overline{F}, \overline{G}, \overline{\eta}, \overline{\epsilon})$ applied to $F(C)$. Finally, the central square commutes by naturality of ϵ and functoriality of $\overline{F}$. $\square$

We will encounter several examples of adjunctions throughout the text.

Definition 1.1.16. ([ML, Section I.5])(Monic, epi, split monic, split epi)
Let $\mathcal{C}$ be a category. A morphism $f \colon C \to D$ in $\hom(\mathcal{C})$ is *monic* when for any two

parallel arrows $g, h\colon D \to C$ the equality $f \circ g = f \circ h$ implies $g = h$. In other words, if f is *left cancellable*.

A morphism $f\colon C \to D$ in $\hom(\mathcal{C})$ is *epi* when for any two parallel arrows $g, h\colon D \to C$ the equality $g \circ f = h \circ f$ implies $g = h$. In other words, if f is *right cancellable*.

For a morphism $f\colon C \to D$, a *right inverse* or *section* is a morphism $r\colon D \to C$ with $f \circ r = \mathrm{Id}_D$. A *left inverse* or *retraction* is a morphism $l\colon D \to C$ such that $l \circ f = \mathrm{Id}_C$.

If $f\colon C \to D$ and $g\colon D \to C$ are two morphisms in $\hom \mathcal{C}$ such that $g \circ f = \mathrm{Id}_C$, then f is a *split monic*, g is a *split epi*, and the composite $h := f \circ g$ is defined and is an *idempotent* (i.e., $h \circ h = h$).

Proposition 1.1.17. *Let (F, G, η, ϵ) be an adjunction between $\mathcal{C}$ and $\mathcal{D}$. F is faithful if and only if the unit η_C is monic, for all $C \in Ob(\mathcal{C})$.*

Proof. (F, G) is an adjunction if and only if $\mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D))$ via φ defined by

$$\varphi_{C,D}(f) = G(f) \circ \eta_C$$

for all $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f \in \mathcal{D}(F(C), D)$.
Let us consider the following diagram:

$$\begin{tikzcd}
\mathcal{C}(B,C) \arrow[rr, "F"] \arrow[dr, "\eta_C \circ -"'] & & \mathcal{D}(F(B),F(C)) \arrow[dl, "\varphi"] \\
& \mathcal{C}(B, GF(C)) &
\end{tikzcd}$$

It is commutative by naturality of η:

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & F(f) \arrow[d, mapsto] \\
\eta_C \circ f \arrow[r, equal] & GF(f) \circ \eta_B
\end{tikzcd}$$

As a consequence we have that if F is faithful, then it is injective between hom-sets and so $\eta_C \circ -$ is injective as well, by composition, for all $C \in \mathcal{C}$. That means that for all $f, g \in \mathcal{C}(B, C)$, $\eta_C \circ f = \eta_C \circ g$ implies $f = g$, i.e. η_C is monic.

Conversely, again by commutativity of the diagram, if η_C is monic, then $\eta_C \circ -$ is injective and so is F between hom-sets (since φ is bijective). $\square$

Proposition 1.1.18. *Let (F, G, η, ϵ) be an adjunction. F is full if and only if η_C is split epi for all $C \in Ob(\mathcal{C})$.*

Proof. Let η_C be split epi for all $C \in \mathcal{C}$. That means that there exists $\gamma_C\colon GF(C) \to C$ such that $\eta_C \circ \gamma_C = \mathrm{Id}_{GF(C)}$.
Let $g\colon F(B) \to F(C)$. Since $\varphi_{B,F(C)}$ is bijective, there exists some $f\colon B \to GF(C)$ such that $g = (\varphi_{B,F(C)})^{-1}(f)$. Moreover, by (1.4) in Theorem 1.1.9:

$$(\varphi_{B,F(C)})^{-1}(f) = \epsilon_{F(C)} \circ F(f) \tag{1.8}$$

On the other hand, by (1.2),

$$\mathrm{Id}_{F(C)} = \epsilon_{F(C)} \circ F(\eta_C).$$

If we compose on the right, on both sides, with $F(\gamma_C)$, we get $F(\gamma_C) = \epsilon_{F(C)}$. Substituting in (1.8):

$$g = (\varphi_{B,F(C)})^{-1}(f) = \epsilon_{F(C)} \circ F(f) = F(\gamma_C) \circ F(f) = F(\gamma_C \circ f).$$

Therefore, F is surjective on hom-sets (full).
Conversely, let F be full. For all $C \in \mathcal{C}$ consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{C}(GF(C), C) \arrow[r, "F"] \arrow[dr, "\eta_C \circ -"'] & \mathcal{D}(FGF(C), F(C)) \arrow[d, "\varphi"] \\
& \mathcal{C}(GF(C), GF(C))
\end{tikzcd}$$

Since $\varphi_{GF(C),F(C)}$ is iso, there exists $\gamma\colon GF(C) \to C$ such that

$$\mathrm{Id}_{GF(C)} = \varphi_{GF(C),F(C)}(F(\gamma)) = \eta_C \circ \gamma,$$

i.e., η_C is split epi for all $C \in \mathcal{C}$. $\square$

Corollary 1.1.19. *Let (F, G, η, ϵ) be an adjunction. F is fully faithful if and only if η_C is iso for all $C \in \mathcal{C}$.*

1.2 Monoidal Categories

From now on we will indicate the generic objects of a category with letters M, N, P, $Q, \ldots$, in order to avoid confusion when we will introduce the coalgebras.

We define the *product of two categories* $\mathcal{C}$ and $\mathcal{D}$ to be the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects (M, N), where $M \in \mathrm{Ob}(\mathcal{C})$ and $N \in \mathrm{Ob}(\mathcal{D})$, and whose morphisms are given by couples of morphism of $\mathcal{C}$ and $\mathcal{D}$. In details, for all M, M' in $\mathcal{C}$ and N, N' in $\mathcal{D}$:

$$\hom_{\mathcal{C}\times\mathcal{D}}((M, N), (M', N')) = \hom_{\mathcal{C}}(M, M') \times \hom_{\mathcal{D}}(N, N').$$

Definition 1.2.1. (Tensor product)
Let $\mathcal{C}$ be a category. Any functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is called a *tensor product.*

Remark 1.2.2. To have a tensor product means that one has:

- an object $M \otimes N$ associated to each couple (M, N) of objects of $\mathcal{C}$;
- a morphism $f \otimes g$ associated to each pair (f, g) of morphisms of $\mathcal{C}$ such that

$$\mathrm{dom}(f \otimes g) = \mathrm{dom}(f) \otimes \mathrm{dom}(g) \quad \text{and} \quad \mathrm{cod}(f \otimes g) = \mathrm{cod}(f) \otimes \mathrm{cod}(g).$$

and the following identities are satisfied:

$$\mathrm{Id}_{M\otimes N} = \mathrm{Id}_M \otimes \mathrm{Id}_N \quad \text{and} \quad (f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g),$$

the latter whenever the composites $f'\circ f$ and $g'\circ g$ are defined in $\mathcal{C}$. Note also that the last relation implies that

$$(\mathrm{Id}_{\mathrm{cod}(f)}\otimes g)\circ(f\otimes \mathrm{Id}_{\mathrm{dom}(g)}) = f\otimes g = (f\otimes \mathrm{Id}_{\mathrm{cod}(g)})\circ(\mathrm{Id}_{\mathrm{dom}(f)}\otimes g)$$

for all f, g in $\hom(\mathcal{C})$.

Definition 1.2.3. (Monoidal category)
A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ is a category $\mathcal{C}$ equipped with a tensor product $\otimes$ and with a distinguished object $\mathbb{I}$, called the *unit*, such that $\otimes$ is associative 'up to' a natural isomorphism a, $\mathbb{I}$ is a left and right unit for $\otimes$ 'up to' natural isomorphisms l and r respectively and 'all' diagrams involving a, l and r must commute. Formally, this means that we have three natural isomorphisms:

$$\begin{aligned} a\colon\ \otimes(\otimes\times \mathrm{Id}_{\mathcal{C}}) &\to \otimes(\mathrm{Id}_{\mathcal{C}}\times\otimes) \quad \textit{associativity constraint}\\ l\colon\ \otimes(\mathbb{I}\times \mathrm{Id}_{\mathcal{C}}) &\to \mathrm{Id}_{\mathcal{C}} \quad \textit{left unit constraint}\\ r\colon\ \otimes(\mathrm{Id}_{\mathcal{C}}\times \mathbb{I}) &\to \mathrm{Id}_{\mathcal{C}} \quad \textit{right unit constraint}\end{aligned}$$

that satisfy the *Pentagon Axiom*:

$$\begin{tikzcd} & ((M\otimes N)\otimes P)\otimes Q \arrow[r, "a_{M,N,P}\otimes Q"] \arrow[dl, "a_{M\otimes N,P,Q}"'] & (M\otimes(N\otimes P))\otimes Q \arrow[dr, "a_{M,N\otimes P,Q}"] & \\ (M\otimes N)\otimes(P\otimes Q) \arrow[drr, "a_{M,N,P\otimes Q}"'] & & & M\otimes((N\otimes P)\otimes Q) \arrow[dl, "M\otimes a_{N,P,Q}"] \\ & & M\otimes(N\otimes(P\otimes Q)) & \end{tikzcd} \tag{1.9}$$

and the *Triangle Axiom*:

$$\begin{tikzcd} (M\otimes\mathbb{I})\otimes N \arrow[rr, "a_{M,\mathbb{I},N}"] \arrow[dr, "r_M\otimes N"'] & \arrow[d, phantom, "\circlearrowleft"] & M\otimes(\mathbb{I}\otimes N) \arrow[dl, "M\otimes l_N"] \\ & M\otimes N & \end{tikzcd} \tag{1.10}$$

for all M, N, P, Q in $\mathcal{C}$.

A monoidal category is said to be *strict* if the associativity and unit constraints are all identities of the category.

Observe that Pentagon Axiom states that the two ways we have to go from $(((M\otimes N)\otimes P)\otimes Q)$ to $(M\otimes(N\otimes(P\otimes Q)))$ must coincide and Triangle Axiom forces the unit $\mathbb{I}$ to 'behave well' with respect to associativity. In what follows could happen that we refer to Pentagon and Triangle Axiom as simply 'the Axioms', for the sake of brevity. Our first goal now is to prove that these two axioms are enough to solve all the problems of this kind. The result that will take care of this is the so called 'Mac Lane's Coherence Theorem', but to come there we do not follow Mac Lane original work ([ML]). Instead, we prefer Kassel's approach ([Ka]), and this is why we will introduce soon the concept of *monoidal equivalence*. For the moment, let us show that the Axioms imply commutativity of other two triangular diagrams.

Proposition 1.2.4. ([Ka, Lemma XI.2.2]) *Let $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category. The triangles*

$$\begin{tikzcd}
(\mathbb{I}\otimes M)\otimes N \arrow[rr, "a_{\mathbb{I},M,N}"] \arrow[dr, "l_M\otimes N"'] & & \mathbb{I}\otimes(M\otimes N) \arrow[dl, "l_{M\otimes N}"] \\
& M\otimes N &
\end{tikzcd} \tag{1.11}$$

and

$$\begin{tikzcd}
(M\otimes N)\otimes\mathbb{I} \arrow[rr, "a_{M,N,\mathbb{I}}"] \arrow[dr, "r_{M\otimes N}"'] & & M\otimes(N\otimes\mathbb{I}) \arrow[dl, "M\otimes r_N"] \\
& M\otimes N &
\end{tikzcd} \tag{1.12}$$

commute for all M, N in $\mathcal{C}$.

Proof. The commutativity of the first triangle is proven in [Ka, Lemma XI.2.2], so let us show that the second one commutes as well. Consider the following diagram, for M, N and P in $\mathcal{C}$:

$$\begin{tikzcd}
& & ((M\otimes N)\otimes\mathbb{I})\otimes P \arrow[dll, "a"'] \arrow[d, "r\otimes P"] \arrow[drr, "a\otimes P"] & & \\
(M\otimes N)\otimes(\mathbb{I}\otimes P) \arrow[rr, "(M\otimes N)\otimes l"] \arrow[dddr, "a"'] & & (M\otimes N)\otimes P \arrow[dd, "a"] & & (M\otimes(N\otimes\mathbb{I}))\otimes P \arrow[ll, "(M\otimes r)\otimes P"'] \arrow[dddl, "a"] \\
& & & & \\
& & M\otimes(N\otimes P) & & \\
& M\otimes(N\otimes(\mathbb{I}\otimes P)) \arrow[ur, "M\otimes(N\otimes l)"] & & M\otimes((N\otimes\mathbb{I})\otimes P) \arrow[ul, "M\otimes(r\otimes P)"'] \arrow[ll, "M\otimes a"] &
\end{tikzcd}$$

where we dropped the subscripts for the sake of simplicity (it is always clear which subscripts are needed). The outside pentagon commutes by the Pentagon Axiom (1.9). The two squares commute both by naturality of a. In view of the Triangle Axiom (1.10), the lower and the upper left triangles commute. Thus the upper right triangle commutes as well and we have that

$$((M\otimes r_N)\circ a)\otimes P = ((M\otimes r_N)\otimes P)\circ(a\otimes P) = r_{M\otimes N}\otimes P.$$

Take $P = \mathbb{I}$ and recall that the right unit constraint is natural, i.e.:

$$\begin{tikzcd}
Q \otimes \mathbb{I} \arrow[r, "r_Q"] \arrow[d, "f\otimes\mathbb{I}"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "f"] \\
Q' \otimes \mathbb{I} \arrow[r, "r_{Q'}"'] & Q'
\end{tikzcd}$$

commutes for each $Q, Q' \in \mathcal{C}$ and every $f\colon Q \to Q'$ in $\hom(\mathcal{C})$. Thus, since r_Q is always an isomorphism, we get that

$$f = r_{Q'} \circ (f \otimes \mathbb{I}) \circ r_Q^{-1}$$

for all f as above, and so

$$((M \otimes r_N) \circ a) \otimes \mathbb{I} = r_{M\otimes N} \otimes \mathbb{I}$$

implies $(M \otimes r_N) \circ a = r_{M\otimes N}$ for all $M, N \in \mathcal{C}$, as desired. $\square$

Definition 1.2.5. ([Ka, Definition XI.4.1]) (Monoidal functor, natural monoidal transformation, monoidal equivalence)
Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \oslash, \mathbb{I}', a', l', r')$ be monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from $\mathbb{I}'$ to $F(\mathbb{I})$, and

$$\varphi_2(M,N)\colon F(M) \oslash F(N) \to F(M \otimes N)$$

is a family of natural isomorphisms indexed by all couples (M, N) of objects of $\mathcal{C}$, such that the diagrams

$$\begin{tikzcd}
\big(F(M) \oslash F(N)\big) \oslash F(P) \arrow[r, "a'_{F(M),F(N),F(P)}"] \arrow[d, "\varphi_2(M,N)\oslash F(P)"'] & F(M) \oslash \big(F(N) \oslash F(P)\big) \arrow[d, "F(M)\oslash\varphi_2(N,P)"] \\
F(M \otimes N) \oslash F(P) \arrow[d, "\varphi_2(M\otimes N,P)"'] & F(M) \oslash F(N \otimes P) \arrow[d, "\varphi_2(M,N\otimes P)"] \\
F((M \otimes N) \otimes P) \arrow[r, "F(a_{M,N,P})"'] & F(M \otimes (N \otimes P))
\end{tikzcd} \tag{1.13}$$

$$\begin{tikzcd}
\mathbb{I}' \oslash F(M) \arrow[r, "l'_{F(M)}"] \arrow[d, "\varphi_0\oslash F(M)"'] & F(M) \\
F(\mathbb{I}) \oslash F(M) \arrow[r, "\varphi_2(\mathbb{I},M)"'] & F(\mathbb{I} \otimes M) \arrow[u, "F(l_M)"']
\end{tikzcd} \tag{1.14}$$

and

$$\begin{tikzcd} F(M) \oslash \mathbb{I}' \arrow[r, "r'_{F(M)}"] \arrow[d, "F(M)\oslash\varphi_0"'] & F(M) \\ F(M) \oslash F(\mathbb{I}) \arrow[r, "\varphi_2(M,\mathbb{I})"'] & F(M \otimes \mathbb{I}) \arrow[u, "F(r_M)"'] \end{tikzcd} \tag{1.15}$$

commute for all objects M, N, P in $\mathcal{C}$. The tensor functor $(F, \varphi_0, \varphi_2)$ is said to be *strict* if the isomorphisms φ_0 and φ_2 are identities of $\mathcal{D}$.

A *natural monoidal transformation* $\eta\colon (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ between monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta\colon F \to F'$ such that the following diagrams commute for each couple (M, N) of objects in $\mathcal{C}$:

$$\begin{tikzcd} & F(\mathbb{I}) \arrow[dd, "\eta_{\mathbb{I}}"] \\ \mathbb{I}' \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\ & F'(\mathbb{I}) \end{tikzcd} \qquad \begin{tikzcd} F(M) \oslash F(N) \arrow[r, "\varphi_2(M,N)"] \arrow[d, "\eta_M \oslash \eta_N"'] & F(M \otimes N) \arrow[d, "\eta_{M\otimes N}"] \\ F'(M) \oslash F'(N) \arrow[r, "\varphi_2'(M,N)"'] & F'(M \otimes N) \end{tikzcd} \tag{1.16}$$

A *natural monoidal isomorphism* is a natural monoidal transformation that is also a natural isomorphism.

A *monoidal equivalence* between monoidal categories is a monoidal functor F from $\mathcal{C}$ to $\mathcal{D}$ such that there exist another monoidal functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural monoidal isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$. If, moreover, both composition are actually identity functors, then it is called an *isomorphism* of monoidal categories.

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$, $\mathcal{D} = (\mathcal{D}, \oslash, \mathbb{I}', a', l', r')$ and $\mathcal{E} = (\mathcal{E}, \odot, \mathbb{I}'', a'', l'', r'')$ be monoidal categories. Assume that $(F, \varphi_0, \varphi_2)$ and (G, ψ_0, ψ_2) are two monoidal functors, $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. Then we can consider the composite functor $GF\colon \mathcal{C} \to \mathcal{E}$. Moreover, if we apply G to $\varphi_0\colon \mathbb{I}' \to F(\mathbb{I})$ we obtain a map in $\mathcal{E}$ that we can compose with $\psi_0\colon \mathbb{I}'' \to G(\mathbb{I}')$:

$$\xi_0 = \left(\mathbb{I}'' \xrightarrow{\psi_0} G(\mathbb{I}') \xrightarrow{G\varphi_0} GF(\mathbb{I}) \right)$$

and analogously, for all $M, N \in \mathcal{C}$:

$$\xi_2(M,N) = \left(GF(M) \odot GF(N) \xrightarrow{\psi_2(F(M),F(N))} G(F(M) \oslash F(N)) \xrightarrow{G\varphi_2(M,N)} GF(M \otimes N) \right)$$

Lemma 1.2.6. *In the previous context, the triple (GF, ξ_0, ξ_2) is a monoidal functor.*

Proof. Clearly, GF is a functor as composition of two functors and ξ_0 is an isomorphism as composition of isomorphisms. Furthermore, ξ_2 is a natural isomorphism as composition of natural isomorphisms. Let us verify that the diagrams (1.13), (1.14) and (1.15) commute.

Consider the following diagram:

$$\begin{tikzcd}
 & (GF(M)\odot GF(N))\odot GF(P) \arrow[rrr, "a''"] \arrow[d, "\psi_2(F(M),F(N))\odot GF(P)"] & & & GF(M)\odot(GF(N)\odot GF(P)) \arrow[d, "GF(M)\odot\psi_2(F(N),F(P))"] & \\
 & G(F(M)\oslash F(N))\odot GF(P) \arrow[dl, "G\varphi_2(M,N)\odot GF(P)"'] \arrow[ddr, "\psi_2(F(M)\oslash F(N),F(P))"] & & & GF(M)\odot G(F(N)\oslash F(P)) \arrow[dl, "\psi_2(F(M),F(N)\oslash F(P))"'] \arrow[ddr, "GF(M)\odot G\varphi_2(N,P)"] & \\
GF(M\otimes N)\odot GF(P) \arrow[ddr, "\psi_2(F(M\otimes N),F(P))"'] & & & G(F(M)\oslash(F(N)\oslash F(P))) \arrow[ddr, "G(F(M)\oslash\varphi_2(N,P))"'] & & \\
 & & G((F(M)\oslash F(N))\oslash F(P)) \arrow[ur, "G(a')"] \arrow[dl, "G(\varphi_2(M,N)\oslash F(P))"] & & & GF(M)\odot GF(N\otimes P) \arrow[dl, "\psi_2(F(M),F(N\otimes P))"] \\
 & G(F(M\otimes N)\oslash F(P)) \arrow[d, "G\varphi_2(M\otimes N,P)"] & & & G(F(M)\oslash F(N\otimes P)) \arrow[d, "G\varphi_2(M,N\otimes P)"] & \\
 & GF((M\otimes N)\otimes P) \arrow[rrr, "GF(a)"'] & & & GF(M\otimes(N\otimes P)) &
\end{tikzcd}$$

The exterior path represents the compatibility of ξ_2 with the associativity constraints (i.e., diagram (1.13)). The upper hexagon commutes by compatibility of ψ_2, while the lower one by compatibility of φ_2 and functoriality of G. The two rectangles, instead, commute by naturality of ψ_2. Next consider:

$$\begin{tikzcd}
 & GF(M)\odot\mathbb{I}'' \arrow[rr, "r''_{GF(M)}"] \arrow[d, "GF(M)\odot\psi_0"] & & GF(M) \\
 & GF(M)\odot G(\mathbb{I}') \arrow[dl, "GF(M)\odot G\varphi_0"'] \arrow[dr, "\psi_2(F(M),\mathbb{I}')"] & & \\
GF(M)\odot GF(\mathbb{I}) \arrow[dr, "\psi_2(F(M),F(\mathbb{I}))"'] & & G(F(M)\oslash\mathbb{I}') \arrow[dl, "G(F(M)\oslash\varphi_0)"] \arrow[uur, "G\left(r'_{F(M)}\right)"] & \\
 & G(F(M)\oslash F(\mathbb{I})) \arrow[rr, "G(\varphi_2(M,\mathbb{I}))"'] & & GF(M\otimes\mathbb{I}) \arrow[uuu, "GF(r_M)"']
\end{tikzcd}$$

The exterior path represents again the compatibility of ξ_0 with the right unit constraint (i.e., diagram (1.15)). The upper square commutes by compatibility of ψ_0, while the lower right one by compatibility of φ_0 and functoriality of G. The remaining square, the left one, commutes by naturality of ψ_2. Analogously, one can prove that also the third diagram commutes. $\square$

Monoidal equivalences are quite a powerful tool. Indeed, as a result of the axioms of monoidal functors, we have that if two monoidal categories are monoidal equivalent, then a diagram involving constraints, identities, tensors and compositions commutes in one category if and only if its image commutes in the other. To get an idea of what it means, recall (1.11) and let $(F, \varphi_0, \varphi_2)\colon \mathcal{C} \to \mathcal{D}$ be a monoidal equivalence between $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ and $(\mathcal{D}, \oslash, \mathbb{I}', a', l', r')$. Look at the following diagram, where we dropped any reference to objects while writing the arrows in order to lighten the notation (it should be always clear which objects are involved):

$$\begin{tikzcd}
(F(\mathbb{I})\oslash F(V))\oslash F(W) \arrow[rrrrrr, "a'"] \arrow[ddr, "\varphi_2\oslash F(W)"] & & & \arrow[d, phantom, "\circlearrowleft"] & & & F(\mathbb{I})\oslash(F(V)\oslash F(W)) \arrow[ddl, "F(\mathbb{I})\oslash\varphi_2"'] \\
& & F((\mathbb{I}\otimes V)\otimes W) \arrow[rr, "F(a)"] \arrow[dr, "F(l\otimes W)"] & & F(\mathbb{I}\otimes(V\otimes W)) \arrow[dl, "F(l)"'] & & \\
& F(\mathbb{I}\otimes V)\oslash F(W) \arrow[ur, "\varphi_2"] \arrow[dr, "F(l)\oslash F(W)"] & \circlearrowleft & F(V\otimes W) & \circlearrowleft & F(\mathbb{I})\oslash F(V\otimes W) \arrow[ul, "\varphi_2"'] & \\
& \circlearrowleft & F(V)\oslash F(W) \arrow[ur, "\varphi_2"] & \circlearrowleft & \mathbb{I}'\oslash F(V\otimes W) \arrow[ul, "l'"'] \arrow[ur, "\varphi_0\oslash F(V\otimes W)"] & \circlearrowleft & \\
(\mathbb{I}'\oslash F(V))\oslash F(W) \arrow[uuuu, "(\varphi_0\oslash F(V))\oslash F(W)"] \arrow[urr, "l'\oslash F(W)"] \arrow[rrrrrr, "a'"'] & & & & & & \mathbb{I}'\oslash(F(V)\oslash F(W)) \arrow[uuuu, "\varphi_0\oslash(F(V)\oslash F(W))"'] \arrow[ullll, "l'"] \arrow[ull, "\mathbb{I}'\oslash\varphi_2"']
\end{tikzcd}$$

The upper hexagon commutes by (1.13). The external square commutes by naturality of a'. The central left square by naturality of φ_2. The central right square is (1.14), as well as the leftmost square. The central lower 'square' commutes by naturality of l' and the rightmost one because the two compositions are actually the same map.

Hence we can conclude that the central triangle commutes if and only if the lower one commutes. That is: (1.11) commutes in $\mathcal{C}$ if and only if it commutes in $\mathcal{D}$. However, this cannot be used, in view of Mac Lane's Coherence Theorem, to prove that (1.11) commutes in all monoidal categories because it commutes in the associated strict monoidal category, since we will need the commutativity of (1.11) in order to prove Mac Lane's Theorem. But it is a simple example of how monoidal equivalence works.

We are now ready to set out on the journey that will lead us to prove Mac Lane's Coherence Theorem. As we said, this proof is due to Kassel. For details, we refer to [Ka].

Let $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category and start by considering the class of all finite sequences of objects of $\mathcal{C}$, including the empty sequence $\emptyset$. Denote it by $\mathcal{S}$. An object S in $\mathcal{S}$ is just

$$S = (C_1, C_2, \ldots, C_k)$$

where $C_i \in \mathrm{Ob}(\mathcal{C})$ for all $i = 1, \ldots, k$. The integer k is the *length* of S.

We can define the *concatenation* of two non empty sequences $S = (C_1, C_2, \ldots, C_k)$ and $S' = (C_{k+1}, C_{k+2}, \ldots, C_{k+n})$ by simply placing the first after the second:

$$S * S' := (C_1, C_2, \ldots, C_k, C_{k+1}, \ldots, C_{k+n})\,.$$

We set also $\emptyset * S = S = S * \emptyset$. Now, to each sequence $S = (C_1, C_2, \ldots, C_k)$ we can assign an object $F(S)$ in $\mathcal{C}$ that is the tensor product of all the objects in S with parenthesis associated on the left. That is, inductively:

$$F(\emptyset) = \mathbb{I}, \qquad F((C)) = C, \qquad F(S * (C)) = F(S) \otimes C \tag{1.17}$$

for each $S \in \mathcal{S}$ and $C \in \mathrm{Ob}(\mathcal{C})$. I.e.,

$$F(C_1, C_2, \ldots, C_k) = ((\cdots((C_1 \otimes C_2) \otimes C_3) \otimes \cdots) \otimes C_{k-1}) \otimes C_k.$$

Next we associate to the monoidal category $\mathcal{C}$ another category, denoted by $\mathcal{C}^{\mathrm{str}}$, whose objects are the sequences of $\mathcal{S}$ and whose morphisms are defined by:

$$\hom_{\mathcal{C}^{\mathrm{str}}}(S, S') := \hom_{\mathcal{C}}(F(S), F(S')).$$

Proposition 1.2.7. ([Ka, Proposition XI.5.1]) *The category $\mathcal{C}^{\mathrm{str}}$ is equivalent to $\mathcal{C}$ via $F\colon \mathcal{C}^{\mathrm{str}} \to \mathcal{C}$.*

Proof. Consider the assignment

$$F\colon \mathcal{C}^{\mathrm{str}} \longrightarrow \mathcal{C}\colon S \longmapsto F(S)$$

as defined above and such that it is the identity on morphisms (we are allowed to do so, by how we defined morphisms in $\mathcal{C}^{\mathrm{str}}$). Let us denote by $\hat{f}\colon S \to S'$ the morphism in $\mathcal{C}^{\mathrm{str}}$ obtained by $f\colon F(S) \to F(S')$.

The functor F is clearly fully faithful. Indeed, it is faithful by definition, and it is full because if $f\colon C \to D$ is a morphism in $\mathcal{C}$ then $f = F\left(\hat{f}\right)$, where $\hat{f}\colon (C) \to (D)$ is a morphism in $\mathcal{C}^{\mathrm{str}}$. Furthermore, for every $C \in \mathrm{Ob}(\mathcal{C})$, $C = F((C))$ and so it also essentially surjective on objects (actually, it is surjective). In view of Theorem 1.1.14, $\mathcal{C}$ and $\mathcal{C}^{\mathrm{str}}$ are equivalent. $\square$

Consider the assignment

$$G\colon \mathcal{C} \to \mathcal{C}^{\mathrm{str}}\colon C \mapsto (C) \tag{1.18}$$

that operates as the identity on morphism (i.e., $G(f) = \hat{f}$). Clearly $FG(C) = F((C)) = C$, while $GF(S) = (F(S))$. However, since

$$\hom_{\mathcal{C}^{\mathrm{str}}}(S, (F(S))) = \hom_{\mathcal{C}}(F(S), F(S))$$

we can consider the natural isomorphism

$$\eta_S := \widehat{\mathrm{Id}}_{F(S)}\colon S \to GF(S). \tag{1.19}$$

Hence, G defines a functor which is the inverse equivalence to F.

Now, we give to $\mathcal{C}^{\mathrm{str}}$ a monoidal structure. The tensor product is given by concatenation:

$$S \otimes S' := S * S'.$$

This implies that, if we can show that it is actually a tensor product, $(\mathcal{C}^{\mathrm{str}}, *, \emptyset)$ is a strict monoidal category ($*$ is trivially associative). Therefore, we need to define the $*$ of two maps in order to be able to verify that $*$ is a functor. And obviously we are going to define it, in order to have that.

First of all, we construct a natural isomorphism

$$\varphi(S, S')\colon F(S) \otimes F(S') \to F(S * S')$$

for every pair (S, S') of objects in $\mathcal{C}^{\mathrm{str}}$ and we define it by induction on the length of S'. Set

$$\varphi(\emptyset, S') := l_{F(S')} \quad \text{and} \quad \varphi(S, \emptyset) := r_{F(S)}. \tag{1.20}$$

Next, set

$$\varphi(S, (C)) := \mathrm{Id}_{F(S)\otimes C} \colon F(S) \otimes C \to F(S * (C)) \tag{1.21}$$

and finally:

$$\varphi(S, S' * (C)) := (\varphi(S, S') \otimes C) \circ a^{-1}_{F(S),F(S'),C}, \tag{1.22}$$

i.e.,

$$\begin{tikzcd}
F(S) \otimes F(S' * (C)) \arrow[rr, dashed, "\varphi"] \arrow[dd, equal, "(1.17)"'] & & F(S * S' * (C)) \arrow[d, equal, "(1.17)"] \\
& & F(S * S') \otimes C \\
F(S) \otimes (F(S') \otimes C) \arrow[rr, "a^{-1}_{F(S),F(S'),C}"'] & & (F(S) \otimes F(S')) \otimes C \arrow[u, "\varphi\otimes C"']
\end{tikzcd}$$

Note that it is an isomorphism because globally it is just the composition of morphisms built from the associativity constraint and the identities by tensoring, in order to reassociate all parenthesis on the left.

Lemma 1.2.8. *The natural isomorphism φ defined above satisfies (1.13). That is, if S, S' and S'' are sequences in $\mathcal{C}^{\mathrm{str}}$, then*

$$\varphi(S, S' * S'') \circ (F(S) \otimes \varphi(S', S'')) \circ a_{F(S),F(S'),F(S'')} = \varphi(S * S', S'') \circ (\varphi(S, S') \otimes F(S'')).$$

Proof. Let us prove it by induction on the length of S''. If $S'' = \emptyset$, then $F(S'') = \mathbb{I}$ and $S' * S'' = S'$. Thus, we have that:

$$\begin{aligned}
\varphi(S, S') \circ (F(S) \otimes \varphi(S', \emptyset)) \circ a_{F(S),F(S'),\mathbb{I}} &\overset{(1.20)}{=} \\
= \varphi(S, S') \circ (F(S) \otimes r_{F(S')}) \circ a_{F(S),F(S'),\mathbb{I}} &\overset{(1.12)}{=} \\
= \varphi(S, S') \circ r_{F(S)\otimes F(S')} &\overset{(\triangle)}{=} \\
= r_{F(S*S')} \circ (\varphi(S, S') \otimes \mathbb{I}) &\overset{(1.20)}{=} \\
= \varphi(S * S', \emptyset) \circ (\varphi(S, S') \otimes \mathbb{I})&
\end{aligned}$$

where in $(\triangle)$ we used naturality of r:

$$\begin{tikzcd}
(F(S) \otimes F(S')) \otimes \mathbb{I} \arrow[r, "r_{F(S)\otimes F(S')}"] \arrow[d, "\varphi(S,S')\otimes\mathbb{I}"'] & F(S) \otimes F(S') \arrow[d, "\varphi(S,S')"] \\
F(S * S') \otimes \mathbb{I} \arrow[r, "r_{F(S*S')}"'] & F(S * S')
\end{tikzcd}$$

Let $C \in \mathrm{Ob}(\mathcal{C})$ and let us prove that if the equality holds for the triple (S, S', S'') then it holds for $(S, S', S'' * (C))$.

$$
\begin{aligned}
&\varphi(S, S' * S'' * (C)) \circ (F(S) \otimes \varphi(S', S'' * (C))) \circ a_{F(S),F(S'),F(S''*(C))} \overset{(1.22)}{=} \\
&= \left[\begin{array}{l} (\varphi(S, S' * S'') \otimes C) \circ \underline{a^{-1}_{F(S),F(S'*S''),C} \circ (F(S) \otimes (\varphi(S', S'') \otimes C))\circ} \\ \circ(F(S) \otimes a^{-1}_{F(S'),F(S''),C}) \circ a_{F(S),F(S'),F(S'')\otimes C} \end{array} \right] \overset{(\blacktriangle)}{=} \\
&= \left[\begin{array}{l} (\varphi(S, S' * S'') \otimes C) \circ ((F(S) \otimes \varphi(S', S'')) \otimes C)\circ \\ \underline{\circ a^{-1}_{F(S),F(S')\otimes F(S''),C} \circ (F(S) \otimes a^{-1}_{F(S'),F(S''),C}) \circ a_{F(S),F(S'),F(S'')\otimes C}} \end{array} \right] \overset{(1.9)}{=} \\
&= \left[\begin{array}{l} \underline{(\varphi(S, S' * S'') \otimes C) \circ ((F(S) \otimes \varphi(S', S'')) \otimes C)\circ} \\ \underline{\circ(a_{F(S),F(S'),F(S'')} \otimes C)} \circ a^{-1}_{F(S)\otimes F(S'),F(S''),C} \end{array} \right] = \\
&= \left[\begin{array}{l} ((\varphi(S, S' * S'') \circ (F(S) \otimes \varphi(S', S'')) \circ a_{F(S),F(S'),F(S'')}) \otimes C)\circ \\ \circ a^{-1}_{F(S)\otimes F(S'),F(S''),C} \end{array} \right] \overset{(\bullet)}{=} \\
&= (\varphi(S * S', S'') \otimes C) \circ ((\varphi(S, S') \otimes F(S'')) \otimes C) \circ a^{-1}_{F(S)\otimes F(S'),F(S''),C} \overset{(\blacktriangle)}{=} \\
&= (\varphi(S * S', S'') \otimes C) \circ a^{-1}_{F(S*S'),F(S''),C} \circ (\varphi(S, S') \otimes (F(S'') \otimes C)) \overset{(1.22)}{=} \\
&= \varphi(S * S', S'' * (C)) \circ (\varphi(S, S') \otimes F(S'' * (C)))
\end{aligned}
$$

where in $(\blacktriangle)$ we used naturality of a and in $(\bullet)$ the induction hypothesis. $\square$

Now, take two morphisms $\hat{f}\colon S \to S'$ and $\hat{g}\colon T \to T'$. By definition, f is a morphism from $F(S)$ to $F(S')$ and g is a morphism from $F(T)$ to $F(T')$ in $\mathcal{C}$. Therefore, we define $\hat{f} * \hat{g}$ as the map that makes the following diagram commutative:

$$
\begin{tikzcd}
F(S) \otimes F(T) \arrow[r, "\varphi(S,T)"] \arrow[d, "f\otimes g"'] & F(S * T) \arrow[d, dashed, "F(\hat{f}*\hat{g})"] \\
F(S') \otimes F(T') \arrow[r, "\varphi(S',T')"'] & F(S' * T')
\end{tikzcd}
$$

There are two things that should be noted here:

1. that such a map $\hat{f} * \hat{g}$ is unique, since φ is a natural isomorphism and F is faithful,
2. that φ is natural in both components, where the naturality is expressed by the above diagram itself.

Theorem 1.2.9. ([Ka, Theorem XI.5.3]) *Equipped with the tensor product $*$, $\mathcal{C}^{\mathrm{str}}$ is a strict monoidal category. Furthermore, $\mathcal{C}$ and $\mathcal{C}^{\mathrm{str}}$ are monoidal equivalent.*

Proof. To complete the proof of the first statement it is enough to show that $*$ preserves the composition of maps. Hence consider

$$
S \xrightarrow{\hat{f}} S' \xrightarrow{\hat{f}'} S''
$$

$$
T \xrightarrow{\hat{g}} T' \xrightarrow{\hat{g}'} T''
$$

and look at the diagram

$$\begin{tikzcd}
F(S)\otimes F(T) \arrow[r, "\varphi(S,T)"] \arrow[d, "f\otimes g"] \arrow[dd, bend right=60, "(f'\circ f)\otimes(g'\circ g)"'] & F(S*T) \arrow[d, "F(\hat{f}*\hat{g})"'] \arrow[dd, bend left=60, "F((\hat{f}'\circ\hat{f})*(\hat{g}'\circ\hat{g}))"] \\
F(S')\otimes F(T') \arrow[r, "\varphi(S',T')"] \arrow[d, "f'\otimes g'"] & F(S'*T') \arrow[d, "F(\hat{f}'*\hat{g}')"'] \\
F(S'')\otimes F(T'') \arrow[r, "\varphi(S'',T'')"] & F(S''*T'')
\end{tikzcd}$$

We have two maps that make the external square commutes: $(\hat{f}' * \hat{g}') \circ (\hat{f} * \hat{g})$ and $(\hat{f}' \circ \hat{f}) * (\hat{g}' \circ \hat{g})$. Thus they must coincide and $*$ is a well defined functor.

Moreover, in view of Lemma 1.2.8 and the last observation, we have that

$$(F, \varphi_0 := \mathrm{Id}_{\mathbb{I}}, \varphi_2 := \varphi)$$

is a monoidal functor. Indeed, the only properties that are left unverified are (1.14) and (1.15), but

$$\begin{tikzcd}
\mathbb{I}\otimes F(S) \arrow[r, "l_{F(S)}=\varphi(\emptyset,S)"] \arrow[d, "\mathrm{Id}_{\mathbb{I}}\otimes F(S)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(S) \\
F(\emptyset)\otimes F(S) \arrow[r, "\varphi(\emptyset,S)"'] & F(\emptyset * S) \arrow[u, "F\left(\widehat{\mathrm{Id}}_{F(S)}\right)"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(S)\otimes\mathbb{I} \arrow[r, "r_{F(S)}=\varphi(S,\emptyset)"] \arrow[d, "F(S)\otimes\mathrm{Id}_{\mathbb{I}}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(S) \\
F(S)\otimes F(\emptyset) \arrow[r, "\varphi(S,\emptyset)"'] & F(S*\emptyset) \arrow[u, "F\left(\widehat{\mathrm{Id}}_{F(S)}\right)"']
\end{tikzcd}$$

Next, recall that we also know that F is an equivalence with inverse equivalence given by G as defined in (1.18). From

$$\hom_{\mathcal{C}^{\mathrm{str}}}(\emptyset, (\mathbb{I})) = \hom_{\mathcal{C}}(\mathbb{I}, \mathbb{I})$$

we can set $\psi_0 := \widehat{\mathrm{Id}_{\mathbb{I}}} = \mathrm{Id}_\emptyset$ and from

$$\hom_{\mathcal{C}^{\mathrm{str}}}((C, C'), (C \otimes C')) = \hom_{\mathcal{C}}(C \otimes C', C \otimes C')$$

we can set $\psi_2(C, C') := \widehat{\mathrm{Id}}_{C\otimes C'}$. With this definitions, (G, ψ_0, ψ_2) is a (strict) monoidal functor. Therefore, to conclude the proof we need to verify that the natural isomorphism $\eta\colon \mathrm{Id}_{\mathcal{C}^{\mathrm{str}}} \to GF$ defined in (1.19) is a natural monoidal transformation between $(\mathrm{Id}_{\mathcal{C}^{\mathrm{str}}}, \mathrm{Id}_\emptyset, \mathrm{Id}_*)$ and (GF, ξ_0, ξ_2), where

$$\xi_0 = G(\varphi_0) \circ \psi_0 \quad \text{and} \quad \xi_2(S, T) = G(\varphi_2(S, T)) \circ \psi_2(F(S), F(T))$$

as in Lemma 1.2.6.

Let us begin by verifying the commutativity of the rightmost diagram in (1.16). If we write it down and we apply F we get:

$$\begin{tikzcd} S*T \arrow[r, "\mathrm{Id}_{S*T}"] \arrow[d, "\eta_S*\eta_T"] & S*T \arrow[d, "\eta_{S*T}"] \\ (F(S),F(T)) \arrow[r, "\widehat{\varphi_2}(S,T)"'] & (F(S*T)) \end{tikzcd} \xrightarrow{F} \begin{tikzcd} F(S*T) \arrow[r, "\mathrm{Id}_{F(S*T)}"] \arrow[d, "F(\eta_S*\eta_T)"] & F(S*T) \arrow[d, "F(\eta_{S*T})"] \\ F(S)\otimes F(T) \arrow[r, "\varphi_2(S,T)"'] & F(S*T) \end{tikzcd} \tag{1.23}$$

Recall that, by definition of $\eta_S * \eta_T$, the following diagram commutes:

$$\begin{tikzcd} F(S)\otimes F(T) \arrow[r, "\varphi(S,T)"] \arrow[d, "\mathrm{Id}_{F(S)}\otimes\mathrm{Id}_{F(T)}"'] & F(S*T) \arrow[d, "F(\eta_S*\eta_T)"] \\ F(GF(S))\otimes F(GF(T)) \arrow[r, "\varphi(GF(S),GF(T))"'] & F(GF(S)*GF(T)) \end{tikzcd} \tag{1.24}$$

However, since we have that:

- $F(GF(S)) \otimes F(GF(T)) = F(S) \otimes F(T)$,
- $\varphi(GF(S), GF(T)) = \mathrm{Id}_{F((F(S)))\otimes F(T)} = \mathrm{Id}_{F(S)\otimes F(T)}$ by (1.21),
- $F(GF(S) * GF(T)) = F(S) \otimes F(T)$ by (1.17)

the commutativity of diagram (1.24) implies that $F(\eta_S * \eta_T) \circ \varphi(S,T) = \mathrm{Id}_{F(S)\otimes F(T)}$, from which we deduce that

$$F(\eta_S * \eta_T) = \varphi(S,T)^{-1}.$$

Now, by recalling that $\eta_{S*T} = \widehat{\mathrm{Id}}_{F(S*T)}$, we have that the right hand diagram of (1.23) commutes and, by faithfulness of F, we conclude that also the leftmost one commutes.

The remaining commutativity, i.e., the one of the leftmost diagram in (1.16), follows since all maps involved are $\widehat{\mathrm{Id}}_{\mathbb{I}}$. □

As a corollary, we can now state Mac Lane's result.

Theorem 1.2.10. (Mac Lane's Coherence Theorem)
Let $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category. Every diagram in $\mathcal{C}$ whose vertices are 'words' W of the same length n representing functors $W\colon \mathcal{C}^n \to \mathcal{C}$ and whose edges are natural transformations $Id_{\mathbb{I}}$, $Id_{Id_{\mathcal{C}}}$, a, l, r and their $\otimes$-products commutes, where the functors in question are $\mathbb{I}$, $Id_{\mathcal{C}}$, $-\otimes-$ and their composites.

Remark 1.2.11. Bluntly speaking, Mac Lane's Coherence Theorem states that Pentagon Axiom and Triangle Axiom are necessary and sufficient to ensure that all diagrams built from the constraints and the identities by composing and tensoring commutes.

Actually, Mac Lane's and Kassel's results are equivalent. However, this exceeds our purposes, but the interested reader can find a proof in [ML, Section XI.3].

Definition 1.2.12. (Algebra, coalgebra)
Let $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category. An *algebra* (A, m, u) in $\mathcal{C}$ is an object $A \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $m \colon A \otimes A \to A$ (the *multiplication*) and $u \colon \mathbb{I} \to A$ (the *unit*) such that the following diagrams commute:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes A"'] & & A \otimes (A \otimes A) \arrow[d, "A\otimes m"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd} \tag{1.25}$$

$$\begin{tikzcd}
\mathbb{I} \otimes A \arrow[r, "u\otimes A"] \arrow[dr, "l_A"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{I} \arrow[l, "A\otimes u"'] \arrow[dl, "r_A"] \\
& A &
\end{tikzcd} \tag{1.26}$$

We express these commutativities by saying that A is *associative* and *unital* respectively. A *morphism of algebras* $f \colon A \to A'$ between (A, m, u) and (A', m', u') is a morphism in $\hom(\mathcal{C})$ such that the following diagrams commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & A' \otimes A' \arrow[d, "m'"] \\
A \arrow[r, "f"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] & & A' \\
& \mathbb{I} \arrow[ul, "u"] \arrow[ur, "u'"'] &
\end{tikzcd}$$

A *coalgebra* (C, Δ, ε) is an object $C \in \mathcal{C}$ together with two morphisms $\Delta \colon C \to C \otimes C$ (the *comultiplication*) and $\varepsilon \colon C \to \mathbb{I}$ (the *counit*) such that the following diagrams commute:

$$\begin{tikzcd}
C \otimes C \arrow[d, "\Delta\otimes C"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & C \otimes C \arrow[d, "C\otimes\Delta"] \\
(C \otimes C) \otimes C \arrow[rr, "a_{C,C,C}"'] & & C \otimes (C \otimes C)
\end{tikzcd} \tag{1.27}$$

$$\begin{tikzcd}
\mathbb{I} \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes C"'] \arrow[r, "C\otimes\varepsilon"] & C \otimes \mathbb{I} \\
& C \arrow[ul, "l_C^{-1}"] \arrow[u, "\Delta"] \arrow[ur, "r_C^{-1}"'] &
\end{tikzcd} \tag{1.28}$$

As above, we express these commutativities by saying that C is *coassociative* and *counital* respectively. A *morphism of coalgebras* $g \colon C \to C'$ between (C, Δ, ε) and $(C', \Delta', \varepsilon')$ is a morphism in $\hom(\mathcal{C})$ such that the following diagrams commutes

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\
C \otimes C \arrow[r, "g\otimes g"'] & C' \otimes C'
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\
& \mathbb{I} &
\end{tikzcd}$$

Example 1.2.13. $\mathbb{I}$ is always both an algebra and a coalgebra.

Let us start by showing that it is an algebra. For each $M \in \mathcal{C}$, apply naturality of l and r to l_M and r_M respectively:

$$\begin{tikzcd}
\mathbb{I} \otimes (\mathbb{I} \otimes M) \arrow[r, "l_{\mathbb{I}\otimes M}"] \arrow[d, "\mathbb{I}\otimes l_M"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{I} \otimes M \arrow[d, "l_M"] \\
\mathbb{I} \otimes M \arrow[r, "l_M"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \otimes \mathbb{I}) \otimes \mathbb{I} \arrow[r, "r_{M\otimes\mathbb{I}}"] \arrow[d, "r_M\otimes\mathbb{I}"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes \mathbb{I} \arrow[d, "r_M"] \\
M \otimes \mathbb{I} \arrow[r, "r_M"'] & M
\end{tikzcd}$$

Recalling that the unit constraints are isomorphisms we get that

$$l_{\mathbb{I}\otimes M} = \mathbb{I} \otimes l_M \tag{1.29a}$$

$$r_{M\otimes\mathbb{I}} = r_M \otimes \mathbb{I} \tag{1.29b}$$

Next, set $M = N = \mathbb{I}$ in (1.10) to get that:

$$\begin{tikzcd}
(\mathbb{I} \otimes \mathbb{I}) \otimes \mathbb{I} \arrow[rr, "a"] \arrow[dr, "r_{\mathbb{I}}\otimes\mathbb{I}"'] & & \mathbb{I} \otimes (\mathbb{I} \otimes \mathbb{I}) \arrow[dl, "\mathbb{I}\otimes l_{\mathbb{I}}"] \\
& \mathbb{I} \otimes \mathbb{I} &
\end{tikzcd}$$

thus

$$(\mathbb{I} \otimes r_{\mathbb{I}}) \circ a \overset{(1.12)}{=} r_{\mathbb{I}\otimes\mathbb{I}} \overset{(1.29b)}{=} r_{\mathbb{I}} \otimes \mathbb{I} = (\mathbb{I} \otimes l_{\mathbb{I}}) \circ a.$$

Since a is invertible, we can erase it and obtain: $\mathbb{I} \otimes r_{\mathbb{I}} = \mathbb{I} \otimes l_{\mathbb{I}}$. However, recall that l is a natural isomorphism: for each $f \colon M \to N$ in $\mathcal{C}$ we have that

$$\begin{tikzcd}
\mathbb{I} \otimes M \arrow[r, "l_M"] \arrow[d, "\mathbb{I}\otimes f"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\
\mathbb{I} \otimes N \arrow[r, "l_N"'] & N
\end{tikzcd}$$

and so $f = l_N \circ (\mathbb{I} \otimes f) \circ l_M^{-1}$. We can conclude, then, that $r_{\mathbb{I}} = l_{\mathbb{I}}$. Therefore, we can set $m := r_{\mathbb{I}} = l_{\mathbb{I}}$ and $u = \mathrm{Id}_{\mathbb{I}}$ and they are both well defined maps in $\mathcal{C}$. Now:

$$\begin{tikzcd}
& (\mathbb{I} \otimes \mathbb{I}) \otimes \mathbb{I} \arrow[rr, "a"] \arrow[dl, "l_I\otimes\mathbb{I}"'] \arrow[drrr, dashed, "r_{\mathbb{I}}\otimes\mathbb{I}"] & & \mathbb{I} \otimes (\mathbb{I} \otimes \mathbb{I}) \arrow[dr, "\mathbb{I}\otimes l_{\mathbb{I}}"] & \\
\mathbb{I} \otimes \mathbb{I} \arrow[drr, "l_{\mathbb{I}}"'] & & & & \mathbb{I} \otimes \mathbb{I} \arrow[dll, "l_{\mathbb{I}}"] \\
& & \mathbb{I} & &
\end{tikzcd}$$

commutes by the Triangle Axiom (1.10) and

$$\begin{tikzcd}
\mathbb{I} \otimes \mathbb{I} \arrow[r, "u\otimes\mathbb{I}"] \arrow[dr, "l_{\mathbb{I}}"'] & \mathbb{I} \otimes \mathbb{I} \arrow[d, "l_{\mathbb{I}}"', "r_{\mathbb{I}}"] & \mathbb{I} \otimes \mathbb{I} \arrow[l, "\mathbb{I}\otimes u"'] \arrow[dl, "r_{\mathbb{I}}"] \\
& \mathbb{I} &
\end{tikzcd}$$

obviously commutes.

Next, proving that it is also a coalgebra is now really easy. Define $\Delta := r_{\mathbb{I}}^{-1} = l_{\mathbb{I}}^{-1}$ and $\varepsilon = \mathrm{Id}_{\mathbb{I}}$. Again in view of (1.10), we have that both the following diagrams commute:

$$\begin{tikzcd}
& \mathbb{I} \arrow[rr, "r_{\mathbb{I}}^{-1}"] \arrow[dl, "r_{\mathbb{I}}^{-1}"'] & & \mathbb{I}\otimes\mathbb{I} \arrow[dr, "r_{\mathbb{I}}^{-1}\otimes\mathbb{I}"] & \\
\mathbb{I}\otimes\mathbb{I} \arrow[rrrr, dashed, "r_{\mathbb{I}}^{-1}\otimes\mathbb{I}"] \arrow[drr, "\mathbb{I}\otimes l_{\mathbb{I}}^{-1}"'] & & & & (\mathbb{I}\otimes\mathbb{I})\otimes\mathbb{I} \\
& & \mathbb{I}\otimes(\mathbb{I}\otimes\mathbb{I}) \arrow[urr, "a^{-1}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{I} & \mathbb{I}\otimes\mathbb{I} \arrow[l, "\varepsilon\otimes\mathbb{I}"'] \arrow[r, "\mathbb{I}\otimes\varepsilon"] & \mathbb{I}\otimes\mathbb{I} \\
& \mathbb{I} \arrow[ul, "l_{\mathbb{I}}^{-1}"] \arrow[u, "l_{\mathbb{I}}^{-1}", "r_{\mathbb{I}}^{-1}"'] \arrow[ur, "r_{\mathbb{I}}^{-1}"'] &
\end{tikzcd}$$

and this concludes the proof.

Let $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category. A *right action* of an algebra (A, m, u) on an object $M \in \mathrm{Ob}(\mathcal{C})$ is an arrow $\mu\colon M \otimes A \to M$ of $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
(M\otimes A)\otimes A \arrow[r, "a_{M,A,A}"] \arrow[d, "\mu\otimes A"'] & M\otimes(A\otimes A) \arrow[r, "M\otimes m"] & M\otimes A \arrow[d, "\mu"] & M\otimes\mathbb{I} \arrow[l, "M\otimes u"'] \arrow[dl, "r_M"] \\
M\otimes A \arrow[rr, "\mu"] & & M &
\end{tikzcd} \tag{1.30}$$

In the same way, a *right coaction* of a coalgebra (C, Δ, ε) on an object $N \in \mathrm{Ob}(\mathcal{C})$ is an arrow $\rho\colon N \to N \otimes C$ of $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
(N\otimes C)\otimes C \arrow[r, "a_{N,C,C}"] & N\otimes(C\otimes C) & N\otimes C \arrow[l, "N\otimes\Delta"'] \arrow[r, "N\otimes\varepsilon"] & N\otimes\mathbb{I} \\
N\otimes A \arrow[u, "\rho\otimes C"] & & N \arrow[ll, "\rho"'] \arrow[u, "\rho"'] \arrow[ur, "r_N^{-1}"'] &
\end{tikzcd} \tag{1.31}$$

Definition 1.2.14. (Modules, comodules and morphisms)
If (A, m, u) is an algebra, a *right A-module* is an object $M \in \mathrm{Ob}(\mathcal{C})$ together with a right action of A on M. A morphism of right A-modules (M, μ_M) and (N, μ_N) is a map $f\colon M \to N \in \mathcal{C}(M, N)$ such that the following commutes:

$$\begin{tikzcd}
M\otimes A \arrow[r, "f\otimes A"] \arrow[d, "\mu_M"'] \arrow[dr, phantom, "\circlearrowleft"] & N\otimes A \arrow[d, "\mu_N"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

Let us denote with $\mathcal{C}_A$ the category of right A-modules.

If (C, Δ, ε) is a coalgebra, a *right C-comodule* is an object $M \in \mathrm{Ob}(\mathcal{C})$ together with a right coaction of C on M. A morphism of right C-comodules (M, ρ_M) and (N, ρ_N) is a map $f\colon M \to N \in \mathcal{C}(M,N)$ such that the following commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"'] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[d, "\rho_N"] \\ M \otimes C \arrow[r, "f\otimes C"'] & N \otimes C \end{tikzcd}$$

Let us denote with $\mathcal{C}^C$ the category of right C-comodules.

Definition 1.2.15. (Bimodules)
Let (A, m, u) and (A', m', u') be algebras in $(\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$. An object M of $\mathcal{C}$ is an (A, A')-bimodule if there exists a right A'-action and a left A-action on M that are compatible, i.e., two maps $\mu\colon A \otimes M \to M$ and $\mu'\colon M \otimes A' \to M$ such that:

$$\begin{tikzcd} (M \otimes A') \otimes A' \arrow[r, "a_{M,A',A'}"] \arrow[d, "\mu'\otimes A'"'] & M \otimes (A' \otimes A') \arrow[r, "M\otimes m'"] & M \otimes A' \arrow[d, "\mu'"] & M \otimes \mathbb{I} \arrow[l, "M\otimes u'"'] \arrow[dl, "r_M"] \\ M \otimes A' \arrow[rr, "\mu'"'] & & M & \end{tikzcd}$$

commutes,

$$\begin{tikzcd} (A \otimes A) \otimes M \arrow[r, "{a_{M,A,A}}^{-1}"] \arrow[d, "m\otimes M"'] & A \otimes (A \otimes M) \arrow[r, "A\otimes\mu"] & A \otimes M \arrow[d, "\mu"] & \mathbb{I} \otimes M \arrow[l, "u\otimes M"'] \arrow[dl, "l_M"] \\ A \otimes M \arrow[rr, "\mu"'] & & M & \end{tikzcd}$$

commutes and

$$\begin{tikzcd} (A \otimes M) \otimes A' \arrow[rr, "a_{A,M,A'}"] \arrow[d, "\mu\otimes A'"'] & & A \otimes (M \otimes A') \arrow[d, "A\otimes\mu'"] \\ M \otimes A' \arrow[r, "\mu'"'] & M & A \otimes M \arrow[l, "\mu"] \end{tikzcd}$$

commutes.

1.3 The Monoidal Category of vector spaces

We shall henceforth consider the monoidal category $(\mathcal{M} = \mathrm{Vect}(\Bbbk), \otimes, \Bbbk, a, l, r)$ of $\Bbbk$-vector spaces, with tensor structure $\otimes$ given by the tensor product over $\Bbbk$ and with unit object $\mathbb{I}$ the ground field $\Bbbk$ itself. An algebra in $\mathcal{M} = \mathrm{Vect}(\Bbbk)$ is an ordinary associative unital $\Bbbk$-algebra and the same for coalgebras. Let us spend just a few words to recall some facts about these objects.

Definition 1.3.1. (Tensor product)
Let U, V be $\Bbbk$-vector spaces and G be an abelian group. A map $b_G\colon U \times V \to G$ is said

to be $\Bbbk$-*biadditive* if, for all $u, u' \in U$, $v, v' \in V$ and $k \in \Bbbk$ we have

$$\begin{aligned} b_G(u+u', v) &= b_G(u,v) + b_G(u',v); \\ b_G(u, v+v') &= b_G(u,v) + b_G(u,v'); \\ b_G(uk, v) &= b_G(u, kv). \end{aligned}$$

A *tensor product* of U and V is an ordered pair (T, b_T) where T is an abelian group and $b_T \colon U \times V \to T$ is a $\Bbbk$-biadditive map that satisfies the following universal property: for every abelian group G and every $\Bbbk$-biadditive map $b_G \colon U \times V \to G$ there exists a unique morphism of abelian groups $\widetilde{b_G} \colon T \to G$ such that the following diagram commutes

$$\begin{tikzcd} U \times V \arrow[r, "b_T"] \arrow[d, "b_G"'] & T \arrow[dl, dotted, "\widetilde{b_G}"] \\ G & \end{tikzcd}$$

Remark 1.3.2. If a tensor product exists, then it is unique up to isomorphism. Indeed, assume that (T, b_T) and (S, b_S) are two tensor products. By the universal property, we can fill in the following diagrams in a unique way:

$$\begin{tikzcd} U \times V \arrow[r, "b_T"] \arrow[d, "b_S"'] & T \arrow[dl, dotted, "f"] \\ S & \end{tikzcd} \qquad \begin{tikzcd} U \times V \arrow[r, "b_S"] \arrow[d, "b_T"'] & S \arrow[dl, dotted, "g"] \\ T & \end{tikzcd}$$

thus, we have that both Id_T and $g \circ f$ make the following diagram commutative

$$\begin{tikzcd} U \times V \arrow[r, "b_T"] \arrow[d, "b_T"'] & T \arrow[dl, dotted, shift right, "g \circ f"'] \arrow[dl, dotted, shift left, "\mathrm{Id}_T"] \\ T & \end{tikzcd}$$

and so, by uniqueness, $g \circ f = \mathrm{Id}_T$. In the same way, one proves that also $f \circ g = \mathrm{Id}_S$.

Actually, it can be proven that the tensor product of two vector spaces always exists. Let us recall just how it is constructed. Let $S = U \times V$ and consider the free abelian group generated by S: $\mathbb{Z}^S$. Define L as the subgroup generated by all the elements of the following type:

$$\begin{gathered} (u, v+v') - (u,v) - (u,v') \\ (u+u', v) - (u,v) - (u',v) \\ (uk, v) - (u, kv) \end{gathered}$$

for $u, u' \in U$, $v, v' \in V$, $k \in \Bbbk$. Thus $U \otimes V := \frac{\mathbb{Z}^S}{L}$ and

$$b_{U \otimes V} \colon U \times V \longrightarrow U \otimes V \colon (u,v) \longmapsto u \otimes v$$

where $u \otimes v$ is just the class of the generator (u, v) in the quotient. Note that $U \otimes V$ is *generated* by elements of the form $u \otimes v$, so that the generic element $x \in U \otimes V$ has the form

$$x = \sum_{i}^{<\infty} u_i \otimes v_i$$

for $u_i \in U$ and $v_i \in V$, and this expression is *not* unique. For further details, we refer to [Ro, Section 8.4].

Since $\Bbbk$ is a field, it is commutative, and therefore a $\Bbbk$-vector space U is actually a $(\Bbbk, \Bbbk)$-bimodule. Moreover, if $u \in U$ and $k \in \Bbbk$, we have that $ku = uk$. This implies that also $U \otimes V$ is a $\Bbbk$-vector space, for U, V $\Bbbk$-vector spaces, with scalar multiplication given by

$$\Bbbk \times (U \otimes V) \longrightarrow U \otimes V : (k, u \otimes v) \longmapsto ku \otimes v$$

and that the following identities hold:

$$k(u \otimes v) = ku \otimes v = uk \otimes v = u \otimes kv = u \otimes vk = (u \otimes v)k \tag{1.32}$$

for all $u \in U$, $v \in V$ and $k \in \Bbbk$.

Let U, V and W be $\Bbbk$-vector spaces. Recall that a function $f \colon U \times V \to W$ is called a $\Bbbk$-*bilinear* map if for each $u \in U$ and $v \in V$ the functions

$$f_u \colon V \longrightarrow W : v \longmapsto f(u, v)$$
$$f_v \colon U \longrightarrow W : u \longmapsto f(u, v)$$

are $\Bbbk$-linear maps. Let us denote with $\hom^{(2)}(U, V; W)$ the space of $\Bbbk$-bilinear maps from $U \times V$ to W. Summing up what we have seen until now, the tensor product of two vector spaces can be characterized as follows.

Theorem 1.3.3. ([Ka, Theorem II.1.1]) *Given $\Bbbk$-vector spaces U and V there exists a $\Bbbk$-vector space $U \otimes V$ and a bilinear map $b_{U \otimes V} \colon U \times V \to U \otimes V$ such that, for all $\Bbbk$-vector spaces W, the linear map*

$$(- \circ b_{U \otimes V}) \colon \hom(U \otimes V, W) \longrightarrow \hom^{(2)}(U, V; W) \colon f \longmapsto f \circ b_{U \otimes V}$$

is an isomorphism of $\Bbbk$-vector spaces. The vector space $U \otimes V$ is called the tensor product *of U and V and it is unique up to isomorphism.*

Proof. We already know that $U \otimes V$ and $b_{U \otimes V}$ exist and that $U \otimes V$ is a $\Bbbk$-vector space and $b_{U \otimes V}$ is a $\Bbbk$-biadditive map. Let us denote with simply b the map $b_{U \otimes V}$. In view of (1.32) we have that b is $\Bbbk$-bilinear, as claimed.

Next, if $f \colon U \times V \to W$ is $\Bbbk$-bilinear, then it is $\Bbbk$-biadditive and so there exists a unique morphism of abelian groups $\widetilde{f} \colon U \otimes V \to W$ such that $\widetilde{f} \circ b = f$, by the universal property of the tensor product. This $\widetilde{f}$ is also $\Bbbk$-linear, for f is $\Bbbk$-bilinear:

$$\widetilde{f}(k(u \otimes v)) = \widetilde{f}(b(ku, v)) = f(ku, v) = kf(u, v) = k\widetilde{f}(u \otimes v).$$

Hence, in order to conclude, it is enough to observe that if $g \colon U \otimes V \to W$ is another $\Bbbk$-linear map such that

$$\begin{tikzcd} U \times V \arrow[r, "b"] \arrow[d, "f"'] & U \otimes V \arrow[dl, "g"] \\ W & \end{tikzcd}$$

commutes, then it is obviously $\mathbb{Z}$-linear and so it has to coincide with $\widetilde{f}$. $\square$

Proposition 1.3.4. *Let $f\colon U \to U'$ and $g\colon V \to V'$ be $\Bbbk$-linear maps. Then we have a $\Bbbk$-linear map*

$$f \otimes g\colon U \otimes V \longrightarrow U' \otimes V'\colon u \otimes v \longmapsto f(u) \otimes g(v).$$

Furthermore, if $f'\colon U' \to U''$ and $g'\colon V' \to V''$ are other two $\Bbbk$-linear maps, we have that

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g).$$

Proof. It's enough to observe that the map

$$h\colon U \times V \longrightarrow U' \otimes V'\colon (u, v) \longmapsto f(u) \otimes g(v)$$

is $\Bbbk$-bilinear to have that there exists a unique $\Bbbk$-linear map $\widetilde{h}\colon U \otimes V \to U' \otimes V'$ such that $\widetilde{h} \circ b = h$, i.e., $\widetilde{h}(u \otimes v) = f(u) \otimes g(v)$.

Next, consider the following diagram, where $h'\colon U' \times V' \to U'' \otimes V''$ is defined as h above:

$$\begin{tikzcd}
U \times V \arrow[rr, "b"] \arrow[d, "(f,g)"'] \arrow[dr, "b'\circ(f,g)"] & & U \otimes V \arrow[dl, dashed, "\widetilde{b'\circ(f,g)}"] \\
U' \times V' \arrow[r, "b'"] \arrow[d, "h'"'] & U' \otimes V' \arrow[dl, dashed, "\widetilde{h'}"] & \\
U'' \otimes V'' & &
\end{tikzcd}$$

Every slashed arrow exists and it is unique by the universal property of the tensor product, and makes the corresponding diagram commute. Hence the map

$$\xi := \widetilde{h'} \circ \left(\widetilde{b' \circ (f, g)}\right)\colon U \otimes V \longrightarrow U'' \otimes V''\colon u \otimes v \longmapsto ((f' \otimes g') \circ (f \otimes g))(u \otimes v)$$

makes commutative the following diagram:

$$\begin{tikzcd}
U \times V \arrow[r, "b"] \arrow[d, "h'\circ(f,g)"'] & U \otimes V \arrow[dl, "\xi"] \\
U'' \otimes V'' &
\end{tikzcd}$$

By the universal property of the tensor product again, we have that $\xi = \widetilde{h' \circ (f, g)}$ and, by definition,

$$\widetilde{h' \circ (f, g)}(u, v) = ((f' \circ f) \otimes (g' \circ g))(u, v)$$

for all $u \in U$, $v \in V$. $\square$

Corollary 1.3.5. *The tensor product of $\Bbbk$-vector spaces is a functor*

$$\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}.$$

Proposition 1.3.6. ([Ka, Proposition II.1.3]) *Let U, V, W be $\Bbbk$-vector spaces. There are isomorphisms*

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

determined by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$,

$$\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$$

determined by $k \otimes v \mapsto kv$ and $v \mapsto v \otimes 1$, and

$$V \otimes W \cong W \otimes V$$

given by the flip $\tau_{V,W}$ defined by $\tau_{V,W}(v \otimes w) = w \otimes v$.

Proof. It is self evident that all the maps are bijections. Thus we only need to show that they are well defined.

For all $w \in W$ define a map

$$\begin{array}{rccc} f_w: & U \times V & \longrightarrow & U \otimes (V \otimes W) \\ & (u,v) & \longmapsto & u \otimes (v \otimes w) \end{array}$$

It is clearly $\Bbbk$-bilinear, thus there exists a unique $\Bbbk$-linear map that fills in the commutative diagram:

$$\begin{tikzcd} U \times V \arrow[r, "b_{U\otimes V}"] \arrow[d, "f_w"'] & U \otimes V \arrow[dl, dashed, "\widetilde{f_w}"] \\ U \otimes (V \otimes W) & \end{tikzcd}$$

Hence, we can define a map

$$\begin{array}{rccc} a: & (U \otimes V) \times W & \longrightarrow & U \otimes (V \otimes W) \\ & (z,w) & \longmapsto & \widetilde{f_w}(z) \end{array}$$

that is $\Bbbk$-bilinear, too. Indeed, if we write $z = \sum_i u_i \otimes v_i$, then for all $z \in U \otimes V$, $w \in W$, $h, k \in \Bbbk$

$$\begin{aligned} a\left(k\left(\sum_i u_i \otimes v_i\right), hw\right) &= a\left(\sum_i ku_i \otimes v_i, hw\right) = \widetilde{f_{hw}}\left(\sum_i ku_i \otimes v_i\right) = \\ &= \sum_i k\widetilde{f_{hw}}(u_i \otimes v_i) = \sum_i kf_{hw}(u_i, v_i) = \\ &= \sum_i k(u_i \otimes (v_i \otimes hw)) = kh\sum_i (u_i \otimes (v_i \otimes w)) = \\ &= kh\, a\left(\sum_i u_i \otimes v_i, w\right) \end{aligned}$$

Therefore, there exists a (unique) $\Bbbk$-linear map

$$\widetilde{a}\colon (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)\colon (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w).$$

Next, define a function

$$
\begin{array}{rccc}
t\colon & V \times W & \longrightarrow & W \otimes V \\
 & (m,n) & \longmapsto & n \otimes m
\end{array}
$$

It is $\Bbbk$-bilinear and thus factors through the tensor product:

$$\tau := \widetilde{t}\colon V \otimes W \longrightarrow W \otimes V : m \otimes n \longmapsto n \otimes m.$$

Finally, consider the $\Bbbk$-linear morphism

$$i_2\colon V \longrightarrow \Bbbk \otimes V : v \longmapsto 1 \otimes v.$$

If we can prove that the function

$$p_2\colon \Bbbk \otimes V \longrightarrow V : k \otimes v \longmapsto kv$$

is well defined, then it follows easily that $p_2 \circ i_2 = \mathrm{Id}_V$ and $i_2 \circ p_2 = \mathrm{Id}_{\Bbbk \otimes V}$. However, the assignment $(k,v) \mapsto kv$ for $k \in \Bbbk$ and $v \in V$ clearly defines a $\Bbbk$-bilinear map from $\Bbbk \times V$ to V that factors through the tensor product and so p_2 exists. $\square$

Corollary 1.3.7. *The three canonical isomorphism of the previous proposition:*

$$
\begin{aligned}
a_{U,V,W}\colon (U \otimes V) \otimes W &\longrightarrow U \otimes (V \otimes W) : (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w) \\
l_V\colon \Bbbk \otimes V &\longrightarrow V : k \otimes v \longmapsto kv \\
r_V\colon V \otimes \Bbbk &\longrightarrow V : v \otimes k \longmapsto kv
\end{aligned}
$$

are natural in all components and satisfy the Axioms (1.9) *and* (1.10).

Summing up, we have just proved that $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$ is a monoidal category, where the constraints are the ones given in the previous corollary. By Mac Lane's Coherence Theorem 1.2.10, we may omit all brackets from iterated tensor products and we may also omit the constraints in any computation involving morphisms in $\mathcal{M}$.

Now, fix a $\Bbbk$-vector space V. Then the assignment

$$(- \otimes V)\colon \mathcal{M} \to \mathcal{M}$$

that maps a vector space U into $U \otimes V$ and a $\Bbbk$-linear map $g\colon U \to U'$ into the $\Bbbk$-linear map $g \otimes V\colon U \otimes V \to U' \otimes V : u \otimes v \mapsto g(u) \otimes v$ defines a functor. Actually, in view of Theorem 1.1.9, this functor is left adjoint to the representable functor

$$\hom(V,-)\colon \mathcal{M} \longrightarrow \mathcal{M} : W \longmapsto \hom(V,W)$$

where the $\Bbbk$-vector space structure on $\hom(V,W)$ is given by

$$(k\, g)(v) = k\, g(v) = g(k\, v) \tag{1.33}$$

for all $k \in \Bbbk$, $g \in \hom(V,W)$ and $v \in V$. Indeed, we have the following result.

Proposition 1.3.8. ([Ka, Corollary II.1.2]) *For any triple (U,V,W) of $\Bbbk$-vector spaces there is a natural isomorphism of $\Bbbk$-vector spaces*

$$\hom(U \otimes V, W) \cong \hom(U, \hom(V,W)).$$

Proof. Recall that, by Theorem 1.3.3,

$$\hom(U \otimes V, W) \cong \hom^{(2)}(U, V; W)$$

via the $\Bbbk$-linear morphism that assigns to each $f: U \otimes V \to W$ in $\hom(U \otimes V, W)$, the composition $f \circ b_{U\otimes V}$, where $b_{U\otimes V}: U \times V \to U \otimes V$ is the canonical map.

Next, pick a $\Bbbk$-bilinear map $g: U \times V \to W$. We already know that for each $u \in U$,

$$g_u: V \longrightarrow W: v \longmapsto g(u, v)$$

is a $\Bbbk$-linear map, thus we can define a function:

$$g^*: U \longrightarrow \hom(V, W): u \longmapsto g_u$$

that is $\Bbbk$-linear, too. Indeed, for all $v \in V$,

$$g^*(ku)(v) = g_{ku}(v) = g(ku, v) = k\,g(u, v) = k\,g_u(v) = k\,g^*(u)(v) \overset{(1.33)}{=} (k\,g^*(u))(v).$$

This observation allows us to define another map:

$$\psi: \hom^{(2)}(U, V; W) \longrightarrow \hom(U, \hom(V, W)): g \longmapsto g^*. \tag{1.34}$$

As one can expect, the $\Bbbk$-vector space structure on $\hom^{(2)}(U, V; W)$ is given by:

$$(k\,g)(u, v) = k\,g(u, v) \qquad (\forall\, k \in \Bbbk, u \in U, v \in V, g \in \hom^{(2)}(U, V; W)). \tag{1.35}$$

Therefore, even ψ is $\Bbbk$-linear, since for all $u \in U$ and $v \in V$

$$\begin{aligned}(k\,g)^*(u)(v) = (k\,g)(u, v) &\overset{(1.35)}{=} k\,g(u, v) = \\ &= k\,g^*(u)(v) \overset{(1.33)}{=} (k\,g^*(u))(v) \overset{(1.33)}{=} \\ &= (k\,g^*)(u)(v),\end{aligned}$$

for every $k \in \Bbbk, u \in U, v \in V, g \in \hom^{(2)}(U, V; W)$.

Conversely, consider

$$\phi:\ \hom(U, \hom(V, W)) \to \hom^{(2)}(U, V; W)$$

that maps $g: U \to \hom(V, W)$ to

$$\begin{array}{rccc}\phi(g): & U \times V & \longrightarrow & W \\ & (u, v) & \longmapsto & g(u)(v)\end{array} \tag{1.36}$$

Let us show that ϕ is the inverse of ψ. First of all, let $g \in \hom(U, \hom(V, W))$. For all $u \in U$ and $v \in V$ we have that

$$\psi(\phi(g))(u)(v) \overset{(1.34)}{=} (\phi(g))^*(u)(v) = (\phi(g))(u, v) \overset{(1.36)}{=} g(u)(v)$$

and so, for all $u \in U$ and $g \in \hom(U, \hom(V, W))$, $\psi(\phi(g))(u) = g(u)$. That is, $\psi(\phi(g)) = g$ for every $g \in \hom(U, \hom(V, W))$.

On the other hand, for all $f \in \hom^{(2)}(U, V; W)$, $u \in U$, $v \in V$,

$$\phi(\psi(f))(u, v) \overset{(1.36)}{=} \psi(f)(u)(v) \overset{(1.34)}{=} f^*(u)(v) = f(u, v).$$

Hence, $\phi \circ \psi = \mathrm{Id}_{\hom^{(2)}(U,V;W)}$.

What we got, is that we have an isomorphism of $\Bbbk$-vector spaces

$$\hom(U \otimes V, W) \cong \hom(U, \hom(V, W)),$$

that is given by the composition of the two isomorphisms $(- \circ b_{U \otimes V})$ and ψ. Let us denote it by $\xi_{U,W}$.

Explicitly, we have that: if $f\colon U \otimes V \to W$ is a morphism in $\hom(U \otimes V, W)$, then $\xi_{U,W}(f)\colon U \to \hom(V, W)$ is the $\Bbbk$-linear map that assigns to each $u \in U$ the function

$$\xi_{U,W}(f)(u): \quad \begin{array}{ccc} V & \longrightarrow & W \\ v & \longmapsto & f(u \otimes v) \end{array} \tag{1.37}$$

It remains to prove that ξ is natural. To do that, pick two morphisms of $\Bbbk$-vector spaces

$$\alpha\colon U \to U'$$
$$\beta\colon W \to W'$$

and consider the diagram:

$$\begin{tikzcd}
\hom(U' \otimes V, W) \arrow[r, "\xi_{U',W}"] \arrow[d, "\beta\circ(-)\circ(\alpha\otimes V)"'] & \hom(U', \hom(V, W)) \arrow[d, "\hom(V,\beta)\circ(-)\circ\alpha"] \\
\hom(U \otimes V, W') \arrow[r, "\xi_{U,W'}"'] & \hom(U, \hom(V, W'))
\end{tikzcd}$$

where $\hom(V, \beta)$ is the function that assigns to each morphism g in $\hom(V, W)$ the morphism $\beta \circ g$ in $\hom(V, W')$. Let u be an element in U, v be an element in V, φ be in $\hom(U' \otimes V, W)$ and set

$$X(-) := (\hom(V, \beta) \circ (\xi_{U',W}(\varphi)) \circ \alpha)(-)$$
$$Y(-) := (\xi_{U,W'})(\beta \circ \varphi \circ (\alpha \otimes V))(-)$$

Hence, we have that on one hand:

$$\begin{aligned} X(u)(v) &= \left[(\hom(V, \beta) \circ (\xi_{U',W}(\varphi)) \circ \alpha)(u)\right](v) = \\ &= \left\{\hom(V, \beta)\left[(\xi_{U',W}(\varphi))(\alpha(u))\right]\right\}(v) = \\ &= \left[\beta \circ (\xi_{U',W}(\varphi))(\alpha(u))\right](v) = \\ &= \beta\left[(\xi_{U',W}(\varphi))(\alpha(u))(v)\right] \overset{(1.37)}{=} \\ &= \beta(\varphi(\alpha(u) \otimes v)) \end{aligned}$$

on the other hand:

$$\begin{aligned}
Y(u)(v) &= \xi_{U,W'}(\beta\circ\varphi\circ(\alpha\otimes V))(u)(v) \overset{(1.37)}{=} \\
&= (\beta\circ\varphi\circ(\alpha\otimes V))(u\otimes v) = \\
&= \beta\left[\varphi(\alpha(u)\otimes v)\right]
\end{aligned}$$

and so the diagram commutes and ξ is natural. $\square$

The previous proposition allows us to prove two more important properties of tensor products. The first one is that it commutes with arbitrary direct sums. The second one is that the functor $-\otimes V$ is right exact.

Proposition 1.3.9. ([Ka, Proposition II.1.4]) *Let $\{U_i\}_{i\in I}$ be a family of $\Bbbk$-vector spaces and V another $\Bbbk$-vector space. Then we have that*

$$\left(\bigoplus_{i\in I} U_i\right)\otimes V \cong \bigoplus_{i\in I}(U_i\otimes V)$$

Proof. By the universal property of the direct sum and Proposition 1.3.8 we have that

$$\begin{aligned}
\hom\left(\left(\bigoplus_{i\in I} U_i\right)\otimes V, W\right) &\cong \hom\left(\bigoplus_{i\in I} U_i, \hom(V,W)\right) \cong \\
&\cong \prod_{i\in I}\hom(U_i,\hom(V,W)) \cong \\
&\cong \prod_{i\in I}\hom(U_i\otimes V, W) \cong \\
&\cong \hom\left(\bigoplus_{i\in I}(U_i\otimes V), W\right)
\end{aligned}$$

Call α the natural isomorphism given by the composition, i.e., for W $\Bbbk$-vector space:

$$\alpha_W\colon \hom\left(\left(\bigoplus_{i\in I} U_i\right)\otimes V, W\right) \to \hom\left(\bigoplus_{i\in I}(U_i\otimes V), W\right)$$

First, consider the case $W = \left(\bigoplus_{i\in I} U_i\right)\otimes V$ and set $\varphi := \alpha\left(\mathrm{Id}_{(\bigoplus_{i\in I} U_i)\otimes V}\right)$. Then, consider the case $W = \bigoplus_{i\in I}(U_i\otimes V)$ and set $\psi := \alpha^{-1}\left(\mathrm{Id}_{\bigoplus_{i\in I}(U_i\otimes V)}\right)$. The subscripts are omitted in order to lighten the notation. We claim that φ and ψ are inverses one another. Indeed, by naturality of α:

$$\psi\circ\varphi = \psi\circ\alpha(\mathrm{Id}) = \alpha(\psi) = \mathrm{Id},$$

$$\varphi\circ\psi = \varphi\circ\alpha^{-1}(\mathrm{Id}) = \alpha^{-1}(\varphi) = \mathrm{Id}.$$

$\square$

Corollary 1.3.10. *Let $\{u_i \mid i \in I\}$ be a basis of the vector space U and $\{v_j \mid j \in J\}$ be a basis of V. Then the set $\{u_i \otimes v_j \mid (i,j) \in I \times J\}$ is a basis of the tensor product $U \otimes V$. Consequently, we have $\dim(U \otimes V) = \dim(U)\dim(V)$.*

Proof. In view of Proposition 1.3.6 and Proposition 1.3.9 we have that

$$U \otimes V \cong \left(\bigoplus_{i\in I} \Bbbk u_i\right) \otimes \left(\bigoplus_{j\in J} \Bbbk v_j\right) \cong \bigoplus_{(i,j)\in I\times J} \Bbbk(u_i \otimes v_j).$$

□

Proposition 1.3.11. *Let $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3 \longrightarrow 0$ be an exact sequence of $\Bbbk$-vector spaces. Then, for each vector space V, the following sequence is exact too:*

$$U_1 \otimes V \xrightarrow{f\otimes V} U_2 \otimes V \xrightarrow{g\otimes V} U_3 \otimes V \longrightarrow 0$$

Proof. There exists a result that states that if R is a commutative ring and

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is a sequence of R-modules, then it is exact if and only if, for each R-module N, is exact the sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \hom(M_3, N) \xrightarrow{\hom(g,N)} \hom(M_2, N) \xrightarrow{\hom(f,N)} \hom(M_1, N),$$

where $\hom(f, N)$ represents the morphism of groups that maps $h \in \hom(M_2, N)$ into $h \circ f \in \hom(M_1, N)$ (for details, refer to [Ro, Section 7.3]).

Obviously, $\Bbbk$ is a commutative ring, thus

$$U_1 \otimes V \xrightarrow{f\otimes V} U_2 \otimes V \xrightarrow{g\otimes V} U_3 \otimes V \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow \hom(U_3 \otimes V, W) \xrightarrow{\hom(g\otimes V,W)} \hom(U_2 \otimes V, W) \xrightarrow{\hom(f\otimes V,W)} \hom(U_1 \otimes V, W)$$

is exact for every $\Bbbk$-vector space W. However, if you set $Z = \hom(V, W)$ and look at the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \hom(U_3 \otimes V, W) \arrow[r, "\hom(g\otimes V,W)"] \arrow[d, "\xi_{U_3,W}"'] & \hom(U_2 \otimes V, W) \arrow[r, "\hom(f\otimes V,W)"] \arrow[d, "\xi_{U_2,W}"'] & \hom(U_1 \otimes V, W) \arrow[d, "\xi_{U_1,W}"'] \\ 0 \arrow[r] & \hom(U_3, Z) \arrow[r, "\hom(g,Z)"] & \hom(U_2, Z) \arrow[r, "\hom(f,Z)"] & \hom(U_1, Z) \end{tikzcd}$$

it is commutative by naturality of ξ and the lower sequence is exact, since the sequence $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3 \longrightarrow 0$ is by hypothesis. □

Actually, everything we have seen until now about tensor product holds true in the more general context of R-modules over a commutative ring R. However, $\Bbbk$-vector spaces have something more than a simple R-module: they are all *free* $\Bbbk$-modules. This guarantees that for each vector space V, the functor $- \otimes V$ is *exact.*

Proposition 1.3.12. *Every $\Bbbk$-vector space is* flat*. This means that, if V is a $\Bbbk$-vector space and*

$$0 \longrightarrow U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3 \longrightarrow 0$$

is a short exact sequence of $\Bbbk$-vector spaces, then

$$0 \longrightarrow U_1 \otimes V \xrightarrow{f \otimes V} U_2 \otimes V \xrightarrow{g \otimes V} U_3 \otimes V \longrightarrow 0$$

is a short exact sequence as well.

Proof. We just have to prove that if $f\colon U \to V$ is an injective morphism, then also $f \otimes W\colon U \otimes W \to V \otimes W$ is, for each $W \in \mathrm{Vect}(\Bbbk)$. However, if $\{e_i \mid i \in I\}$ is a $\Bbbk$-basis for W, then

$$W \cong \bigoplus_{i \in I} \Bbbk$$

as $\Bbbk$-vector spaces. In light of Proposition 1.3.6 and Proposition 1.3.9, we have the following commutative diagram:

$$\begin{tikzcd}
U \otimes W \arrow[r, "f \otimes W"] \arrow[d, leftrightarrow, "\cong"'] & V \otimes W \arrow[d, leftrightarrow, "\cong"] \\
U \otimes (\bigoplus_{i \in I} \Bbbk) \arrow[r, "f \otimes \mathrm{Id}"] \arrow[d, leftrightarrow, "\cong"'] & V \otimes (\bigoplus_{i \in I} \Bbbk) \arrow[d, leftrightarrow, "\cong"] \\
\bigoplus_{i \in I} (U \otimes \Bbbk) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \bigoplus_{i \in I} (V \otimes \Bbbk) \arrow[d, leftrightarrow, "\cong"] \\
\bigoplus_{i \in I} U \arrow[r, "\varphi"'] & \bigoplus_{i \in I} V
\end{tikzcd}$$

that means that $f \otimes W$ is injective if and only if φ is injective, where φ is the codiagonal morphism of the family $f_i\colon U_i \to V_i$, $i \in I$, defined by $f_i(u) = f(u)$ for all $u \in U_i = U$.

Now, it is clear that φ is injective if and only if all f_i are injective. Indeed, by definition of φ the following diagram commutes:

$$\begin{tikzcd}
U_s \arrow[r, "f_s"] \arrow[d, "\varepsilon_s"'] & V_s \arrow[d, "\varepsilon_s"] \\
\bigoplus_{i \in I} U \arrow[r, "\varphi"'] & \bigoplus_{i \in I} V
\end{tikzcd}$$

where ε_s is the canonical inclusion of the s-factor. That implies that if φ is injective, then also f_s is, for all $s \in I$. Conversely, if

$$(f_i(u_i))_{i \in I} = \varphi((u_i)_{i \in I}) = \varphi((v_i)_{i \in I}) = (f_i(v_i))_{i \in I}$$

then, taking the projection on the s-factor, we find that $f_s(u_s) = f_s(v_s)$ for all $s \in I$. By hypothesis on the f_s, we have that $u_s = v_s$ for all $s \in I$ and so φ is injective.

As f is injective by hypothesis, φ is injective and so $f \otimes W$ is. $\square$

Remark 1.3.13. As a general fact, any functor preserves split morphisms: if f and g are morphisms in a certain category $\mathcal{C}$ such that $g \circ f = \mathrm{Id}_{\mathrm{dom}(f)}$, and if $F \colon \mathcal{C} \to \mathcal{D}$ is any functor, then

$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{Id}_{\mathrm{dom}(f)}) = \mathrm{Id}_{F(\mathrm{dom}(f))}.$$

In particular the tensor product functor does (in any monoidal category). In our case of vector spaces then, an alternative proof of Proposition 1.3.12 could be given by employing such a fact. Indeed, any monomorphism (it is just a synonym for monic) in the category of vector spaces splits.

Let us just recall briefly why: let $f \colon U \to V$ be a monomorphism of vector spaces. Pick a basis $\{e_i \mid i \in I\}$ for U and consider its image in V: $\mathcal{B} = \{f(e_i) \mid i \in I\}$. Complete it to a basis for V: $\mathcal{B} \cup \mathcal{B}'$, and define $g \colon V \to U$ by setting $g(f(e_i)) := e_i$ for all i in I and $g(\mathcal{B}') = 0$. This g is a well defined linear map and it is such that $g \circ f = \mathrm{Id}_U$.

Now, since we have proved that $\mathcal{M} = \mathrm{Vect}(\Bbbk)$ is a monoidal category, we can consider algebras and coalgebras in $\mathcal{M}$. But as we said at the beginning of this section, algebras and coalgebras in $\mathrm{Vect}(\Bbbk)$ are nothing more than the usual $\Bbbk$-algebras and $\Bbbk$-coalgebras. Moreover, we can consider modules and comodules as defined in Definition 1.2.14 and these are the ordinary modules and comodules over (not necessarily commutative or cocommutative) $\Bbbk$-algebras and coalgebras. However, a quite interesting new concept that we are going to introduce shortly is the one of bialgebras. Before, let us fix some notations.

Remark 1.3.14. (Sweedler' Sigma Notation) Consider the category of $\Bbbk$-vector spaces $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$. Let (C, Δ, ε) be a coalgebra in $\mathcal{M}$ and let $x \in C$ be an element. *Sweedler' Sigma Notation* is a formal writing to denote the image through the comultiplication of a general element of the coalgebra:

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \in C \otimes C.$$

Since Δ is coassociative and ε is a counit for Δ, the following equalities hold:

$$\begin{aligned} \sum_{(x)} \left(\sum_{(x_{(1)})} \left(x_{(1)}\right)_{(1)} \otimes \left(x_{(1)}\right)_{(2)} \right) \otimes x_{(2)} &= \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \\ \sum_{(x)} x_{(1)} \otimes \left(\sum_{(x_{(2)})} \left(x_{(2)}\right)_{(1)} \otimes \left(x_{(2)}\right)_{(2)} \right) &= \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \end{aligned} \tag{1.38a}$$

$$\sum_{(x)} \varepsilon(x_{(1)})\, x_{(2)} = x = \sum_{(x)} x_{(1)}\, \varepsilon(x_{(2)}) \tag{1.38b}$$

Nevertheless, we will prefer a slightly less heavy variation of the 'Sweedler' Sigma notation':

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_1 \otimes x_2 \quad (\forall\, x \in C),$$

without summation symbols and too many parenthesis.

In the same way, let (N, ρ) be a right C-comodule as defined in 1.2.14. For all $n \in N$, the following identities are immediate consequences of the definition and the formal 'Sigma notation' we introduced:

$$\rho(n) = \sum_{(n)} n_{(0)} \otimes n_{(1)} = n_0 \otimes n_1 \tag{1.39a}$$

$$n_0 \otimes ((n_1)_1 \otimes (n_1)_2) = ((n_0)_0 \otimes (n_0)_1) \otimes n_1 = n_0 \otimes n_1 \otimes n_2 \tag{1.39b}$$

$$n_0\, \varepsilon(n_1) = n \tag{1.39c}$$

where $n_0 \otimes ((n_1)_1 \otimes (n_1)_2) = (N \otimes \Delta)(\rho(n))$ and $((n_0)_0 \otimes (n_0)_1) \otimes n_1 = (\rho \otimes C)(\rho(n))$.

Let (A, m, u) be an algebra in $\mathcal{M}$. We know that $A \otimes A$ is an object in $\mathcal{M}$ as well. Actually, it is an algebra too.

Proposition 1.3.15. *Let (A, m, u) be an algebra in $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$ and define:*

$$m_\otimes = (m \otimes m) \circ (a_{A,A,A\otimes A})^{-1} \circ (A \otimes a_{A,A,A}) \circ (A \otimes \tau \otimes A) \circ \left(A \otimes (a_{A,A,A})^{-1}\right) \circ a_{A,A,A\otimes A}$$

and $u_\otimes = (u \otimes u) \circ \Delta_\Bbbk$, where τ is the twist: $\tau(a \otimes b) = b \otimes a$. Then $(A \otimes A, m_\otimes, u_\otimes)$ is an algebra in $\mathcal{M}$.

Proof. Indeed, since:

$$m_\otimes((a \otimes b) \otimes (c \otimes d)) = ac \otimes bd \quad \text{and} \quad u_\otimes(1_\Bbbk \otimes 1_\Bbbk) = 1_A \otimes 1_A,$$

it is self evident that $m_\otimes$ inherits all the properties of m:

$$\begin{aligned} &m_\otimes(m_\otimes((a \otimes b) \otimes (c \otimes d)) \otimes (x \otimes y)) = (ac)x \otimes (bd)y = \\ &\quad = a(cx) \otimes b(dy) = m_\otimes((a \otimes b) \otimes m_\otimes((c \otimes d) \otimes (x \otimes y))), \end{aligned} \qquad \text{(associativity)}$$

$$\left.\begin{aligned} m_\otimes((u_\otimes \otimes (A \otimes A))(k \otimes (a \otimes b))) &= k(a \otimes b), \\ m_\otimes(((A \otimes A) \otimes u_\otimes)((a \otimes b) \otimes k)) &= (a \otimes b)k. \end{aligned}\right\} \quad \text{(unity)}$$

□

Dually, the same holds true for coalgebras.

Proposition 1.3.16. *Let (C, Δ, ε) be a coalgebra in $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$ and define*

$$\Delta_\otimes = (a_{A,A,A\otimes A})^{-1} \circ (A \otimes a_{A,A,A}) \circ (A \otimes \tau \otimes A) \circ \left(A \otimes (a_{A,A,A})^{-1}\right) \circ a_{A,A,A\otimes A} \circ (\Delta \otimes \Delta)$$

and $\varepsilon_\otimes = m_\Bbbk \circ (\varepsilon \otimes \varepsilon)$. Then $(C \otimes C, \Delta_\otimes, \varepsilon_\otimes)$ is a coalgebra in $\mathcal{M}$.

Proof. $\Delta_\otimes$ and $\varepsilon_\otimes$ are morphisms in $\mathcal{M}$, since they are composition of morphisms in $\mathcal{M}$. The coassociativity of Δ guarantees that:

$$\begin{aligned} (\Delta_\otimes \otimes C)\,(\Delta_\otimes(x \otimes y)) &= (((x_1)_1 \otimes (y_1)_1) \otimes ((x_1)_2 \otimes (y_1)_2)) \otimes (x_2 \otimes y_2) = \\ &= (x_1 \otimes y_1) \otimes (x_2 \otimes y_2) \otimes (x_3 \otimes y_3) = \\ &= (x_1 \otimes y_1) \otimes (((x_2)_1 \otimes (y_2)_1) \otimes ((x_2)_2 \otimes (y_2)_2)) = \\ &= (C \otimes \Delta_\otimes)\,(\Delta_\otimes(x \otimes y)). \end{aligned}$$

Moreover:

$$(\varepsilon_\otimes \otimes C)(\Delta_\otimes(x \otimes y)) = \varepsilon(x_1)\varepsilon(y_1) \otimes x_2 \otimes y_2 = 1_{\Bbbk} \otimes x \otimes y = (l_{C\otimes C})^{-1}(x \otimes y)$$

and

$$(C \otimes \varepsilon_\otimes)(\Delta_\otimes(x \otimes y)) = x_1 \otimes y_1 \otimes \varepsilon(x_2)\varepsilon(y_2) = x \otimes y \otimes 1_{\Bbbk} = (r_{C\otimes C})^{-1}(x \otimes y).$$

□

Remark 1.3.17. That the tensor product of two algebras (or coalgebras) is an algebra (or a coalgebra), is in fact mainly a consequence of the symmetry (given by the flip map τ) of the tensor product of vector spaces. The same holds true in the monoidal category of (say left) modules over a commutative ring. But unfortunately, there is no direct way to endow a tensor product of two algebras (resp. coalgebras) with a structure of algebra (resp. coalgebra), if the handled monoidal category is no longer symmetric (or at least braided). This happens for instance in case of the category of bimodules over a non commutative ring.

Assume now that $A \in \mathcal{M}$ is an object equipped simultaneously with an algebra structure (A, m, u) and a coalgebra structure (C, Δ, ε).

Theorem 1.3.18. ([Ka, Theorem III.2.1], [Sw, Proposition 3.1.1]) *The following are equivalent:*

(1) Δ and ε are morphisms of algebras.

(2) m and u are morphisms of coalgebras.

Proof. Observe that m and u are morphisms of coalgebras if and only if the following diagrams commutes:

$$\begin{tikzcd} A\otimes A \arrow[rr, "m_A"] \arrow[d, "\Delta_{A\otimes A}"'] & & A \arrow[d, "\Delta_A"] \\ (A\otimes A)\otimes(A\otimes A) \arrow[rr, "m_A\otimes m_A"'] \arrow[urr, phantom, "(a)"] & & A\otimes A \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[rr, "m_A"] \arrow[dr, "\varepsilon_{A\otimes A}"'] & \arrow[d, phantom, "(b)"] & A \arrow[dl, "\varepsilon_A"] \\ & \Bbbk & \end{tikzcd}$$

$$\begin{tikzcd} \Bbbk \arrow[rr, "u_A"] \arrow[d, "\Delta_{\Bbbk}"'] & & A \arrow[d, "\Delta_A"] \\ \Bbbk\otimes\Bbbk \arrow[rr, "u_A\otimes u_A"'] \arrow[urr, phantom, "(c)"] & & A\otimes A \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[rr, "u_A"] \arrow[dr, "\varepsilon_{\Bbbk}"'] & \arrow[d, phantom, "(d)"] & A \arrow[dl, "\varepsilon_A"] \\ & \Bbbk & \end{tikzcd}$$

On the other hand, Δ and ε are morphisms of algebras if and only if the following diagrams commute:

$$\begin{tikzcd} A\otimes A \arrow[rr, "\Delta_A\otimes\Delta_A"] \arrow[d, "m_A"'] & & (A\otimes A)\otimes(A\otimes A) \arrow[d, "m_{A\otimes A}"] \\ A \arrow[rr, "\Delta_A"'] \arrow[urr, phantom, "(i)"] & & A\otimes A \end{tikzcd} \qquad \begin{tikzcd} & \Bbbk \arrow[dl, "u_A"'] \arrow[dr, "u_{A\otimes A}"] \arrow[d, phantom, "(ii)"] & \\ A \arrow[rr, "\Delta_A"'] & \, & A\otimes A \end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "\varepsilon_A \otimes \varepsilon_A"] \arrow[d, "m_A"'] \arrow[dr, phantom, "(iii)"] & \Bbbk \otimes \Bbbk \arrow[d, "m_{\Bbbk}"] \\
A \arrow[r, "\varepsilon_A"'] & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Bbbk \arrow[dl, "u_A"'] \arrow[dr, "u_{\Bbbk}"] \arrow[d, phantom, "(iv)"] & \\
A \arrow[rr, "\varepsilon_A"'] & & \Bbbk
\end{tikzcd}$$

Omitting the associative constraints we have that:

- (a) commutes if and only if

  $$\Delta_A \circ m_A = (m_A \otimes m_A) \circ \Delta_{A\otimes A} = (m_A \otimes m_A) \circ (\mathrm{Id}_A \otimes \tau \otimes \mathrm{Id}_A) \circ (\Delta_A \otimes \Delta_A) = \\ = m_{A\otimes A} \circ (\Delta_A \otimes \Delta_A),$$

  if and only if (i) commutes.

- (b) commutes if and only if $\varepsilon_A \circ m_A = \varepsilon_{A\otimes A} = m_{\Bbbk} \circ (\varepsilon_A \otimes \varepsilon_A)$, if and only if (iii) commutes.

- (c) commutes if and only if $\Delta_A \circ u_A = (u_A \otimes u_A) \circ \Delta_{\Bbbk} = u_{A\otimes A}$, if and only if (ii) commutes.

- (d) commutes if and only if $\varepsilon_A \circ u_A = \varepsilon_{\Bbbk} = \mathrm{Id}_{\Bbbk}$, if and only if (iv) commutes.

□

Definition 1.3.19. (Bialgebra)
A bialgebra is a quintuple $(B, m, u, \Delta, \varepsilon)$ where (B, m, u) is an algebra and (B, Δ, ε) is a coalgebra verifying the equivalent conditions of Theorem 1.3.18.

Note that in a bialgebra $(B, m, u, \Delta, \varepsilon)$ the following identities hold:

$$(xy)_1 \otimes (xy)_2 = x_1y_1 \otimes x_2y_2$$

and

$$\Delta(1_A) = 1_A \otimes 1_A, \quad \varepsilon(xy) = \varepsilon(x)\,\varepsilon(y), \quad \varepsilon(1_A) = 1_{\Bbbk}.$$

Remark 1.3.20. Let (A, m, u) be an algebra with a morphism of algebras $\Delta\colon A \to A \otimes A$, called the *comultiplication*, and a morphism of algebras $\varepsilon\colon A \to \Bbbk$, called the *counit*. If (M, μ_M) and (N, μ_N) are right A-modules, then $M \otimes N$ comes with a natural structure of right $A \otimes A$-module:

$$(m \otimes n) \cdot (a \otimes b) = m \cdot a \otimes n \cdot b,$$

extended by linearity. Indeed:

$$(m \otimes n) \cdot ((a \otimes b)(c \otimes d)) = m(ac) \otimes n(bd) = ((m \otimes n) \cdot (a \otimes b)) \cdot (c \otimes d)),$$

$$(m \otimes n) \cdot (1 \otimes 1) = m \otimes n.$$

The comultiplication allows us to convert this structure into an A-module structure:

$$(m \otimes n) \cdot a = (m \otimes n) \cdot \Delta(a) \quad \forall\, m \in M, n \in N, a \in A,$$

extended, again, by linearity. In view of the fact that Δ is a morphism of algebras, we have that:

$$(m \otimes n) \cdot ab = (m \otimes n) \cdot \Delta(ab) = ((m \otimes n) \cdot \Delta(a)) \cdot \Delta(b) = ((m \otimes n) \cdot a) \cdot b.$$

Via ε we can give an A-module structure to $\Bbbk$ too:

$$k \cdot a = k\,\varepsilon(a).$$

Thus the tensor product over $\Bbbk$ restricts to a tensor structure on the category $\mathcal{M}_A$, even if Δ is not coassociative or ε is not a counit for Δ.

The last Remark suggests a way to characterize bialgebras in terms of their categories of modules.

Proposition 1.3.21. ([Ka, Proposition XI.3.1]) *Let (A, m, u) be an algebra with comultiplication Δ and counit ε. $(A, m, u, \Delta, \varepsilon)$ is a bialgebra if and only if $(\mathcal{M}_A, \otimes, \Bbbk, a, l, r)$ is a monoidal category, where a, l and r are the same constraints of $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$.*

Proof. Start assuming that $(A, m, u, \Delta, \varepsilon)$ is a bialgebra. Obviously, a, l and r are natural and satisfy the Axioms (1.9) and (1.10). Hence we are left to prove that $a_{M,N,P}$, l_M and r_M are morphisms of right A-modules for all M, N and P in $\mathcal{M}_A$. However, A-linearity is a straightforward consequence of the coassociativity and counity of Δ and ε. In particular, in view of (1.38a) and (1.38b), for all $m \in M$, $n \in N$, $p \in P$ and $x \in A$:

$$\begin{aligned} a_{M,N,P}([(m \otimes n) \otimes p] \cdot x) &= a_{M,N,P}((m \cdot (x_1)_1 \otimes n \cdot (x_1)_2) \otimes p \cdot x_2) = \\ &= m \cdot (x_1)_1 \otimes (n \cdot (x_1)_2 \otimes p \cdot x_2) \overset{(1.38a)}{=} \\ &= m \cdot x_1 \otimes (n \cdot (x_2)_1 \otimes p \cdot (x_2)_2) = \\ &= a_{M,N,P}((m \otimes n) \otimes p) \cdot x, \end{aligned}$$

$$l_M((1 \otimes m) \cdot x) = l_M(\varepsilon(x_1) \otimes m \cdot x_2) = m \cdot \varepsilon(x_1)x_2 \overset{(1.38b)}{=} l_M(1 \otimes m) \cdot x$$

and

$$r_M((m \otimes 1) \cdot x) = r_M(m \cdot x_1 \otimes \varepsilon(x_2)) = m \cdot x_1\,\varepsilon(x_2) \overset{(1.38b)}{=} r_M(m \otimes 1) \cdot x$$

Proving the converse is immediate, too. Let $m \in M$, $n \in N$, $p \in P$ and $x \in A$. We know that the associative constraint is right A-linear, thus

$$a_{M,N,P}([(m \otimes n) \otimes p] \cdot x) = a_{M,N,P}((m \otimes n) \otimes p) \cdot x.$$

By definition of the right A-action on the tensor product, this last relation can be rewritten as:

$$(m \otimes (n \otimes p))(\Delta \otimes A)(\Delta(x)) = (m \otimes (n \otimes p))(A \otimes \Delta)(\Delta(x)).$$

Setting $M = N = P = A$ and $m = n = p = 1$ gives coassociativity of Δ. In the same way, since also the left and right unit constraints are A-linear, we have that:

$$\begin{aligned} m \cdot \varepsilon(x_1)x_2 &= l_M((1 \otimes m) \cdot x) = l_M(1 \otimes m) \cdot x = m \cdot x, \\ m \cdot x_1\,\varepsilon(x_2) &= r_M((m \otimes 1) \cdot x) = r_M(m \otimes 1) \cdot x = m \cdot x. \end{aligned}$$

Setting again $M = A$ and $m = 1$ we find that ε is a counit. $\square$

Example 1.3.22. Let X be a set and $V := \Bbbk X = \bigoplus_{x\in X} \Bbbk x$ be the $\Bbbk$-vector space with basis X. Define

$$\Delta(x) = x \otimes x \quad \text{and} \quad \varepsilon(x) = 1_{\Bbbk}$$

for all $x \in X$ and extend them by $\Bbbk$-linearity. Thus we have that

$$[a_{V,V,V} \circ (\Delta \otimes V) \circ \Delta](x) = x \otimes (x \otimes x) = (V \otimes \Delta)(\Delta(x)),$$
$$(\varepsilon \otimes V)(\Delta(x)) = x = (V \otimes \varepsilon)(\Delta(x))$$

and so $(\Bbbk X, \Delta, \varepsilon)$ is a coalgebra. Furthermore, assume that X is equipped with a unital monoid structure, i.e., with an associative map $\mu\colon X \times X \to X$ having a left and right unit u. Then $(\Bbbk X, m, u, \Delta, \varepsilon)$ is a bialgebra, where m denotes the $\Bbbk$-linear extension of μ to $\Bbbk X$.

Indeed, if we define the $\Bbbk$-linear function

$$\mu\colon \Bbbk X \times \Bbbk X \longrightarrow \Bbbk X\colon \left(\sum_i^{<\infty} k_i x_i, \sum_j^{<\infty} h_j y_j\right) \longmapsto \sum_{i,j}^{<\infty} k_i h_j \mu(x_i, y_j).$$

then it is clearly $\Bbbk$-bilinear and thus there exists a unique $\Bbbk$-linear map:

$$m\colon \Bbbk X \otimes \Bbbk X \longrightarrow \Bbbk X\colon \sum_i^{<\infty} k_i (x_i \otimes y_i) \longmapsto \sum_i^{<\infty} k_i \mu(x_i, y_i).$$

Since μ is associative and unital, also m becomes associative and unital and so $(\Bbbk X, m, u)$ is an algebra. For simplicity's sake, denote $\mu(x, y) = xy$. Since

$$\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$$
$$\varepsilon(xy) = 1 = \varepsilon(x)\,\varepsilon(y)$$

we have that, actually, $(\Bbbk X, m, u, \Delta, \varepsilon)$ is a bialgebra as claimed.

The best known example of such a construction is the *group algebra* $\Bbbk G$ defined on a group G. We will come back to the group algebra later.

Example 1.3.23. Consider $\Bbbk[T]$, the polynomial algebra of one indeterminate T. Besides the bialgebra structure inherited by the previous example (it can be seen as the vector space with basis the monoid $\{T^n \mid n \in \mathbb{N}\}$), it can be equipped with another bialgebra structure. Define

$$\Delta(T) = T \otimes 1 + 1 \otimes T \quad \text{and} \quad \varepsilon(T) = 0$$

and extend them by induction using polynomial multiplication:

$$\Delta(T^{n+1}) = \Delta(T^n)\Delta(T) \quad \text{and} \quad \varepsilon(T^n) = 0,$$

for all $n \in \mathbb{N}$. By construction, Δ and ε are morphisms of algebras, coassociative and counital respectively:

$$(\Bbbk[T] \otimes \Delta)\,(\Delta(T)) = T \otimes 1 \otimes 1 + 1 \otimes T \otimes 1 + 1 \otimes 1 \otimes T = (\Delta \otimes \Bbbk[T])\,(\Delta(T))$$
$$(r \circ (\Bbbk[T] \otimes \varepsilon) \circ \Delta)(T) = T = (l \circ (\varepsilon \otimes \Bbbk[T]) \circ \Delta)(T)$$

Hence, $\Bbbk[T]$ with this structure maps is again a bialgebra.

Moreover, take $\Bbbk[T, T^{-1}]$, the Laurent polynomial algebra of one indeterminate T. It is a bialgebra as it was seen in Example 1.3.22, by taking the cyclic free abelian group generated by $\{T\}$.

Chapter 2

The Structure Theorem for Hopf modules

Throughout we will assume that $\mathcal{M} = (\mathcal{M}, \otimes, \Bbbk, a, l, r)$ is the monoidal category of $\Bbbk$-vector spaces and that $B = (B, m, u, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{M}$.

2.1 An equivalence between $\mathcal{M}_B^B$ and $\mathcal{M}$

Lemma 2.1.1. *(B, m) is a right B-module and $((B, m), \Delta, \varepsilon)$ is a coalgebra within the monoidal category $(\mathcal{M}_B, \otimes, \Bbbk, a, l, r)$.*

Proof. First of all, (B, m) is trivially a right B-module and then an object in $\mathcal{M}_B$. Secondly, we have that Δ and ε are both right B-module morphisms, since they are morphisms of algebras:

$$\begin{aligned}\Delta(m) \cdot b = m_1 b_1 \otimes m_2 b_2 = (mb)_1 \otimes (mb)_2 = \Delta(m \cdot b),\\ \varepsilon(m) \cdot b = \varepsilon(m)\,\varepsilon(b) = \varepsilon(m \cdot b)\end{aligned}$$

and, finally, we have that ε is a counit for Δ and Δ is coassociative, because B is a bialgebra. $\square$

Thus we can construct the category $\mathcal{M}_B^B := (\mathcal{M}_B)^B$.

Remark 2.1.2. By virtue of the symmetry that arises from Theorem 1.3.18, we can also consider the algebra $((B, \Delta), m, u)$ within the monoidal category $(\mathcal{M}^B, \otimes, \Bbbk, a, l, r)$ and then define $\mathcal{M}_B^B := (\mathcal{M}^B)_B$. Nevertheless, we will see in Chapter 3 that our choice is a matter of consistency.

Definition 2.1.3. (Hopf modules)
An object M in $\mathrm{Ob}\left(\mathcal{M}_B^B\right)$ with two structures $\mu_M \in \hom(\mathcal{M})$, $\mu_M \colon M \otimes B \to M$, and $\rho_M \in \hom(\mathcal{M}_B)$, $\rho_M \colon M \to M \otimes B$, is called a *(right) Hopf B-module* (where the B-module structure on $M \otimes B$ is given through Δ, as in Remark 1.3.20). We will usually refer to M as simply an *Hopf module*, without further specifications, and the right B-action will be denoted by:

$$\mu_M(m \otimes b) := m \cdot b$$

for the sake of simplicity.

Remark 2.1.4. Observe that if (M, μ_M, ρ_M) is an Hopf module, where we denote with μ_M the right B-action and with ρ_M the right B-coaction, then ρ_M has to be a B-module morphism, so that the following should commute:

$$\begin{tikzcd} M \otimes B \arrow[r, "\rho_M \otimes B"] \arrow[d, "\mu_M"'] & (M \otimes B) \otimes B \arrow[d, "\mu_{M\otimes B}"] \\ M \arrow[r, "\rho_M"'] & M \otimes B \end{tikzcd}$$

i.e.,

$$\rho_M(m \cdot b) = (m_0 \otimes m_1) \cdot b = (m_0 \otimes m_1) \cdot \Delta(b) = m_0 \cdot b_1 \otimes m_1 b_2. \tag{2.1}$$

Lemma 2.1.5. *Let M be a $\Bbbk$-vector space. Then $M \otimes B$ becomes an Hopf module by setting, for $m \in M$ and $b, x \in B$*

$$(m \otimes b) \cdot x := m \otimes bx \tag{2.2}$$

$$\rho_{M\otimes B}(m \otimes b) := (m \otimes b_1) \otimes b_2 \tag{2.3}$$

Proof. Note that the following diagrams:

$$\begin{tikzcd} M \otimes B \otimes B \otimes B \arrow[r, "M\otimes B\otimes m"] \arrow[d, "\mu_{M\otimes B}\otimes B"'] & M \otimes B \otimes B \arrow[d, "\mu_{M\otimes B}"] \\ M \otimes B \otimes B \arrow[r, "\mu_{M\otimes B}"'] & M \otimes B \end{tikzcd} \qquad \begin{tikzcd} M \otimes B \otimes \Bbbk \arrow[r, "M\otimes B\otimes u"] \arrow[dr, "r_{M\otimes B}"'] & M \otimes B \otimes B \arrow[d, "\mu_{M\otimes B}"] \\ & M \otimes B \end{tikzcd}$$

$$\begin{tikzcd} M \otimes B \arrow[r, "\rho_{M\otimes B}"] \arrow[d, "\rho_{M\otimes B}"'] & M \otimes B \otimes B \arrow[d, "\rho_{M\otimes B}\otimes B"] \\ M \otimes B \otimes B \arrow[r, "M\otimes B\otimes\Delta"'] & M \otimes B \otimes B \otimes B \end{tikzcd} \qquad \begin{tikzcd} M \otimes B \arrow[d, "\rho_{M\otimes B}"'] \arrow[dr, "r^{-1}_{M\otimes B}"] & \\ M \otimes B \otimes B \arrow[r, "M\otimes B\otimes\varepsilon"'] & M \otimes B \otimes \Bbbk \end{tikzcd}$$

are simply the diagrams that express that m and u are associative and unital, and Δ and ε are coassociative and counital, tensorized by M on the left. Moreover:

$$\begin{aligned} \rho_{M\otimes B}((m \otimes b) \cdot x) &= \rho_{M\otimes B}(m \otimes bx) = m \otimes \Delta(bx) = \\ &= m \otimes \Delta(b)\Delta(x) = (m \otimes \Delta(b)) \cdot x = \\ &= \rho_{M\otimes B}(m \otimes b) \cdot x, \end{aligned}$$

so that $\rho_{M\otimes B}$ is a right B-module map. $\square$

Define the *space of coinvariants* of an Hopf module M as the equalizer in $\mathcal{M}$ of:

$$\begin{tikzcd} 0 \arrow[r] & M^{\mathrm{Co}B} \arrow[r] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "i_1"'] & M \otimes B \end{tikzcd}$$

where

$$\begin{array}{rccc} i_1: & M & \longrightarrow & M \otimes B \\ & m & \longmapsto & m \otimes 1 \end{array}$$

i.e.,

$$M^{\mathrm{Co}B} := \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

Lemma 2.1.6. *Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra in $\mathcal{M}$ and M be an Hopf module over B. There is an isomorphism of $\Bbbk$-vector spaces:*

$$\begin{array}{rccc} \psi: & \hom_{\mathcal{M}^B}(\Bbbk, M) & \longrightarrow & M^{\mathrm{Co}B} \\ & \sigma & \longmapsto & \sigma(1) \end{array}$$

where the B-coaction on $\Bbbk$ is given through the unit u, dually with respect to the B-module structure:

$$\rho_{\Bbbk} \colon \Bbbk \longrightarrow \Bbbk \otimes B \colon 1_{\Bbbk} \longmapsto 1_{\Bbbk} \otimes 1_B$$

Proof. First of all, let us check that ψ is well defined. Since σ is a morphism of B-comodules we have that

$$\rho_M \circ \sigma = (\sigma \otimes B) \circ \rho_{\Bbbk},$$

i.e., $\rho_M(\sigma(1)) = \sigma(1) \otimes 1$. Hence $\sigma(1) \in M^{\mathrm{Co}B}$ and ψ is obviously $\Bbbk$-linear. To show that it is an isomorphism, let us exhibit an explicit inverse:

$$\begin{array}{rccc} \phi: & M^{\mathrm{Co}B} & \longrightarrow & \hom_{\mathcal{M}^B}(\Bbbk, M) \\ & m & \longmapsto & \sigma_m \end{array}$$

where σ_m is defined by $\sigma_m(1) = m$ and extended by $\Bbbk$-linearity. It is well defined because, for all $k \in \Bbbk$,

$$\rho_M(\sigma_m(k)) = k\rho_M(m) = km \otimes 1 = (\sigma_m \otimes B)(k \otimes 1) = (\sigma_m \otimes B)\rho_K(k).$$

Moreover:

$$\begin{gathered} \phi(\psi(\sigma)) = \sigma_{\sigma(1)} \colon 1 \mapsto \sigma(1), \\ \psi(\phi(m)) = \sigma_m(1) = m, \end{gathered}$$

and so it is the inverse map of ψ. $\square$

Next, consider the assignments

$$\begin{gathered} L: \mathcal{M} \longrightarrow \mathcal{M}_B^B: \ M \longmapsto M \otimes B \\ R: \mathcal{M}_B^B \longrightarrow \mathcal{M}: \ P \longmapsto P^{\mathrm{Co}B} \end{gathered}$$

that, on morphisms, operate as:

$$\begin{gathered} L(f) \colon M \otimes B \longrightarrow N \otimes B \colon m \otimes b \longmapsto f(m) \otimes b \quad (\forall\, f \in \hom_{\mathcal{M}}(M, N)) \\ R(g) \colon P^{\mathrm{Co}B} \longrightarrow Q^{\mathrm{Co}B} \colon p \longmapsto g(p) \quad \left(\forall\, g \in \hom_{\mathcal{M}_B^B}(P, Q)\right) \end{gathered}$$

Theorem 2.1.7. *The pair (L, R) is an adjunction with unit*

$$\eta_M \colon M \longrightarrow (M \otimes B)^{\mathrm{Co}B} \colon m \longmapsto m \otimes 1 \tag{2.4}$$

and counit

$$\epsilon_M \colon P^{\mathrm{Co}B} \otimes B \longrightarrow P \colon p \otimes b \longmapsto pb \tag{2.5}$$

Proof. L is trivially a functor. In order to prove that also R is a functor, it's enough to verify that $R(g)$ maps $P^{\mathrm{Co}B}$ into $Q^{\mathrm{Co}B}$ for all $g \in \hom_{\mathcal{M}_B^B}(P, Q)$ and for all $P, Q \in \mathcal{M}_B^B$. Thus, pick $p \in P^{\mathrm{Co}B}$ and consider $g(p)$. Since g is in particular a morphism of B-comodules:

$$\rho_Q(g(p)) = (g \otimes B)(\rho_P(p)) = (g \otimes B)(p \otimes 1) = g(p) \otimes 1$$

and so $g(p) \in Q^{\mathrm{Co}B}$.

Next, the counit ϵ_P is clearly well defined for each $P \in \mathcal{M}_B^B$. On the other hand, we need to show that the unit η_M actually maps M into $(M \otimes B)^{\mathrm{Co}B}$ for every $M \in \mathcal{M}$. Hence, let $m \in M$ and consider

$$\rho_{M\otimes B}(m \otimes 1) = (M \otimes \Delta)(m \otimes 1) = m \otimes 1 \otimes 1,$$

then $m \otimes 1 \in (M \otimes B)^{\mathrm{Co}B}$. Moreover, η_M is obviously $\Bbbk$-linear for all $M \in \mathcal{M}$, but we have to show that ϵ_P is an Hopf module map, for all $P \in \mathcal{M}_B^B$:

$$\begin{aligned}
\epsilon_P((p \otimes b) \cdot x) &= \epsilon_P(p \otimes bx) = p(bx) = (pb)x = \epsilon_P(p \otimes b) \cdot x \\
\rho_P(\epsilon_P(p \otimes b)) &= \rho_P(pb) = \rho_P(p) \cdot b = pb_1 \otimes b_2 = \\
&= (\epsilon_P \otimes B)(p \otimes b_1 \otimes b_2) = \\
&= (\epsilon_P \otimes B)(\rho_{P^{\mathrm{Co}B}\otimes B}(p \otimes b))
\end{aligned}$$

for all $p \in P^{\mathrm{Co}B}$ and $b, x \in B$. Let us prove now that they are both natural:

- Let $f \colon M \to N$ be a morphism in $\mathcal{M}$, we have that:

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "f"'] & (M \otimes B)^{\mathrm{Co}B} \arrow[d, "R(f\otimes B)"] \\
N \arrow[r, "\eta_N"'] & (N \otimes B)^{\mathrm{Co}B}
\end{tikzcd}
\qquad
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & m \otimes 1 \arrow[d, mapsto] \\
f(m) \arrow[r, mapsto] & f(m) \otimes 1
\end{tikzcd}$$

  since $R(f \otimes B) = f \otimes B$. Hence η is natural.

- Let $g \colon P \to Q$ be a morphism in $\mathcal{M}_B^B$. We have that:

$$\epsilon_Q \circ (R(g) \otimes B) = \mu_Q \circ (g \otimes B) = g \circ \mu_P = R(g) \circ \epsilon_P$$

  since g is a morphism of B-modules, so that ϵ is natural.

It remains to prove that the Triangular Identities are satisfied:

$$\begin{tikzcd}
R(P) \arrow[r, "\eta_{R(P)}"] \arrow[d, equal] & RLR(P) \arrow[r, "R(\epsilon_P)"] \arrow[d, equal] & R(P) \arrow[d, equal] \\
P^{\mathrm{Co}B} \arrow[r, "\eta_{R(P)}"] & (P^{\mathrm{Co}B} \otimes B)^{\mathrm{Co}B} \arrow[r, "\epsilon_P"] & P^{\mathrm{Co}B}
\end{tikzcd}$$

$$p \longmapsto p \otimes 1 \longmapsto p$$

for each $P \in \mathcal{M}_B^B$. Furthermore:

$$\begin{tikzcd}
L(M) \arrow[r, "L(\eta_M)"] \arrow[d, equal] & LRL(M) \arrow[r, "\epsilon_{L(M)}"] \arrow[d, equal] & L(M) \arrow[d, equal] \\
M \otimes B \arrow[r, "\eta_M \otimes B"] & (M \otimes B)^{\mathrm{Co}B} \otimes B \arrow[r, "\epsilon_{L(M)}"] & M \otimes B
\end{tikzcd}$$

$$m \otimes b \longmapsto (m \otimes 1) \otimes b \longmapsto (m \otimes 1) \cdot b$$

but $(m \otimes 1) \cdot b = m \otimes b$ and so even this last composition is the identity map, for all $M \in \mathcal{M}$. $\square$

Proposition 2.1.8. *The unit η of the adjunction (L, R, η, ϵ) of Theorem 2.1.7 is always a natural isomorphism.*

Proof. Let M be a $\Bbbk$-vector space and consider $\eta_M \colon M \to (M \otimes B)^{\mathrm{Co}B}$. Let

$$m := \sum_i^{<\infty} m_i \otimes b_i \in (M \otimes B)^{\mathrm{Co}B}.$$

We know that $\rho_{M \otimes B}(m) = m \otimes 1$. I.e.,

$$\sum_i^{<\infty} m_i \otimes (b_i)_1 \otimes (b_i)_2 = \sum_i^{<\infty} m_i \otimes b_i \otimes 1. \tag{2.6}$$

Apply $M \otimes \varepsilon \otimes B$ to both sides of (2.6) to get that:

$$\sum_i^{<\infty} m_i \otimes b_i = \sum_i^{<\infty} m_i \, \varepsilon(b_i) \otimes 1.$$

Hence $m = \eta_M \left(\sum_i^{<\infty} m_i \, \varepsilon(b_i) \right)$ and we showed that η_M is surjective. To prove that it is also injective consider the composition:

$$\psi_M = \left((M \otimes B)^{\mathrm{Co}B} \hookrightarrow M \otimes B \xrightarrow{M \otimes \varepsilon} M \otimes \Bbbk \cong M \right)$$

If $\eta_M(m) = m \otimes 1 = n \otimes 1 = \eta_M(n)$, then $m = \psi_M(m \otimes 1) = \psi_M(n \otimes 1) = n$ and so η_M is injective. Note that actually the map ψ_M is the inverse map of η_M in $\mathcal{M}$. $\square$

What we ask now is if (and when) (L, R, η, ϵ) is an equivalence of categories. The answer to this question is the so called 'Structure Theorem for Hopf modules' and involves the concept of *Hopf algebra* that we are going to introduce.

2.2 Hopf algebras

Let (C, Δ, ε) and (A, m, u) be a coalgebra and an algebra respectively and consider the $\Bbbk$-vector space $\mathcal{H} = \hom_{\mathcal{M}}(C, A)$. We can equip $\mathcal{H}$ with a structure of $\Bbbk$ algebra in the following way ([Sw, Section 4.0]). Let $f, g \in \mathcal{H}$ and consider the composition:

$$f * g = \left(C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A \right)$$

The map $f * g := m \circ (f \otimes g) \circ \Delta$ is called the *convolution product* of f and g. We can also consider the special map $u \circ \varepsilon \colon C \to A$.

Lemma 2.2.1. *Within the above context, $(\mathcal{H}, *, u \circ \varepsilon)$ is an associative unital algebra.*

Proof. Let us start with the associativity of $*$. We should show that the following commutes:

$$\begin{tikzcd}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \arrow[r, "* \otimes \mathcal{H}"] \arrow[d, "\mathcal{H} \otimes *"'] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "*"] \\
\mathcal{H} \otimes \mathcal{H} \arrow[r, "*"'] & \mathcal{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
f \otimes g \otimes h \arrow[r, mapsto] \arrow[dd, mapsto] & f * g \otimes h \arrow[d, mapsto] \\
& (f * g) * h \\
f \otimes g * h \arrow[r, mapsto] & f * (g * h)
\end{tikzcd}$$

but, for all $c \in C$ we have that:

$$\begin{aligned}
((f * g) * h)(c) = (f * g)(c_1)h(c_2) = [f((c_1)_1)g((c_1)_2)]h(c_2) &\overset{(1.38a)}{=} \\
= f(c_1)g(c_2)h(c_3) \overset{(1.38a)}{=} f(c_1)[g((c_2)_1)h((c_2)_2)] &= (f * (g * h))(c).
\end{aligned}$$

Hence $*$ is associative. Furthermore,

$$\begin{aligned}
(f * (u \circ \varepsilon))(c) &= f(c_1)\, \varepsilon(c_2)u(1_{\Bbbk}) = f(c_1\, \varepsilon(c_2)) \overset{(1.38b)}{=} f(c) \\
((u \circ \varepsilon) * f)(c) &= \varepsilon(c_1)u(1_{\Bbbk})f(c_2) = f(\varepsilon(c_1)\, c_2) \overset{(1.38b)}{=} f(c)
\end{aligned}$$

for all $c \in C$ and $f \in \mathcal{H}$, so that it is also unital with unit $u \circ \varepsilon$. $\square$

Definition 2.2.2. (Antipode, Hopf algebra)
Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra and let $\hom_{\mathcal{M}}(B, B)$ be equipped with the structure of algebra described in the previous lemma. An element $s \in \hom_{\mathcal{M}}(B, B)$ such that

$$s * \mathrm{Id} = u \circ \varepsilon = \mathrm{Id} * s \tag{2.7}$$

is called an *antipode* for B. An *Hopf algebra* is a bialgebra B that admits an antipode. Usually we indicate Hopf algebras with the capital letter H.

Remark 2.2.3. If B has an antipode, then it is unique, being a two-sided inverse. Moreover, a $\Bbbk$-linear map $s \colon B \to B$ is the antipode of B if and only if

$$b_1 s(b_2) = \varepsilon(b)1_B = s(b_1)b_2$$

for each $b \in B$.

Example 2.2.4. Let G be a group and $\Bbbk G$ be the group algebra on G. We know (Example 1.3.22) that $\Bbbk G$ is a bialgebra in $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$ with structures given by, for $g, h \in G$:

$$m(g \otimes h) = gh \quad u(1_\Bbbk) = 1_G$$

$$\Delta(g) = g \otimes g \qquad \varepsilon(g) = 1_\Bbbk$$

Consider the $\Bbbk$-linear map

$$s \colon \Bbbk G \longrightarrow \Bbbk G \colon g \longmapsto g^{-1} \tag{2.8}$$

We have that, for all $g \in G$, $(u \circ \varepsilon(g)) = 1_G$ and

$$\begin{aligned}(s * \mathrm{Id})(g) &= m((s \otimes \mathrm{Id})(g \otimes g)) = m(g^{-1} \otimes g) = 1_G \\ (\mathrm{Id} * s)(g) &= m((\in \otimes s)(g \otimes g)) = m(g \otimes g^{-1}) = 1_G\end{aligned}$$

so that, by $\Bbbk$-linearity:

$$(s * \mathrm{Id})\left(\sum_i^{<\infty} k_i g_i\right) = \left(\sum_i^{<\infty} k_i\right) 1_g = (u \circ \varepsilon)\left(\sum_i^{<\infty} k_i g_i\right).$$

The first important thing about the antipode is that it is an antiendomorphism of H as a bialgebra, as the following proposition states.

Proposition 2.2.5. *Let $(H, m, u, \Delta, \varepsilon, s)$ be a Hopf algebra. Then:*

(1) $s \circ m = m \circ \tau \circ (s \otimes s)$, (2.9a)

(2) $s \circ u = u$, (2.9b)

(3) $\tau \circ (s \otimes s) \circ \Delta = \Delta \circ s$, (2.9c)

(4) $\varepsilon \circ s = \varepsilon$, (2.9d)

where τ denotes the twist:

$$\begin{array}{cccc}\tau: & H \otimes H & \longrightarrow & H \otimes H \\ & h \otimes l & \longmapsto & l \otimes h\end{array}$$

Proof. The idea that lies behind the proof of (1) and (3) is the same: we will endow $\hom(H \otimes H, H)$ and $\hom(H, H \otimes H)$ with the algebra structure of Lemma 2.2.1, where the structure of algebra on $H \otimes H$ is given in Proposition 1.3.15 and the structure of coalgebra in Proposition 1.3.16, i.e.,

$$\begin{array}{ll}(h \otimes l)(g \otimes f) = hg \otimes lf & u_\otimes(1_\Bbbk) = 1_H \otimes 1_H \\ \Delta_\otimes(h \otimes l) = (h_1 \otimes l_1) \otimes (h_2 \otimes l_2) & \varepsilon_\otimes(h \otimes l) = \varepsilon(h)\varepsilon(l)\end{array} \tag{2.10}$$

Let us indicate with $\star$ such an algebra structure for both $\hom(H \otimes H, H)$ and $\hom(H, H \otimes H)$ indifferently.

(1) Consider the following three maps:

$$
\begin{aligned}
X &\colon H \otimes H \longrightarrow H \colon h \otimes l \longmapsto hl \\
Y &\colon H \otimes H \longrightarrow H \colon h \otimes l \longmapsto s(l)s(h) \\
Z &\colon H \otimes H \longrightarrow H \colon h \otimes l \longmapsto s(hl)
\end{aligned}
$$

We are going to prove that $Z \star X = u \circ \varepsilon_{\otimes} = X \star Y$, from which we deduce that $Z = Y$ by uniqueness of the inverse. For all $h, l \in H$:

$$
\begin{aligned}
(Z \star X)(h \otimes l) &= (m \circ (Z \otimes X) \circ \Delta_{\otimes})(h \otimes l) \overset{(2.10)}{=} \\
&= (m \circ (Z \otimes X))((h_1 \otimes l_1) \otimes (h_2 \otimes l_2)) = \\
&= m(s(h_1 l_1) \otimes h_2 l_2) = \\
&= s(h_1 l_1) h_2 l_2 \overset{(*)}{=} \\
&= s((hl)_1)(hl)_2 \overset{(2.7)}{=} \\
&= (u \circ \varepsilon)(hl) = (u \circ \varepsilon_{\otimes})(h \otimes l)
\end{aligned}
$$

where $(*)$ is a consequence of the fact that Δ is a morphism of algebras. On the other hand, for every $h, l \in H$:

$$
\begin{aligned}
(X \star Y)(h \otimes l) &= (m \circ (X \otimes Y) \circ \Delta_{\otimes})(h \otimes l) \overset{(2.10)}{=} \\
&= (m \circ (X \otimes Y))((h_1 \otimes l_1) \otimes (h_2 \otimes l_2)) = \\
&= m(h_1 l_1 \otimes s(l_2)s(h_2)) = \\
&= h_1 l_1 s(l_2) s(h_2) \overset{(2.7)}{=} \\
&= h_1 s(h_2)\, \varepsilon(l) \overset{(2.7)}{=} \\
&= (u \circ \varepsilon)(hl) = (u \circ \varepsilon_{\otimes})(h \otimes l)
\end{aligned}
$$

(2) Note that $\varepsilon \circ u = \mathrm{Id}_{\Bbbk}$ since ε is a morphism of algebras, then:

$$
\begin{aligned}
u &= u \circ \varepsilon \circ u \overset{(2.7)}{=} \\
&= (\mathrm{Id} * s) \circ u = m \circ (\mathrm{Id} \otimes s) \circ \Delta \circ u \overset{(**)}{=} \\
&= m \circ (\mathrm{Id} \otimes s) \circ u_{\otimes} = \\
&= m \circ (\mathrm{Id} \otimes s) \circ (u \otimes u) \circ \Delta_{\Bbbk} \overset{(\triangle)}{=} \\
&= s \circ u
\end{aligned}
$$

where in $(**)$ we used the fact that Δ is a morphism of algebras and $(\triangle)$ follows from:

$$
(m \circ (\mathrm{Id} \otimes s) \circ (u \otimes u) \circ \Delta_{\Bbbk})(k) = k u(1_{\Bbbk}) s(u(1_{\Bbbk})) = s(u(k)) \quad (\forall\, k \in \Bbbk)
$$

since all maps are $\Bbbk$-linear.

(3) We replicate the idea of (1). Consider the three maps:

$$\begin{aligned} X &\colon H \longrightarrow H \otimes H \colon h \longmapsto h_1 \otimes h_2 \\ Y &\colon H \longrightarrow H \otimes H \colon h \longmapsto s(h_2) \otimes s(h_1) \\ Z &\colon H \longrightarrow H \otimes H \colon h \longmapsto s(h)_1 \otimes s(h)_2 \end{aligned}$$

and let us show that $Z \star X = u_\otimes \circ \varepsilon = X \star Y$. For all $h \in H$:

$$\begin{aligned} (Z \star X)(h) &= (m_\otimes \circ (Z \otimes X) \circ \Delta)(h) = \\ &= (m_\otimes \circ (Z \otimes X))(h_1 \otimes h_2) = \\ &= m_\otimes[(s(h_1)_1 \otimes s(h_1)_2) \otimes ((h_2)_1 \otimes (h_2)_2)] = \\ &= s(h_1)_1(h_2)_1 \otimes s(h_1)_2(h_2)_2 \overset{(**)}{=} \\ &= (s(h_1)h_2)_1 \otimes (s(h_1)h_2)_2 = \\ &= \Delta((s * \mathrm{Id})(h)) \overset{(2.7)}{=} \\ &= \Delta(\varepsilon(h)1_H) = \varepsilon(h)(1 \otimes 1) = (u_\otimes \circ \varepsilon)(h) \end{aligned}$$

$$\begin{aligned} (X \star Y)(h) &= (m_\otimes \circ (X \otimes Y) \circ \Delta)(h) = \\ &= (m_\otimes \circ (X \otimes Y))(h_1 \otimes h_2) = \\ &= m_\otimes[((h_1)_1 \otimes (h_1)_2) \otimes (s((h_2)_2) \otimes s((h_2)_1))] = \\ &= (h_1)_1 s((h_2)_2) \otimes (h_1)_2 s((h_2)_1) \overset{(1.38a)}{=} \\ &= h_1 s(h_4) \otimes h_2 s(h_3) \overset{(1.38a)}{=} \\ &= h_1 s(h_3) \otimes (h_2)_1 s((h_2)_2) \overset{(2.7)}{=} \\ &= h_1 s(h_3) \otimes \varepsilon(h_2)1_H = \\ &= h_1 s(h_2) \otimes 1 \overset{(2.7)}{=} \\ &= \varepsilon(h)(1 \otimes 1) = (u_\otimes \circ \varepsilon)(h) \end{aligned}$$

where $(**)$ follows from the fact that Δ is a morphism of algebras.

(4) Apply ε to both sides of (2.7) to get that:

$$\varepsilon \circ (\mathrm{Id} * s) = \varepsilon \circ u \circ \varepsilon = \varepsilon.$$

Moreover, for all $h \in H$,

$$(\varepsilon \circ (\mathrm{Id} * s))(h) = \varepsilon(h_1 s(h_2)) = \varepsilon(h_1)\varepsilon(s(h_2)) = \varepsilon(s(h))$$

since all maps are $\Bbbk$-linear and $\varepsilon(h_1) \in \Bbbk$.

□

The following lemma, that appears as an exercise in [Sw, Chapter 4], retrieve some additional properties of the antipode.

Lemma 2.2.6. *Let $(H, m, u, \Delta, \varepsilon, s)$ be a Hopf algebra. Then, for all $h \in H$:*

(1) $1 \otimes h = h_1 s(h_2) \otimes h_3$

(2) $1 \otimes h = s(h_1) h_2 \otimes h_3$

(3) $h \otimes 1 = h_1 \otimes h_2 s(h_3)$

(4) $h \otimes 1 = h_1 \otimes s(h_2) h_3$

Proof. All the four properties are just the same check and they are a trivial consequence of (1.38a) and (2.7), e.g.:

$$h_1 s(h_2) \otimes h_3 \overset{(1.38a)}{=} (h_1)_1 s((h_1)_2) \otimes h_2 \overset{(2.7)}{=} \varepsilon(h_1) 1_H \otimes h_2 = 1 \otimes h.$$

□

2.3 The Structure Theorem for Hopf modules

The following theorem, commonly known as the Structure Theorem for Hopf modules, answers to the question when the adjunction (L, R, η, ϵ) of Theorem 2.1.7 is an equivalence of categories. For a less categorical approach refer to [Sw, Theorem 4.1.1] and [Ab, Theorem 3.1.8].

Theorem 2.3.1. *Let $(H, m, u, \Delta, \varepsilon, s)$ be a Hopf algebra. Then the counit*

$$\epsilon_M \colon M^{\mathrm{Co}H} \otimes H \longrightarrow M \colon m \otimes h \longmapsto m \cdot h$$

of the adjunction $(L, R, \eta, \varepsilon)$ is a natural isomorphism. In particular, for each Hopf module M on a Hopf algebra H,

$$M \cong M^{\mathrm{Co}H} \otimes H.$$

Proof. First of all, for every Hopf module M in $\mathcal{M}_H^H$, consider the projection:

$$\tau \colon M \longrightarrow M^{\mathrm{Co}H} \colon m \longmapsto m_0 \cdot s(m_1) \tag{2.11}$$

This map is well-defined as:

$$\begin{aligned}
\rho_M(\tau(m)) &= (m_0 \cdot s(m_1))_0 \otimes (m_0 \cdot s(m_1))_1 \overset{(2.1)}{=} \\
&= (m_0)_0 \cdot s(m_1)_1 \otimes (m_0)_1 s(m_1)_2 \overset{(2.9c)}{=} \\
&= (m_0)_0 \cdot s((m_1)_2) \otimes (m_0)_1 s((m_1)_1) \overset{(1.38a)}{=} \\
&= m_0 \cdot s(m_3) \otimes m_1 s(m_2) \overset{(2.7)}{=} \\
&= m_0 \cdot s(m_2) \otimes \varepsilon(m_1) 1 = \\
&= m_0 \cdot s(m_1) \otimes 1 = \\
&= \tau(m) \otimes 1
\end{aligned}$$

so that τ maps M into $M^{\mathrm{Co}H}$. Now, we show that the map

$$\beta := (\tau \otimes H) \circ \rho_M \colon M \longrightarrow M^{\mathrm{Co}H} \otimes H \colon m \longmapsto m_0 \cdot s(m_1) \otimes m_2 \tag{2.12}$$

is the inverse of the counit ϵ_M. For every $m \in M$

$$\begin{aligned}\epsilon_M(\beta(m)) &= \epsilon_M(m_0 \cdot s(m_1) \otimes m_2) = \\ &= m_0 \cdot s(m_1)m_2 \overset{(2.7)}{=} \\ &= m_0 \cdot \varepsilon(m_1)1 = m.\end{aligned}$$

On the other hand, for all $n \otimes h \in M^{\mathrm{Co}H} \otimes H$,

$$\begin{aligned}\beta(\epsilon_M(n \otimes h)) = \beta(n \cdot h) &= (n \cdot h)_0 \cdot s((n \cdot h)_1) \otimes (n \cdot h)_2 \overset{(2.1)}{=} \\ &= n \cdot h_1 s(h_2) \otimes h_3 \overset{(2.9a)}{=} \\ &= n\,\varepsilon(h_1) \otimes h_2 = \\ &= n \otimes h\end{aligned}$$

where the third equality involves also the fact that $n \in M^{\mathrm{Co}H}$. $\square$

Actually, Theorem 2.3.1 retrieves just one side of an equivalence, in the sense that it admits a converse. But before state it, we need some introductory considerations.

Remark 2.3.2. Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. If we consider the vector space $H \otimes H$, we can equip it with the right H-action given by the multiplication on the second factor:

$$(x \otimes y) \cdot h = x \otimes yh,$$

with the diagonal right H-coaction:

$$\rho^r_\otimes(x \otimes y) = x_1 \otimes y_1 \otimes x_2 y_2,$$

and with the left H-comodule structure given by:

$$\rho^l_\otimes(x \otimes y) = x_1 \otimes x_2 \otimes y,$$

for all $x, y, h \in H$. If we indicate with a full dot the given structures and with an empty dot the trivial structures we can summarize in:

$$^{\bullet}H^{\bullet}_{\circ} \otimes H^{\bullet}_{\bullet}$$

The multiplication on the second factor is trivially an H-action, as the comultiplication of the first factor is an H-coaction. Let us just prove that the diagonal coaction is actually a coaction:

$$\begin{aligned}(H \otimes H \otimes \Delta)(\rho^r_\otimes(x \otimes y)) &= (H \otimes H \otimes \Delta)(x_1 \otimes y_1 \otimes x_2 y_2) = \\ &= x_1 \otimes y_1 \otimes (x_2 y_2)_1 \otimes (x_2 y_2)_2 = \\ &= x_1 \otimes y_1 \otimes (x_2)_1 (y_2)_1 \otimes (x_2)_2 (y_2)_2 \overset{(1.38a)}{=} \\ &= x_1 \otimes y_1 \otimes x_2 y_2 \otimes x_3 y_3 \\ (\rho^r_\otimes \otimes H)(\rho^r_\otimes(x \otimes y)) &= (\rho^r_\otimes \otimes H)(x_1 \otimes y_1 \otimes x_2 y_2) = \\ &= (x_1)_1 \otimes (y_1)_1 \otimes (x_1)_2 (y_1)_2 \otimes x_2 y_2 \overset{(1.38a)}{=} \\ &= x_1 \otimes y_1 \otimes x_2 y_2 \otimes x_3 y_3\end{aligned}$$

and:

$$\begin{aligned} r_{\otimes}((H \otimes H \otimes \varepsilon)(\rho^r_{\otimes}(x \otimes y))) &= r_{\otimes}((H \otimes H \otimes \varepsilon)(x_1 \otimes y_1 \otimes x_2y_2)) = \\ &= r_{\otimes}(x_1 \otimes y_1 \otimes \varepsilon(x_2y_2)) = \\ &= r_{\otimes}(x \otimes y \otimes 1_{\Bbbk}) = x \otimes y, \end{aligned}$$

for all $x, y \in H$. Furthermore, we can equip $(H \otimes H)^{\mathrm{Co}H} \otimes H$ with the following structures:

$$\begin{aligned} ((x \otimes y) \otimes h) \cdot l &= (x \otimes y) \otimes hl \\ \rho^r_{(H\otimes H)^{\mathrm{Co}H} \otimes H}((x \otimes y) \otimes h) &= (x \otimes y) \otimes h_1 \otimes h_2 \\ \rho^l_{(H\otimes H)^{\mathrm{Co}H} \otimes H}((x \otimes y) \otimes h) &= x_1 \otimes (x_2 \otimes y) \otimes h \end{aligned}$$

for all $x, y, h, l \in H$. The first one is clearly a right H-action, as the last two are H-coactions. We should only prove that the image of $\rho^l_{(H\otimes H)^{\mathrm{Co}H} \otimes H}$ is actually within $H \otimes (H \otimes H)^{\mathrm{Co}H} \otimes H$. Observe that it's enough to verify that:

$$\begin{array}{rccc} \xi: & (H \otimes H)^{\mathrm{Co}H} & \longrightarrow & H \otimes (H \otimes H)^{\mathrm{Co}H} \\ & x \otimes y & \longmapsto & x_1 \otimes (x_2 \otimes y) \end{array}$$

is well defined. Hence let us concentrate on this last claim. Initially, consider the following $\Bbbk$-linear map:

$$\psi \colon (H \otimes H) \longrightarrow (H \otimes H) \otimes H \colon x \otimes y \longmapsto \rho^r_{(H\otimes H)}(x \otimes y) - (x \otimes y \otimes 1).$$

Note that $z \in (H \otimes H)^{\mathrm{Co}H}$ if and only if $\psi(z) = 0$, so that $\ker(\psi) = (H \otimes H)^{\mathrm{Co}H}$. Thus we have the following exact sequence:

$$0 \longrightarrow (H \otimes H)^{\mathrm{Co}H} \hookrightarrow H \otimes H \xrightarrow{\psi} (H \otimes H) \otimes H$$

Since H, as $\Bbbk$-vector space, is a free $\Bbbk$-module, it is $\Bbbk$-flat and so the functor $H \otimes -$ is exact. Hence we have another exact sequence:

$$0 \longrightarrow H \otimes (H \otimes H)^{\mathrm{Co}H} \hookrightarrow H \otimes (H \otimes H) \xrightarrow{H \otimes \psi} H \otimes ((H \otimes H) \otimes H),$$

from which we deduce that $w \in H \otimes (H \otimes H)^{\mathrm{Co}H}$ if and only if $w \in H \otimes (H \otimes H)$ and $w \in \ker(H \otimes \psi)$. Thus, let us apply $H \otimes \psi$ to $x_1 \otimes (x_2 \otimes y)$:

$$\begin{aligned} (H \otimes \psi)(x_1 \otimes (x_2 \otimes y)) &= x_1 \otimes (((x_2)_1 \otimes y_1 \otimes (x_2)_2 y_2) - (x_2 \otimes y \otimes 1)) = \\ &= (x_1 \otimes (x_2)_1 \otimes y_1 \otimes (x_2)_2 y_2) - (x_1 \otimes x_2 \otimes y \otimes 1) \overset{(1.38a)}{=} \\ &= ((x_1)_1 \otimes (x_1)_2 \otimes y_1 \otimes x_2 y_2) - (x_1 \otimes x_2 \otimes y \otimes 1) = \\ &= (\Delta \otimes H \otimes H)(\rho^r_{(H\otimes H)}(x \otimes y)) - (\Delta \otimes H \otimes H)(x \otimes y \otimes 1) = \\ &= 0 \end{aligned}$$

since $x \otimes y \in (H \otimes H)^{\mathrm{Co}H}$. Hence $\xi(z) \in H \otimes (H \otimes H)^{\mathrm{Co}H}$ for all $z \in (H \otimes H)^{\mathrm{Co}H}$, by $\Bbbk$-linearity.

Lemma 2.3.3. *The map $\epsilon_{H\otimes H}\colon (H\otimes H)^{\mathrm{Co}H}\otimes H \to H\otimes H$ is a morphism with respect to all the structures that occur in Remark 2.3.2.*

Proof. We will indicate $\epsilon_{H\otimes H}$ with $\epsilon_\otimes$, $\rho^{r/l}_{H\otimes H}$ with $\rho^{r/l}_\otimes$ and $\rho^{r/l}_{(H\otimes H)^{\mathrm{Co}H}\otimes H}$ with $\rho^{r/l}_{\mathrm{Co}H}$ in order to lighten the notation.

That $\epsilon_\otimes$ is linear with respect to the right H-action follows from Theorem 2.1.7. Let us prove the other two linearities:

- Let us start with the right colinearity. We have to show that the following diagram commutes:

$$\begin{tikzcd}
(H\otimes H)^{\mathrm{Co}H}\otimes H \arrow[r, "\epsilon_\otimes"] \arrow[d, "\rho^r_{\mathrm{Co}H}"'] & H\otimes H \arrow[d, "\rho^r_\otimes"] \\
{[(H\otimes H)^{\mathrm{Co}H}\otimes H]\otimes H} \arrow[r, "\epsilon_\otimes\otimes H"'] & (H\otimes H)\otimes H
\end{tikzcd}$$

  Actually, it does. Indeed:

$$\begin{aligned}
(\epsilon_\otimes\otimes H)(\rho^r_{\mathrm{Co}H}((x\otimes y)\otimes h)) &= (\epsilon_\otimes\otimes H)(x\otimes y\otimes h_1\otimes h_2) = \\
&= (x\otimes y)\cdot h_1\otimes h_2 = \\
&= x\otimes yh_1\otimes h_2
\end{aligned}$$

$$\begin{aligned}
\rho^r_\otimes(\epsilon_\otimes((x\otimes y)\otimes h)) = \rho^r_\otimes((x\otimes y)\cdot h) &= \rho^r_\otimes(x\otimes yh) = \\
&= x_1\otimes (yh)_1\otimes x_2(yh)_2 = x_1\otimes y_1h_1\otimes x_2y_2h_2 = \\
&= \rho^r_\otimes(x\otimes y)(H\otimes\Delta)(1\otimes h) = \\
&= x\otimes yh_1\otimes h_2
\end{aligned}$$

  since $x\otimes y\in (H\otimes H)^{\mathrm{Co}H}$ and, by $\Bbbk$-linearity, it holds for all $z\in (H\otimes H)^{\mathrm{Co}H}$ and $h\in H$.

- For the left H-colinearity, the following diagram should commute:

$$\begin{tikzcd}
(H\otimes H)^{\mathrm{Co}H}\otimes H \arrow[r, "\epsilon_\otimes"] \arrow[d, "\rho^l_{\mathrm{Co}H}"'] & H\otimes H \arrow[d, "\rho^l_\otimes"] \\
H\otimes[(H\otimes H)^{\mathrm{Co}H}\otimes H] \arrow[r, "H\otimes\epsilon_\otimes"'] & H\otimes(H\otimes H)
\end{tikzcd}$$

  but if we recall how the left H-coaction are defined, this is obvious:

$$\rho^l_{(H\otimes H)^{\mathrm{Co}H}\otimes H}((x\otimes y)\otimes h) = x_1\otimes(x_2\otimes y)\otimes h$$
$$\rho^l_\otimes(x\otimes y) = x_1\otimes x_2\otimes y$$

and this concludes the proof. □

Remark 2.3.4. Note that $H\otimes H$ is a Hopf module with the right structures defined in Remark 2.3.2:

$$\begin{aligned}
\rho^r_\otimes((x\otimes y)\cdot h) &= \rho^r_\otimes(x\otimes yh) = x_1\otimes(yh)_1\otimes x_2(yh)_2 = \\
&= x_1\otimes y_1h_1\otimes x_2y_2h_2 = (x_1\otimes y_1\otimes x_2y_2)(1\otimes h_1\otimes h_2) = \\
&= \rho^r_\otimes(x\otimes y)\cdot h,
\end{aligned}$$

for all $x, y, h \in H$.

Theorem 2.3.5. *Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra and suppose that the adjunction (L, R, η, ϵ) of Theorem 2.1.7 is a category equivalence. Then the map $s\colon H \to H$ defined by, for all $h \in H$:*

$$s(h) := (\varepsilon \otimes H \otimes \varepsilon)\left(\epsilon_{H\otimes H}^{-1}(h \otimes 1)\right) \tag{2.13}$$

is an antipode. In particular, $(H, m, u, \Delta, \varepsilon, s)$ is a Hopf algebra.

Proof. Denote again $\epsilon_{\otimes} := \epsilon_{H\otimes H}$, $\rho^{r/l}_{\mathrm{Co}H} := \rho^{r/l}_{(H\otimes H)^{\mathrm{Co}H}\otimes H}$, $\rho^{r/l}_{\otimes} := \rho^{r/l}_{H\otimes H}$ and

$$h^1 \otimes h^2 \otimes h^3 := \epsilon_{\otimes}^{-1}(h \otimes 1) \in (H \otimes H)^{\mathrm{Co}H} \otimes H$$

for all $h \in H$ (summation understood). In view of Lemma 2.3.3 we know that $\epsilon_{\otimes}$ is a morphism of 'Hopf bicomodules' (meaning just a Hopf module with an additional left comodule structure) with the structures given by the dots:

$$\epsilon_{\otimes}\colon {}^{\bullet}(H \otimes H)^{\mathrm{Co}H} \otimes H^{\bullet}_{\bullet} \to {}^{\bullet}H^{\bullet}_{\circ} \otimes H^{\bullet}_{\bullet},$$

thus $\epsilon_{\otimes}^{-1}$ is a morphism with respect to the same structures as well. Since it is H-linear,

$$\epsilon_{\otimes}^{-1}(x \otimes y) = \epsilon_{\otimes}^{-1}((x \otimes 1) \cdot y) = \epsilon_{\otimes}^{-1}(x \otimes 1) \cdot y,$$

for all $x, y \in H$, so that it is enough to work on elements of the form $h \otimes 1$. By the right colinearity we get that:

$$\begin{aligned}(h_1)^1 \otimes (h_1)^2 \otimes (h_1)^3 \otimes h_2 &= (\epsilon_{\otimes}^{-1} \otimes H)(\rho^r_{\otimes}(h \otimes 1)) = \\ &= \rho^r_{\mathrm{Co}H}(\epsilon_{\otimes}^{-1}(h \otimes 1)) = h^1 \otimes h^2 \otimes (h^3)_1 \otimes (h^3)_2.\end{aligned} \tag{2.14}$$

for each $h \in H$. On the other hand, by the left colinearity we have that:

$$\begin{aligned}h_1 \otimes (h_2)^1 \otimes (h_2)^2 \otimes (h_2)^3 &= (H \otimes \epsilon_{\otimes}^{-1})(\rho^l_{\otimes}(h \otimes 1)) = \\ &= \rho^l_{\mathrm{Co}H}(\epsilon_{\otimes}^{-1}(h \otimes 1)) = (h^1)_1 \otimes (h^1)_2 \otimes h^2 \otimes h^3.\end{aligned} \tag{2.15}$$

for all $h \in H$. Furthermore, relation (2.13) that defines s now rewrites as:

$$s(h) = \varepsilon(h^1)\, h^2\, \varepsilon(h^3) \quad (\forall\, h \in H\,). \tag{2.16}$$

Keeping in mind these three last identities, we are going to derive some properties of s that will be used to prove that it is, actually, the convolution inverse of the identity.

Start by applying $\varepsilon \otimes H \otimes \varepsilon \otimes H$ to both sides of (2.14). This becomes, for $h \in H$:

$$\begin{aligned}s(h_1) \otimes h_2 &= \varepsilon((h_1)^1)\,(h_1)^2\,\varepsilon((h_1)^3) \otimes h_2 = \\ &= \varepsilon(h^1)h^2 \otimes \varepsilon((h^3)_1)\,(h^3)_2 = \varepsilon(h^1)h^2 \otimes h^3.\end{aligned} \tag{2.17}$$

Next, in view of the previous identity, apply $H \otimes \varepsilon \otimes H \otimes H$ to both sides of (2.15) and obtain, for $h \in H$:

$$\begin{aligned}h_1 \otimes s(h_2) \otimes h_3 &\overset{(2.17)}{=} h_1 \otimes \varepsilon((h_2)^1)\,(h_2)^2 \otimes (h_2)^3 = \\ &= (h^1)_1\,\varepsilon((h^1)_2) \otimes h^2 \otimes h^3 = \epsilon_{\otimes}^{-1}(h \otimes 1)\end{aligned} \tag{2.18}$$

Relation (2.18) is one key to prove that s is an antipode. Indeed:

$$h \otimes 1 = \epsilon_{\otimes}(\epsilon_{\otimes}^{-1}(h \otimes 1)) \overset{(2.18)}{=} h_1 \otimes s(h_2)h_3 \tag{2.19}$$

for each $h \in H$, so that, applying $\varepsilon \otimes H$ to both sides:

$$\varepsilon(h)1_H = s(h_1)h_2. \tag{2.20}$$

Moreover, applying $\varepsilon \circ m$ to both sides of (2.19), we get also that:

$$\varepsilon(h) = \varepsilon(h_1 s(h_2)h_3) = \varepsilon(s(h)) \quad (\forall\, h \in H). \tag{2.21}$$

The other key is the fact that, for $h \in H$:

$$h_1 \otimes s(h_2) \otimes h_3 = \epsilon_{\otimes}^{-1}(h \otimes 1) \in (H \otimes H)^{\mathrm{Co}H} \otimes H,$$

thus

$$(h_1)_1 \otimes (s(h_2))_1 \otimes (h_1)_2(s(h_2))_2 \otimes h_3 = h_1 \otimes s(h_2) \otimes 1 \otimes h_3.$$

Apply $m \otimes H \otimes H$ to both sides:

$$(h_1 s(h_2))_1 \otimes (h_1 s(h_2))_2 \otimes h_3 = h_1 s(h_2) \otimes 1 \otimes h_3$$

and then apply $\varepsilon \otimes H \otimes H$:

$$h_1 s(h_2) \otimes h_3 = \varepsilon(h_1)\, \varepsilon(s(h_2))1 \otimes h_3 \overset{(2.21)}{=} 1 \otimes h \quad (\forall\, h \in H).$$

As for relation (2.20), apply $H \otimes \varepsilon$ to this last identity:

$$h_1 s(h_2) = \varepsilon(h)1_H \quad (\forall\, h \in H),$$

to find out that s is also the right convolution inverse of the identity. □

Following [BW], we refer to the subsequent result as the 'Structure Theorem for Hopf modules', because it is the complete formulation of the original one.

Theorem 2.3.6. ([BW, Theorem 15.5]) (Structure Theorem for Hopf modules) *Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra in $(\mathcal{M}, \otimes, \Bbbk, a, l, r)$. Then, the following assertions are equivalent:*

1. *The bialgebra H is a Hopf algebra.*
2. *For each Hopf H-module $M \in \mathcal{M}_H^H$, $M \cong M^{\mathrm{Co}H} \otimes H$.*

Chapter 3

The Fundamental Structure Theorem for quasi-Hopf bimodules

3.1 Quasi-bialgebras

Recall that if we have an algebra with comultiplication and counit $(A, m, u, \Delta, \varepsilon)$, then we can equip the category of right A-modules with a tensor product (that is the restriction of the tensor product between $\Bbbk$-vector spaces) and a unit (the base field $\Bbbk$ itself): $(\mathcal{M}_A, \otimes, \Bbbk)$. We have seen in Chapter 1, Proposition 1.3.21, that $(\mathcal{M}, \otimes, \Bbbk, a, \ell, \mathtt{r})$ is monoidal if and only if A is a bialgebra (*Just for this section, we are going to indicate with ℓ the left unit constraint and with $\mathtt{r}$ the right unit constraint, in order to avoid confusion*). But if we weaken our requests, e.g. don't asking for coassociativity of Δ, we find out that there exists a larger class of algebras such that the corresponding category of right A-modules is monoidal.

Definition 3.1.1. (Quasi-bialgebra)
Let $(A, m, u, \Delta, \varepsilon)$ be an algebra with comultiplication and counit as introduced in Remark 1.3.20. A is a *quasi-bialgebra* if $(\mathcal{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho)$ is monoidal.

Remark 3.1.2. Pay attention: we are not requesting that the constraints are the same of the monoidal category $(\mathcal{M}, \otimes, \Bbbk, a, \ell, \mathtt{r})$, as for the ordinary bialgebra. We are saying that there exists constraints (α, λ, ρ) such that they are natural isomorphisms of right A-modules and satisfy the Pentagon and Triangle Axioms.

Theorem 3.1.3. *Let $(A, m, u, \Delta, \varepsilon)$ be an algebra with comultiplication and counit. A is a quasi-bialgebra if and only if there exist an invertible element $\Phi \in A \otimes A \otimes A$ and two invertible elements l, r in A such that:*

$$(\Delta \otimes A)(\Delta(x))\Phi = \Phi(A \otimes \Delta)(\Delta(x)) \tag{3.1}$$

$$(\varepsilon \otimes A)(\Delta(x)) = lxl^{-1} \tag{3.2}$$

$$(A \otimes \varepsilon)(\Delta(x)) = rxr^{-1} \tag{3.3}$$

for all $x \in A$, and

$$(\Delta \otimes A \otimes A)(\Phi)(A \otimes A \otimes \Delta)(\Phi) = (\Phi \otimes 1)(A \otimes \Delta \otimes A)(\Phi)(1 \otimes \Phi) \tag{3.4}$$

$$(A \otimes \varepsilon \otimes A)(\Phi) = r \otimes l^{-1} \tag{3.5}$$

Usually, we will write $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$ and $\Phi^{-1} = \phi^1 \otimes \phi^2 \otimes \phi^3$ (summation understood).

Proof. Let us start by the 'if' part and assume that Φ, l and r exist. We can define:

$$\begin{array}{rccc} \alpha_{M,N,P}: & (M \otimes N) \otimes P & \longrightarrow & M \otimes (N \otimes P) \\ & (m \otimes n) \otimes p & \longmapsto & (m \otimes (n \otimes p)) \cdot \Phi \end{array} \tag{3.6}$$

$$\begin{array}{rccc} \lambda_M: & \Bbbk \otimes M & \longrightarrow & M \\ & 1 \otimes m & \longmapsto & m \cdot l \end{array} \tag{3.7}$$

$$\begin{array}{rccc} \rho_M: & M \otimes \Bbbk & \longrightarrow & M \\ & m \otimes 1 & \longmapsto & m \cdot r \end{array} \tag{3.8}$$

These are morphism of right A-modules. Indeed:

$$\begin{aligned} \alpha_{M,N,P}(((m \otimes n) \otimes p) \cdot a) &= \alpha_{M,N,P}(((m \otimes n) \otimes p) \cdot (\Delta \otimes A)(\Delta(a))) = \\ &= (m \otimes (n \otimes p)) \cdot (\Delta \otimes A)(\Delta(a))\Phi \overset{(3.1)}{=} \\ &= (m \otimes (n \otimes p)) \cdot \Phi(A \otimes \Delta)(\Delta(a)) = \\ &= (m \otimes (n \otimes p)) \cdot \Phi \cdot a = \\ &= \alpha_{M,N,P}((m \otimes n) \otimes p) \cdot a, \end{aligned}$$

$$\begin{aligned} \lambda_M((1 \otimes m) \cdot a) &= \lambda_M(\varepsilon(a_1) \otimes (m \cdot a_2)) = \\ &= m \cdot \varepsilon(a_1)a_2 l \overset{(3.2)}{=} m \cdot la = \lambda_M(m) \cdot a, \end{aligned}$$

$$\begin{aligned} \rho_M((m \otimes 1) \cdot a) &= \rho_M((m \cdot a_1) \otimes \varepsilon(a_2)) = \\ &= m \cdot a_1 \varepsilon(a_2) r \overset{(3.3)}{=} m \cdot ra = \rho_M(m) \cdot a. \end{aligned}$$

for all $m \in M$, $n \in N$, $p \in P$, $a \in A$. Moreover, since Φ, l and r are all invertibles, these are bijective with inverses given by:

$$\begin{array}{rccc} \alpha^{-1}_{M,N,P}: & M \otimes (N \otimes P) & \longrightarrow & (M \otimes N) \otimes P \\ & m \otimes (n \otimes p) & \longmapsto & ((m \otimes n) \otimes p) \cdot \Phi^{-1} \end{array}$$

$$\begin{array}{rccc} \lambda^{-1}_M: & M & \longrightarrow & \Bbbk \otimes M \\ & m & \longmapsto & (1 \otimes m) \cdot l^{-1} \end{array}$$

$$\begin{array}{rccc} \rho^{-1}_M: & M & \longrightarrow & M \otimes \Bbbk \\ & m & \longmapsto & (m \otimes 1) \cdot r^{-1} \end{array}$$

Next, we show that they are natural. Pick three morphisms of right A-modules: $f\colon M \to M'$, $g\colon N \to N'$ and $h\colon P \to P'$ and observe that:

$$\begin{aligned} (f \otimes (g \otimes h))((m \otimes (n \otimes p)) \cdot \Phi) &= (f \otimes (g \otimes h))((m \cdot \Phi^1 \otimes (n \cdot \Phi^2 \otimes p \cdot \Phi^3))) = \\ &= (f(m \cdot \Phi^1) \otimes (g(n \cdot \Phi^2) \otimes h(p \cdot \Phi^3))) = \\ &= (f(m) \cdot \Phi^1 \otimes (g(n) \cdot \Phi^2 \otimes h(p) \cdot \Phi^3)) = \\ &= (f \otimes (g \otimes h))((m \otimes (n \otimes p))) \cdot \Phi \end{aligned}$$

for all $m \in M$, $n \in N$, $p \in P$ since f, g and h are right A-linear. Hence the following diagram commutes:

$$\begin{tikzcd}
(M \otimes N) \otimes P \arrow[r, "\alpha_{M,N,P}"] \arrow[d, "(f\otimes g)\otimes h"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes (N \otimes P) \arrow[d, "f\otimes(g\otimes h)"] \\
(M' \otimes N') \otimes P' \arrow[r, "\alpha_{M',N',P'}"'] & M' \otimes (N' \otimes P')
\end{tikzcd}$$

Furthermore, even the following diagrams commute, since f is right A-linear:

$$\begin{tikzcd}
M \otimes \Bbbk \arrow[r, "\rho_M"] \arrow[d, "f\otimes\Bbbk"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\
M' \otimes \Bbbk \arrow[r, "\rho_{M'}"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
m \otimes 1 \arrow[r, mapsto, "\rho_M"] \arrow[d, mapsto, "f\otimes\Bbbk"'] & m \cdot r \arrow[d, mapsto, "f"] \\
f(m) \otimes 1 \arrow[r, mapsto, "\rho_{M'}"'] & f(m) \cdot r
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk \otimes M \arrow[r, "\lambda_M"] \arrow[d, "\Bbbk\otimes f"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\
\Bbbk \otimes M' \arrow[r, "\lambda_{M'}"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes m \arrow[r, mapsto, "\lambda_M"] \arrow[d, mapsto, "\Bbbk\otimes f"'] & m \cdot l \arrow[d, mapsto, "f"] \\
1 \otimes f(m) \arrow[r, mapsto, "\lambda_{M'}"'] & f(m) \cdot l
\end{tikzcd}$$

It remains to prove that α, λ and ρ satisfies the Axioms (1.9) and (1.10).

Pentagon: for all $m \in M$, $n \in N$, $p \in P$, $q \in Q$

$$\begin{tikzcd}
(((m \otimes n) \otimes p) \otimes q) \arrow[r, mapsto, "\alpha_{M\otimes N,P,Q}"] \arrow[ddd, mapsto, "\alpha_{M,N,P}\otimes Q"'] & ((m \otimes n) \otimes (p \otimes q)) \cdot (\Delta \otimes A \otimes A)(\Phi) \arrow[d, mapsto, "\alpha_{M,N,P\otimes Q}"] \\
& (m \otimes (n \otimes (p \otimes q))) \cdot (\Delta \otimes A \otimes A)(\Phi)(A \otimes A \otimes \Delta)(\Phi) \arrow[d, equal, "(3.4)"] \\
& (m \otimes ((n \otimes p) \otimes q)) \cdot (\Phi \otimes 1)(A \otimes \Delta \otimes A)(\Phi)(1 \otimes \Phi) \\
((m \otimes (n \otimes p)) \otimes q) \cdot (\Phi \otimes 1) \arrow[r, mapsto, "\alpha_{M,N\otimes P,Q}"'] & (m \otimes ((n \otimes p) \otimes q)) \cdot (\Phi \otimes 1)(A \otimes \Delta \otimes A)(\Phi) \arrow[u, mapsto, "M\otimes\alpha_{N,P,Q}"']
\end{tikzcd}$$

Triangle: for all $m \in M$, $n \in N$

$$\begin{tikzcd}
(m \otimes 1) \otimes n \arrow[r, mapsto, "\alpha_{M,\Bbbk,N}"] \arrow[d, mapsto, "r_M\otimes N"'] & (m \otimes (1 \otimes n)) \cdot (A \otimes \varepsilon \otimes A)(\Phi) \arrow[d, mapsto, "M\otimes l_N"] \\
(m \otimes n) \cdot (r \otimes 1) \arrow[r, equal, "(3.5)"'] & (m \otimes n) \cdot (A \otimes \varepsilon \otimes A)(\Phi)(1 \otimes l)
\end{tikzcd}$$

For the 'only if' part, assume that $(\mathcal{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho)$ is monoidal and define:

$$\Phi = \alpha_{A,A,A}(1_A \otimes 1_A \otimes 1_A) \tag{3.9a}$$
$$l = \lambda_A(1_\Bbbk \otimes 1_A) \tag{3.9b}$$
$$r = \rho_A(1_A \otimes 1_\Bbbk) \tag{3.9c}$$

Observe that, for each $m \in M$ right A-module, there exists a unique morphism of A-modules

$$\widehat{m} \colon A \longrightarrow M \colon 1_A \longmapsto m$$

since every A-linear map from A into an A-module (right or left is the same) is uniquely determined by the image of 1_A. Hence, by naturality of α, for all $m \in M$, $n \in N$ and $p \in P$ we have a commutative diagram:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[d, "(\widehat{m}\otimes\widehat{n})\otimes\widehat{p}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes (A \otimes A) \arrow[d, "\widehat{m}\otimes(\widehat{n}\otimes\widehat{p})"] \\
(M \otimes N) \otimes P \arrow[r, "\alpha_{M,N,P}"'] & M \otimes (N \otimes P)
\end{tikzcd}$$

If we apply it to $1_A \otimes 1_A \otimes 1_A$ we find that for all $m \in M$, $n \in N$, $p \in P$:

$$(m \otimes (n \otimes p)) \cdot \Phi = (\widehat{m} \otimes (\widehat{n} \otimes \widehat{p}))(\Phi) = \alpha_{M,N,P}((m \otimes n) \otimes p). \tag{3.10}$$

Note that, since α is a natural isomorphism, there exists an element $\phi^1 \otimes \phi^2 \otimes \phi^3$ in $A \otimes A \otimes A$ such that $\alpha_{A,A,A}(\phi^1 \otimes \phi^2 \otimes \phi^3) = 1 \otimes 1 \otimes 1$. Thus:

$$1 \otimes 1 \otimes 1 = \alpha_{A,A,A}(\alpha^{-1}_{A,A,A}(1 \otimes 1 \otimes 1)) = (\phi^1 \otimes (\phi^2 \otimes \phi^3)) \cdot \Phi.$$

On the other hand, also α^{-1} is a natural isomorphism and if we indicate with

$$\phi^1 \otimes \phi^2 \otimes \phi^3 := \alpha^{-1}_{A,A,A}(1 \otimes 1 \otimes 1),$$

then naturality implies that:

$$\alpha^{-1}_{M,N,P}(m \otimes (n \otimes p)) = ((m \otimes n) \otimes p) \cdot (\phi^1 \otimes \phi^2 \otimes \phi^3)$$

for all $m \in M$, $n \in N$, $p \in P$. Therefore:

$$1 \otimes 1 \otimes 1 = \alpha^{-1}_{A,A,A}(\alpha_{A,A,A}(1 \otimes 1 \otimes 1)) = \Phi \cdot (\phi^1 \otimes \phi^2 \otimes \phi^3)$$

and we deduce that Φ is invertible, with two-sided inverse $\Phi^{-1} = \phi^1 \otimes \phi^2 \otimes \phi^3$. Moreover, α is also right A-linear:

$$\begin{tikzcd}
((M \otimes N) \otimes P) \otimes A \arrow[r, "\alpha_{M,N,P} \otimes A"] \arrow[d, "\mu_{(M\otimes N)\otimes P}"'] \arrow[dr, phantom, "\circlearrowleft"] & (M \otimes (N \otimes P)) \otimes A \arrow[d, "\mu_{M\otimes(N\otimes P)}"] \\
((M \otimes N) \otimes P) \arrow[r, "\alpha_{M,N,P}"'] & (M \otimes (N \otimes P))
\end{tikzcd}$$

so that:

$$\begin{aligned}(m \otimes (n \otimes p)) \cdot \Phi(A \otimes \Delta)(\Delta(a)) = \alpha_{M,N,P}((m \otimes n) \otimes p)(A \otimes \Delta)(\Delta(a)) = \\ = \alpha_{M,N,P}(((m \otimes n) \otimes p) \cdot (\Delta \otimes A)(\Delta(a))) = (m \otimes (n \otimes p)) \cdot (\Delta \otimes A)(\Delta(a))\Phi \end{aligned} \tag{3.11}$$

for all $m \in M$, $n \in N$, $p \in P$ and $a \in A$. Taking $M = N = P = A$ and evaluating at $1 \otimes 1 \otimes 1$, (3.11) gives:

$$\Phi(A \otimes \Delta)(\Delta(a)) = (\Delta \otimes A)(\Delta(a))\Phi.$$

The same arguments that work for α, also work for λ and ρ. Hence, naturality of λ gives:

$$m \cdot l = \widehat{m}(\lambda_A(1 \otimes 1)) = \lambda_M((\Bbbk \otimes \widehat{m})(1 \otimes 1)) = \lambda_M(1 \otimes m)$$

for all $m \in M$ and for all $M \in \mathcal{M}_A$. Since λ is a natural isomorphism too, we have that there exists $l^{-1} \in A$ such that

$$1 \otimes l^{-1} := \lambda_A^{-1}(1)$$

and

$$1 \otimes ml^{-1} = (\Bbbk \otimes \widehat{m})((\lambda_A^{-1})(1)) = \lambda_M^{-1}(\widehat{m}(1)) = \lambda_M^{-1}(m)$$

for all $m \in M$. Then, in particular:

$$1 = \lambda_A(\lambda_A^{-1}(1)) = \lambda_A(1 \otimes l^{-1}) = l^{-1}l$$

and

$$1 \otimes 1 = \lambda_A^{-1}(\lambda_A(1 \otimes 1)) = \lambda_A^{-1}(l) = 1 \otimes ll^{-1}$$

ensure that l is invertible with two-sided inverse l^{-1}. Moreover, A-linearity of λ implies that

$$m \cdot la = \lambda_M(1 \otimes m) \cdot a = \lambda_M((1 \otimes m) \cdot (\varepsilon \otimes A)(\Delta(a))) = m \cdot \varepsilon(a_1)a_2 l \tag{3.12}$$

for every $m \in M$ and $a \in A$. Choosing $M = A$ and evaluating (3.12) at $m = 1$ gives:

$$la = (\varepsilon \otimes A)(\Delta(a))l.$$

Analogously:

- for all $m \in M$ and for each $M \in \mathcal{M}_A$, $\rho_M(m \otimes 1) = m \cdot r$;
- exists r^{-1} and it satisfies $1 \otimes r^{-1} := \rho_A^{-1}(1)$;
- for all $m \in M$ and $a \in A$,

$$m \cdot ra = \rho_M(m \otimes 1) \cdot a = \rho_M((m \otimes 1) \cdot (A \otimes \varepsilon)(\Delta(a))) = m \cdot a_1 \varepsilon(a_2) r. \tag{3.13}$$

Evaluating (3.13) at $m = 1$ gives:

$$ra = (A \otimes \varepsilon)(\Delta(a))r.$$

Next, we need to prove that also (3.4) and (3.5) are satisfied and, as one can expect, these follow by the Pentagon and Triangle Axioms. By the Pentagon Axiom we have:

$$(A \otimes \alpha_{A,A,A}) \circ \alpha_{A,A\otimes A,A} \circ (\alpha_{A,A,A} \otimes A) = \alpha_{A,A,A\otimes A} \circ \alpha_{A\otimes A,A,A}$$

and evaluating it at $1 \otimes 1 \otimes 1$ we get:

$$(\Phi \otimes 1)(A \otimes \Delta \otimes A)(\Phi)(1 \otimes \Phi) = (\Delta \otimes A \otimes A)(\Phi)(A \otimes A \otimes \Delta)(\Phi).$$

Instead, by the Triangle Axiom we know that:

$$(A \otimes l_A) \circ \alpha_{A,\Bbbk,A} = r_A \otimes A$$

and evaluating it at $1_A \otimes 1_{\Bbbk} \otimes 1_A$ this gives:

$$(A \otimes \varepsilon \otimes A)(\Phi)(1 \otimes l) = (r \otimes 1),$$

i.e., the last axiom of quasi-bialgebra that misses. $\square$

Remark 3.1.4. The equivalent conditions defining a quasi-bialgebra that we gave here are not the traditional ones. Actually, the most common definition is: an algebra with comultiplication and counit $(A, m, u, \Delta, \varepsilon)$ is a quasi-bialgebra if the category of *left* A-modules is monoidal (cfr. [Ka, Definition XV.1.1]). With this definition, the axioms of Theorem 3.1.3 become (cfr. [Ka, Proposition XV.1.2]):

$$(A \otimes \Delta)(\Delta(a))\Phi = \Phi(\Delta \otimes A)(\Delta(a)) \tag{3.14a}$$
$$(\varepsilon \otimes A)(\Delta(a)) = l^{-1}al \tag{3.14b}$$
$$(A \otimes \varepsilon)(\Delta(a)) = r^{-1}ar \tag{3.14c}$$

for all $a \in A$, and

$$(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes A \otimes A)(\Phi) = (1 \otimes \Phi)(A \otimes \Delta \otimes A)(\Phi)(\Phi \otimes 1) \tag{3.15a}$$
$$(A \otimes \varepsilon \otimes A)(\Phi) = r \otimes l^{-1} \tag{3.15b}$$

Moreover, the constraints α, λ and ρ need to be modified in:

$$\alpha_{M,N,P}((m \otimes n) \otimes p) = \Phi \cdot (m \otimes (n \otimes p))$$
$$\lambda_M(1 \otimes m) = l \cdot m$$
$$\rho_M(m \otimes 1) = r \cdot m$$

Nevertheless, the two definitions are equivalent. Indeed, these last axioms can be obtained by substituting Φ^{-1} to Φ, l^{-1} to l and r^{-1} to r into the previous ones. Anyhow, for coherence's sake, from now on we will use this last ordinary axioms, instead of the ones that appears in Theorem 3.1.3.

Theorem 3.1.3 shows that there is at least one substantial difference within bialgebras and quasi-bialgebras: in a bialgebra we can always reassign parenthesis and renumber the indices of the Sweedler' Sigma Notation. In a quasi-bialgebra we can reassign parenthesis, but we cannot renumber the indices: we should need coassociativity of Δ, that we have no more. What happens in the quasi-bialgebra case is that Δ is quasi-coassociative via the *Drinfel'd reassociator* Φ, i.e.:

$$\Phi \cdot [((a_1)_1 \otimes (a_1)_2) \otimes a_2] = [a_1 \otimes ((a_2)_1 \otimes (a_2)_2)] \cdot \Phi.$$

For the sake of completeness, we give also the definition of what a morphism of quasi-bialgebras is.

Definition 3.1.5. (Quasi-bialgebra morphism)
A $\Bbbk$-linear map $f\colon A \to A'$ between two quasi-bialgebras $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ and $(A', m', u', \Delta', \varepsilon', \Phi', l', r')$ is a *quasi-bialgebra morphism* if it is an algebra morphism such that:

- Preserves the comultiplication and counit, in the sense that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A' \arrow[d, "\Delta'"] \\ A \otimes A \arrow[r, "f \otimes f"'] & A' \otimes A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & \circlearrowleft & A' \arrow[dl, "\varepsilon'"] \\ & \Bbbk & \end{tikzcd}$$

- Preserves Φ, l and r, i.e.,

$$(f \otimes f \otimes f)(\Phi) = \Phi' \qquad f(l) = l' \qquad f(r) = r'$$

It is quite a heavy job to deal with Φ, l and r. Unfortunately, Φ is what distinguish quasi-bialgebras from bialgebras and so we cannot expect that there exists a way to get rid of it, but, as we are going to show now, l and r are not so fundamental and we can do without them.

Theorem 3.1.6. ([Ka, Proposition XV.3.2]) *Let $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ be a quasi-bialgebra and let $F \in A \otimes A$ be an invertible element. Define, for all $a \in A$:*

$$\Delta_F(a) := F\Delta(a)F^{-1} \tag{3.16}$$

and the elements:

$$\Phi_F := (1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) \tag{3.17a}$$
$$l_F := l(\varepsilon \otimes A)(F^{-1}) \tag{3.17b}$$
$$r_F := r(A \otimes \varepsilon)(F^{-1}) \tag{3.17c}$$

Then $(A, m, u, \Delta_F, \varepsilon, \Phi_F, l_F, r_F)$ is a quasi-bialgebra denoted by A_F. We say that A_F is obtained from A by twisting *via the element F (cfr. [Dr1, Remark on page 1422]).*

Proof. We have to verify that Δ_F is a morphism of algebras (ε has been not modified) and that all five axioms of quasi-bialgebra are satisfied: (3.14) and (3.15). Let us begin with Δ_F. Since Δ is a morphism of algebras:

$$\Delta_F(a)\Delta_F(b) = F\Delta(a)F^{-1}F\Delta(b)F^{-1} = F\Delta(a)\Delta(b)F^{-1} = F\Delta(ab)F^{-1} = \Delta_F(ab)$$

and so Δ_F is a morphism of algebras, too. Next, let us show that (3.14) are satisfied.

(3.14a). For all $a \in A$:

$$
\begin{aligned}
&\Phi_F(\Delta_F \otimes A)(\Delta_F(a)) \overset{\substack{(3.16)\\(3.17a)}}{=} \\
&= (1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1)(F \otimes 1)(\Delta \otimes A)(\Delta_F(a))(F^{-1} \otimes 1) = \\
&= (1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(\Delta \otimes A)(F\Delta(a)F^{-1})(F^{-1} \otimes 1) = \\
&= (1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(\Delta(a))(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) \overset{(3.14a)}{=} \\
&= (1 \otimes F)(A \otimes \Delta)(F)(A \otimes \Delta)(\Delta(a))\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) = \\
&= (1 \otimes F)(A \otimes \Delta)(F)(A \otimes \Delta)(\Delta(a)F^{-1})(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) = \\
&= (1 \otimes F)(A \otimes \Delta)(\Delta_F(a))(1 \otimes F^{-1})(1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) = \\
&= (A \otimes \Delta_F)(\Delta_F(a))\Phi_F
\end{aligned}
$$

(3.14b). For all $a \in A$:

$$
\begin{aligned}
l_F(\varepsilon \otimes A)(\Delta_F(a)) &= l(\varepsilon \otimes A)(F^{-1})(\varepsilon \otimes A)(F\Delta(a)F^{-1}) = \\
&= l(\varepsilon \otimes A)(\Delta(a))(\varepsilon \otimes A)(F^{-1}) \overset{(3.14b)}{=} \\
&= al(\varepsilon \otimes A)(F^{-1}) = \\
&= al_F
\end{aligned}
$$

(3.14c). For all $a \in A$:

$$
\begin{aligned}
r_F(A \otimes \varepsilon)(\Delta_F(a)) &= r(A \otimes \varepsilon)(F^{-1})(A \otimes \varepsilon)(F\Delta(a)F^{-1}) = \\
&= r(A \otimes \varepsilon)(\Delta(a))(A \otimes \varepsilon)(F^{-1}) \overset{(3.14c)}{=} \\
&= ar(A \otimes \varepsilon)(F^{-1}) = \\
&= ar_F
\end{aligned}
$$

In order to prove that (3.15a) is satisfied, we need to break it into smaller identities. First of all, note that:

$$
\begin{aligned}
&(A \otimes A \otimes \Delta)(\Phi_F) = (A \otimes A \otimes \Delta)((1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1)) = \\
&= (1 \otimes (A \otimes \Delta)(F))(A \otimes (A \otimes \Delta)\Delta)(F)(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes \Delta)(F^{-1})(F^{-1} \otimes 1 \otimes 1)
\end{aligned}
$$

and that:

$$
\begin{aligned}
&(\Delta \otimes A \otimes A)(\Phi_F) = (\Delta \otimes A \otimes A)((1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1)) = \\
&= (1 \otimes 1 \otimes F)(\Delta \otimes \Delta)(F)(\Delta \otimes A \otimes A)(\Phi)((\Delta \otimes A)\Delta \otimes A)(F^{-1})((\Delta \otimes A)(F^{-1}) \otimes 1)
\end{aligned}
$$

and also that:

$$
(1 \otimes 1 \otimes F^{-1})(F \otimes 1 \otimes 1) = (F \otimes F^{-1}) = (F \otimes 1 \otimes 1)(1 \otimes 1 \otimes F^{-1}).
$$

Thus:

$$
\begin{aligned}
&(A\otimes A\otimes \Delta_F)(\Phi_F)(\Delta_F\otimes A\otimes A)(\Phi_F) = \\
&= (1\otimes 1\otimes F)(A\otimes A\otimes \Delta)(\Phi_F)(F\otimes F^{-1})(\Delta\otimes A\otimes A)(\Phi_F)(F^{-1}\otimes 1\otimes 1) = \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(A\otimes (A\otimes\Delta)\Delta)(F)(A\otimes A\otimes\Delta)(\Phi) \\ (\Delta\otimes A\otimes A)(\Phi)((\Delta\otimes A)\Delta\otimes A)(F^{-1})((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] \overset{(3.15a)}{=} \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(A\otimes (A\otimes\Delta)\Delta)(F)(1\otimes\Phi)(A\otimes\Delta\otimes A)(\Phi) \\ (\Phi\otimes 1)((\Delta\otimes A)\Delta\otimes A)(F^{-1})((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] \overset{(3.14a)}{=} \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(1\otimes\Phi)(A\otimes (\Delta\otimes A)\Delta)(F)(A\otimes\Delta\otimes A)(\Phi) \\ ((A\otimes\Delta)\Delta\otimes A)(F^{-1})(\Phi\otimes 1)((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] = \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(1\otimes\Phi) \\ (A\otimes\Delta\otimes A)((A\otimes\Delta)(F))(A\otimes\Delta\otimes A)(\Phi)(A\otimes\Delta\otimes A)((\Delta\otimes A)(F^{-1})) \\ (\Phi\otimes 1)((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] = \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(1\otimes\Phi) \\ (A\otimes\Delta\otimes A)((1\otimes F^{-1})(1\otimes F)(A\otimes\Delta)(F)\Phi(\Delta\otimes A)(F^{-1})(F^{-1}\otimes 1)(F\otimes 1)) \\ (\Phi\otimes 1)((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] = \\
&= \left[\begin{array}{l} (1\otimes 1\otimes F)(1\otimes (A\otimes\Delta)(F))(1\otimes\Phi)(1\otimes (\Delta\otimes A)(F^{-1})) \\ (A\otimes\Delta\otimes A)(\Phi_F) \\ ((A\otimes\Delta)(F)\otimes 1)(\Phi\otimes 1)((\Delta\otimes A)(F^{-1})\otimes 1)(F^{-1}\otimes 1\otimes 1) \end{array}\right] = \\
&= \left[\begin{array}{l} (1\otimes (1\otimes F)(A\otimes\Delta)(F)\Phi(\Delta\otimes A)(F^{-1})(F^{-1}\otimes 1))(1\otimes F\otimes 1) \\ (A\otimes\Delta\otimes A)(\Phi_F) \\ (1\otimes F^{-1}\otimes 1)((1\otimes F)(A\otimes\Delta)(F)\Phi(\Delta\otimes A)(F^{-1})(F^{-1}\otimes 1)\otimes 1) \end{array}\right] = \\
&= (1\otimes\Phi_F)(A\otimes\Delta_F\otimes A)(\Phi_F)(\Phi_F\otimes 1)
\end{aligned}
$$

For the remaining axiom observe that:

$$
\begin{aligned}
(A\otimes\varepsilon\otimes A)(\Phi_F) &= (A\otimes\varepsilon\otimes A)((1\otimes F)(A\otimes\Delta)(F)\Phi(\Delta\otimes A)(F^{-1})(F^{-1}\otimes 1)) \overset{(3.15b)}{=} \\
&= \left[\begin{array}{l} (1\otimes (\varepsilon\otimes A)(F))(A\otimes (\varepsilon\otimes A)(\Delta))(F)(r\otimes l^{-1}) \\ ((A\otimes\varepsilon)\Delta\otimes A)(F^{-1})((A\otimes\varepsilon)(F^{-1})\otimes 1) \end{array}\right] \overset{(3.14b)}{\overset{(3.14c)}{=}} \\
&= \left[\begin{array}{l} (1\otimes (\varepsilon\otimes A)(F))(1\otimes l^{-1})F(1\otimes l)(r\otimes l^{-1}) \\ (r^{-1}\otimes 1)F^{-1}(r\otimes 1)((A\otimes\varepsilon)(F^{-1})\otimes 1) \end{array}\right] \overset{(3.17b)}{\overset{(3.17c)}{=}} \\
&= (1\otimes l_F^{-1})(r_F\otimes 1) = (r_F\otimes l_F^{-1})
\end{aligned}
$$

□

Definition 3.1.7. (Twist equivalent quasi-bialgebras)
Two quasi-bialgebras $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ and $(A', m', u', \Delta', \varepsilon', \Phi', l', r')$ are *twist equivalent* if there exists $F \in A' \otimes A'$ invertible and an isomorphism of quasi-bialgebras $f\colon A \to A'_F$.

Proposition 3.1.8. *Let $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ be a quasi-bialgebra and let $F \in A\otimes A$ be an invertible element. Then the triple:*

$$(R, \varphi_0, \varphi_2) = (Id_{{}_A\mathcal{M}}, Id_{\Bbbk}, \varphi_2)\colon ({}_A\mathcal{M}, \otimes, \Bbbk, \alpha, \lambda, \rho) \to ({}_{A_F}\mathcal{M}, \oslash, \Bbbk, \alpha_F, \lambda_F, \rho_F),$$

where, here, $\oslash$ is simply $\otimes$ (but it will be useful to remember explicitly in which category the tensor product has been done),

$$\varphi_2(M,N)\colon R(M)\oslash R(N)\longrightarrow R(M\otimes N)\colon m\oslash n\longmapsto F^{-1}\cdot(m\otimes n),$$

and

$$\begin{aligned}\alpha_F\colon (M\oslash N)\oslash P\longrightarrow M\oslash(N\oslash P)\colon (m\oslash n)\oslash p\longmapsto \Phi_F\cdot(m\oslash(n\oslash p))\\ \lambda_F\colon \Bbbk\oslash M\longrightarrow M\colon 1\oslash m\longmapsto l_F\cdot m\\ \rho_F\colon M\oslash\Bbbk\longrightarrow M\colon m\oslash 1\longmapsto r_F\cdot m\end{aligned}$$

defines a monoidal functor between monoidal categories that is also an isomorphism.

Proof. Note that (A,m,u) and (A_F,m,u) are exactly the same algebra in $(\mathcal{M},\otimes,\Bbbk,a,l,r)$, and so the categories ${}_A\mathcal{M}$ and ${}_{A_F}\mathcal{M}$ coincides. Thus $R=\mathrm{Id}$ is trivially a well defined functor and $\varphi_0=\mathrm{Id}_\Bbbk$ is an isomorphism between $R(\Bbbk)=\Bbbk$ and $\Bbbk$. Moreover, φ_2 is a natural isomorphism of left A_F-modules. Indeed, the fact that:

$$\begin{aligned}\varphi_2(M,N)(a\cdot(m\oslash n)) &= \varphi_2(M,N)(\Delta_F(a)\cdot(m\oslash n)) =\\ &= F^{-1}\Delta_F(a)\cdot(m\otimes n) =\\ &= \Delta(a)F^{-1}\cdot(m\otimes n) =\\ &= a\cdot\varphi_2(M,N)(m\oslash n)\end{aligned}$$

for all $m\in M$, $n\in N$, $a\in A$, shows that it is a morphism, and

$$\psi_2(M,N)\colon R(M\otimes N)\longrightarrow R(M)\oslash R(N)\colon m\otimes n\longmapsto F\cdot(m\oslash n)$$

is an explicit inverse for $\varphi_2(M,N)$, for each pair (M,N) in ${}_A\mathcal{M}$. Furthermore, if we let $g\colon M\to M'$ and $h\colon N\to N'$ be two morphisms of left A-modules, then for all $m\in M$ and $n\in N$:

$$\begin{aligned}R(g\otimes h)(\varphi_2(M,N)(m\oslash n)) &= R(g\otimes h)(f^1\cdot m\otimes f^2\cdot n)) =\\ &= g(f^1\cdot m)\otimes h(f^2\cdot n) =\\ &= F^{-1}\cdot(g(m)\otimes h(n)) =\\ &= \varphi_2(M',N')(g(m)\oslash h(n)) =\\ &= \varphi_2(M',N')((R(g)\oslash R(h))(m\oslash n))\end{aligned}$$

where $f^1\otimes f^2 := F^{-1}$, so that the following diagram commutes:

$$\begin{tikzcd} R(M)\oslash R(N) \arrow[r, "\varphi_2(M,N)"] \arrow[d, "R(g)\oslash R(h)"'] & R(M\otimes N) \arrow[d, "R(g\otimes h)"] \\ R(M')\oslash R(N') \arrow[r, "\varphi_2(M',N')"'] & R(M'\otimes N') \end{tikzcd}$$

and φ_2 is natural. Now, observe that (1.13), (1.14) and (1.15) are satisfied since:

$$\begin{tikzcd}
(m\oslash n)\oslash p \arrow[r, mapsto, "\alpha_F(R(M),R(N),R(P))"] \arrow[d, mapsto, "\varphi_2(M,N)\oslash R(P)"'] & \Phi_F\cdot(m\oslash(n\oslash p)) \arrow[d, mapsto, "R(M)\oslash\varphi_2(N,P)"] \\
(F^{-1}\oslash 1)\cdot((m\otimes n)\oslash n) \arrow[dd, mapsto, "\varphi_2(M\otimes N,P)"'] & (1\oslash F^{-1})\Phi_F\cdot(m\oslash(n\otimes p)) \arrow[d, mapsto, "\varphi_2(M,N\otimes P)"] \\
 & (A_F\oslash\Delta)(F^{-1})(1\otimes F^{-1})\Phi_F\cdot(m\otimes(n\otimes p)) \arrow[d, equal, "(3.17a)"] \\
(\Delta\oslash A_F)(F^{-1})(F^{-1}\otimes 1)\cdot((m\otimes n)\otimes n) \arrow[r, "R(\alpha(M,N,P))"'] & \Phi(\Delta\oslash A_F)(F^{-1})(F^{-1}\otimes 1)\cdot((m\otimes n)\otimes n)
\end{tikzcd}$$

$$\begin{tikzcd}
1\oslash m \arrow[r, "\lambda_F(R(M))"] \arrow[d, "\varphi_0\oslash R(M)"'] & l(\varepsilon\otimes A)(F^{-1})\cdot(1\otimes m) \\
1\oslash m \arrow[r, "\varphi_2(\Bbbk,M)"'] & F^{-1}\cdot(1\otimes m) \arrow[u, "R(\lambda_M)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
m\oslash 1 \arrow[r, "\rho_F(R(M))"] \arrow[d, "R(M)\oslash\varphi_0"'] & r(A\otimes\varepsilon)(F^{-1})\cdot(m\otimes 1) \\
m\oslash 1 \arrow[r, "\varphi_2(M,\Bbbk)"'] & F^{-1}\cdot(m\otimes 1) \arrow[u, "R(\rho_M)"']
\end{tikzcd}$$

so that, as claimed, $(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\Bbbk},\varphi_2)$ is a monoidal functor.

Next, recall that (Theorem 3.1.6) $(A_F,m,u,\Delta_F,\varepsilon,\Phi_F,l_F,r_F)$ is a quasi-bialgebra, too. Hence, in the same way as above, we can prove that

$$(L,\psi_0,\psi_2)\colon ({}_{A_F}\mathcal{M},\oslash,\Bbbk,\alpha_F,\lambda_F,\rho_F)\to({}_{(A_F)_{F^{-1}}}\mathcal{M},\otimes,\Bbbk,(\alpha_F)_{F^{-1}},(\lambda_F)_{F^{-1}},(\rho_F)_{F^{-1}})$$

is a monoidal functor between monoidal categories, where $L=\mathrm{Id}_{{}_{A_F}\mathcal{M}}$, $\psi_0=\mathrm{Id}_{\Bbbk}$ and

$$\psi_2(M,N)\colon L(M)\otimes L(N)\longrightarrow L(M\oslash N)\colon m\otimes n\longmapsto F\cdot(m\oslash n).$$

Observe now that $(\Delta_F)_{F^{-1}}=\Delta$, $(\Phi_F)_{F^{-1}}=\Phi$, $(l_F)_{F^{-1}}=l$ and $(r_F)_{F^{-1}}=r$. Therefore, $(A_F)_{F^{-1}}=A$ and

$$({}_{(A_F)_{F^{-1}}}\mathcal{M},\otimes,\Bbbk,(\alpha_F)_{F^{-1}},(\lambda_F)_{F^{-1}},(\rho_F)_{F^{-1}})=({}_A\mathcal{M},\otimes,\Bbbk,\alpha,\lambda,\rho).$$

This implies that we have another monoidal functor that goes the other way with respect to R. Furthermore, both compositions RL and LR are actually the identity of ${}_A\mathcal{M}$, so that R is an isomorphism of monoidal categories. □

Lemma 3.1.9. *Let $(A,m,u,\Delta,\varepsilon,\Phi,l,r)$ be a quasi-bialgebra. Then $(\varepsilon\otimes\varepsilon)\circ\Delta=\varepsilon$.*

Proof. For all $a \in A$:

$$\begin{aligned}(\varepsilon \otimes \varepsilon)(\Delta(a)) &= (\varepsilon \otimes \varepsilon)(a_1 \otimes a_2) = \varepsilon(a_1)\,\varepsilon(a_2) = \varepsilon(a_1\,\varepsilon(a_2)) = \\ &= \varepsilon((A \otimes \varepsilon)(\Delta(a))) \overset{(3.14c)}{=} \varepsilon(r^{-1}ar) = \\ &= \varepsilon(a)\end{aligned}$$

□

Now we are ready for take care of the elements l and r.

Theorem 3.1.10. *Every quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ is twist equivalent to a quasi-bialgebra $(A', m', u', \Delta', \varepsilon', \Phi', l', r')$ such that $r' = l' = 1$.*

Proof. First of all, apply $\varepsilon \otimes \varepsilon \otimes \varepsilon$ to both sides of (3.15a). In view of Lemma 3.1.9, we get that:

$$[(\varepsilon \otimes \varepsilon \otimes \varepsilon)(\Phi)]^3 = [(\varepsilon \otimes \varepsilon \otimes \varepsilon)(\Phi)]^2.$$

Since Φ is invertible:

$$(\varepsilon \otimes \varepsilon \otimes \varepsilon)(\Phi) = 1.$$

Now, if we apply $\varepsilon \otimes \Bbbk \otimes \varepsilon$ to both sides of (3.15b) we find out that:

$$\varepsilon(r)\,\varepsilon(l^{-1}) = (\varepsilon \otimes \varepsilon \otimes \varepsilon)(\Phi) = 1,$$

and so $\delta := \varepsilon(r)^{-1} = \varepsilon(l)^{-1}$. Define $F := \delta(r \otimes l)$. $F \in A \otimes A$ is invertible with inverse $F^{-1} = \delta^{-1}(r^{-1} \otimes l^{-1})$, thus A is twist equivalent to $(A_F, m, u, \Delta_F, \varepsilon, \Phi_F, l_F, r_F)$, by Theorem 3.1.6. Moreover:

$$\begin{aligned} l_F &= l(\varepsilon \otimes A)(F^{-1}) = l\,\varepsilon(r)\varepsilon(r)^{-1}l^{-1} = 1 \\ r_F &= r(A \otimes \varepsilon)(F^{-1}) = r\,\varepsilon(l)r^{-1}\varepsilon(l)^{-1} = 1 \end{aligned}$$

as desired. □

Corollary 3.1.11. *Let $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ be a quasi-bialgebra and let $F = \delta(r \otimes l)$. Then there exists an isomorphism of monoidal categories between $({}_A\mathcal{M}, \otimes, \Bbbk, \alpha, \lambda, \rho)$ and $({}_{A_F}\mathcal{M}, \otimes, \Bbbk, \alpha_F, \ell, \mathtt{r})$, where ℓ and $\mathtt{r}$ are the same constraints of $(\mathcal{M}, \otimes, \Bbbk, a, \ell, \mathtt{r})$.*

Remark 3.1.12. In view of Theorem 3.1.10, we can always assume that in a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ one has $l = 1 = r$. With this assumption, let $(A, m, u, \Delta, \varepsilon, \Phi)$ and $(A', m', u', \Delta', \varepsilon', \Phi')$ be twist equivalent quasi-bialgebras. Then there exists $F \in A \otimes A$ invertible and an isomorphism of quasi-bialgebras $\varphi \colon A' \to A_F$. Note that, by the definitions we gave in Theorem 3.1.6, we have that

$$l_F = l(\varepsilon \otimes A)(F^{-1}) \quad \text{and} \quad r_F = r(A \otimes \varepsilon)(F^{-1}).$$

On the other hand we have also that:

$$l = 1 = r, \qquad l' = 1 = r' \qquad \text{and} \qquad l_F = \varphi(l') = 1 = \varphi(r') = r_F.$$

Thus F satisfies:

$$(A \otimes \varepsilon)(F) = 1 = (\varepsilon \otimes A)(F).$$

Hence we are led to give the following definition (cfr. [Ka, Definition XV.3.1]).

Definition 3.1.13. (Gauge transformation)
Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. A *gauge transformation* on A is an invertible element F of $A \otimes A$ such that

$$(A \otimes \varepsilon)(F) = 1 = (\varepsilon \otimes A)(F). \tag{3.18}$$

Remark 3.1.14. This twisting construction is due ultimately to Drinfel'd (cfr. [Dr1]), but the idea of introducing a proper terminology to refer to the elements that satisfy the conditions of Definition 3.1.13 comes to us from [Ka] and [BCT]. In the literature, a gauge transformation is also referred to as a *twist* simply, however we preferred to distinguish between the action of twisting (the twist) and the element via which we twist (the gauge transformation).

If we twist a quasi-bialgebra with trivial l, r via a general invertible element F in $A \otimes A$, we do not find a quasi-bialgebra with trivial l, r; but if we twist it with a gauge transformation, then we do.

Moreover, let $(B, m, u, \Delta, \varepsilon)$ be an ordinary bialgebra. If we consider $\Phi = 1 \otimes 1 \otimes 1$, then $(B, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. Now, take any gauge transformation F on B. B_F is a quasi-bialgebra (Theorem 3.1.6), but generally it is not an ordinary bialgebra. Indeed:

$$\Phi_F = (1 \otimes F)(A \otimes \Delta)(F)(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1)$$

does not equal $1 \otimes 1 \otimes 1$ in general, and Δ_F is not coassociative. In such cases, B_F is a non trivial example of quasi-bialgebra.

3.2 The fundamental structure theorem for quasi-Hopf bimodules

From now on we will work with quasi-bialgebras such that $r = l = 1$, so that a quasi-bialgebra is the datum of $(A, m, u, \Delta, \varepsilon, \Phi)$ where:

- (A, m, u) is an associative unital $\Bbbk$-algebra,
- $\Delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to \Bbbk$ are morphisms of algebras,
- $\Phi \in A \otimes A \otimes A$ is an invertible element, called the *Drinfel'd associator* or simply *reassociator*, such that:

$$(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes A \otimes A)(\Phi) = (1 \otimes \Phi)(A \otimes \Delta \otimes A)(\Phi)(\Phi \otimes 1) \tag{3.19}$$

$$(A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1 \tag{3.20}$$

- Δ and ε are quasi-coassociative and counital:

$$(A \otimes \Delta)(\Delta(a))\Phi = \Phi(\Delta \otimes A)(\Delta(a)) \tag{3.21}$$

$$(\varepsilon \otimes A)(\Delta(a)) = a \tag{3.22}$$

$$(A \otimes \varepsilon)(\Delta(a)) = a \tag{3.23}$$

for all $a \in A$.

(cfr. [HN, Section 2], [Dr1, Section 1])

Lemma 3.2.1. *Let* $(A, m, u, \Delta, \varepsilon, \Phi)$ *be a quasi-bialgebra. Then:*

$$(\varepsilon \otimes A \otimes A)(\Phi) = 1 \otimes 1 = (A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1 = (A \otimes A \otimes \varepsilon)(\Phi). \tag{3.24}$$

Proof. The central identities come from condition (3.20). For the left identity, apply $\varepsilon \otimes \varepsilon \otimes A \otimes A$ to (3.19) to get:

$$\begin{aligned}(1 \otimes (\varepsilon \otimes A \otimes A)(\Phi))((\varepsilon \otimes (\varepsilon \otimes A)\Delta \otimes A)(\Phi))((\varepsilon \otimes \varepsilon \otimes A)(\Phi) \otimes 1) = \\ = ((\varepsilon \otimes \varepsilon \otimes \Delta)(\Phi))(((\varepsilon \otimes \varepsilon)\Delta \otimes A \otimes A)(\Phi)).\end{aligned} \tag{3.25}$$

Since the following identities hold:

- $(\varepsilon \otimes A)\Delta = \mathrm{Id}$,
- $(\varepsilon \otimes \varepsilon)\Delta = \varepsilon$,
- $(\varepsilon \otimes \varepsilon \otimes A)(\Phi) = (\varepsilon \otimes \Bbbk \otimes A)(A \otimes \varepsilon \otimes A)(\Phi) = 1$,
- $(\varepsilon \otimes \varepsilon \otimes \Delta)(\Phi) = (\varepsilon \otimes \Bbbk \otimes \Delta)(A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1$

relation (3.25) can be written as

$$((\varepsilon \otimes A \otimes A)(\Phi))^2 = (\varepsilon \otimes A \otimes A)(\Phi).$$

Since Φ is invertible, we can simplify to

$$(\varepsilon \otimes A \otimes A)(\Phi) = 1 \otimes 1.$$

Analogously, applying $A \otimes A \otimes \varepsilon \otimes \varepsilon$ to both sides of (3.19):

$$\begin{aligned}(1 \otimes (A \otimes \varepsilon \otimes \varepsilon)(\Phi))((A \otimes (A \otimes \varepsilon)\Delta \otimes \varepsilon)(\Phi))((A \otimes A \otimes \varepsilon)(\Phi) \otimes 1) = \\ = ((A \otimes A \otimes (\varepsilon \otimes \varepsilon)\Delta)(\Phi))((\Delta \otimes \varepsilon \otimes \varepsilon)(\Phi))\end{aligned} \tag{3.26}$$

and simplifying as above:

$$((A \otimes A \otimes \varepsilon)(\Phi))^2 = (A \otimes A \otimes \varepsilon)(\Phi),$$

from which we get:

$$(A \otimes A \otimes \varepsilon)(\Phi) = 1 \otimes 1.$$

□

Conventionally, we will write

$$\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3 \qquad \text{and} \qquad \Phi^{-1} = \phi^1 \otimes \phi^2 \otimes \phi^3,$$

summation understood.

Recall that, if $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra, then $(\mathcal{M}_A, \otimes, \Bbbk, \alpha_A, l, r)$ is a monoidal category (Theorem 3.1.3) in the following way.
Given a right A-module M, we denote by:

$$\mu = \mu^r_M : \quad \begin{array}{ccc} M \otimes A & \longrightarrow & M \\ m \otimes a & \longmapsto & m \cdot a \end{array}$$

its right A-action. The tensor product of two A-modules M and N is a right A-module via the diagonal action, i.e., $\mu((m \otimes n) \otimes a) = (m \cdot a_1 \otimes n \cdot a_2)$. The unit is $\Bbbk$, regarded as a right A-module via the trivial right action $\mu(k \otimes a) = k\,\varepsilon(a)$. The associativity and unit constraints are defined by (cfr Theorem 3.1.3 and Remark 3.1.4):

$$\alpha_A(M, N, P)((m \otimes n) \otimes p) = (m \otimes (n \otimes p)) \cdot \Phi^{-1}$$

$$l_M(k \otimes m) = km \qquad \text{and} \qquad r_M(m \otimes k) = mk.$$

Remark 3.2.2. There are two more monoidal category structures that we can construct on a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$:

1. Consider the category of *left* A-modules ${}_A\mathcal{M}$. Denote by $\mu^l = \mu^l_M : A \otimes M \to M$ the left A-action. If M and N are left A-modules, then their tensor product is a left A-module via the diagonal action: $\mu^l(a \otimes (m \otimes n)) = a_1 \cdot m \otimes a_2 \cdot n$. $\Bbbk$ is a left A-module via the trivial left action: $\mu^l(a \otimes k) = \varepsilon(a)\,k$. In view of Remark 3.1.4 and [Ka, Proposition XV.1.2], we have that $({}_A\mathcal{M}, \otimes, \Bbbk, {}_A\alpha, l, r)$ is a monoidal category, where:

$$_A\alpha(M, N, P)((m \otimes n) \otimes p) = \Phi \cdot (m \otimes (n \otimes p)).$$

2. Consider the category of (A, A)-bimodules ${}_A\mathcal{M}_A$. Putting together the two results above we get that $({}_A\mathcal{M}_A, \otimes, \Bbbk, {}_A\alpha_A, l, r)$ is a monoidal category, where:

$$_A\alpha_A(M, N, P)((m \otimes n) \otimes p) = \Phi \cdot (m \otimes (n \otimes p)) \cdot \Phi^{-1}$$

(cfr. [Sc2, Section 3]).

The following proposition is the analogue of Lemma 2.1.1.

Proposition 3.2.3. *$((A, m, m), \Delta, \varepsilon)$ is a coalgebra in $({}_A\mathcal{M}_A, \otimes, \Bbbk, {}_A\alpha_A, l, r)$.*

Proof. (A, m, m) is either a right A-module and a left A-module and the structures are compatible since (A, m, u) is an associative (unital) algebra.

- Δ is an (A, A)-bimodule morphism. Indeed:

$$\Delta(m) \cdot a = m_1 a_1 \otimes m_2 a_2 = (ma)_1 \otimes (ma)_2 = \Delta(m \cdot a),$$

$$a \cdot \Delta(m) = a_1 m_1 \otimes a_2 m_2 = (am)_1 \otimes (am)_2 = \Delta(a \cdot m),$$

since it is an algebra morphism.

- In the same way, ε is an (A, A)-bimodule morphism. Indeed:

$$\varepsilon(m) \cdot a = \varepsilon(m)\, \varepsilon(a) = \varepsilon(m \cdot a),$$

$$a \cdot \varepsilon(m) = \varepsilon(a)\, \varepsilon(m) = \varepsilon(a \cdot m),$$

since it is an algebra morphism.

- ε is a counit for Δ, in view of (3.22) and (3.23).

- Δ is coassociative. Indeed:

$$\begin{tikzcd}
A \otimes A \arrow[d, "\Delta \otimes A"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & A \otimes A \arrow[d, "A \otimes \Delta"] \\
(A \otimes A) \otimes A \arrow[rr, "{{}_A\alpha_A(A,A,A)}"'] & & A \otimes (A \otimes A)
\end{tikzcd}$$

commutes, since:

$${}_A\alpha_A(A, A, A)((\Delta \otimes A)(\Delta(a))) = \Phi \cdot (\Delta \otimes A)(\Delta(a)) \cdot \Phi^{-1} \overset{(3.21)}{=} (A \otimes \Delta)(\Delta(a)).$$

$\square$

Remark 3.2.4. Coassociativity strictly depends on ${}_A\alpha_A$. Note that A is not a coalgebra in $\mathcal{M}_A$, nor in ${}_A\mathcal{M}$, in general.

Now we can define the correct generalization of Hopf modules to quasi-bialgebras [HN, Definition 3.1].

Definition 3.2.5. (Quasi-Hopf bimodules)
The category of (right) *quasi-Hopf A-bimodules* is defined to be:

$${}_A\mathcal{M}_A^A := ({}_A\mathcal{M}_A)^A$$

A morphism of quasi-Hopf bimodules is just a (A, A)-bimodule morphism that is also right A-colinear.

Remark 3.2.6. Note that:

- For $((M, \mu^l, \mu^r), \rho)$ to be a quasi-Hopf bimodule, ρ has to be a (A, A)-bimodule morphism, that is:

$$\rho(a \cdot m) = a_1 \cdot m_0 \otimes a_2 \cdot m_1 \qquad \text{and} \qquad \rho(m \cdot a) = m_0 \cdot a_1 \otimes m_1 \cdot a_2.$$

- The following relations hold:

$$m_0\, \varepsilon(m_1) = m, \tag{3.27}$$
$$(m_0 \otimes (m_1)_1 \otimes (m_1)_2) \cdot \Phi = \Phi \cdot ((m_0)_0 \otimes (m_0)_1 \otimes m_1). \tag{3.28}$$

(cfr. [HN, Definition 3.1], [BC, Definition 3.1])

3.2.1 An adjunction between ${}_A\mathcal{M}_A^A$ and ${}_A\mathcal{M}$.

Lemma 3.2.7. *Let (C, Δ, ε) be a coalgebra in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, a, l, r)$. The assignment $T\colon \mathcal{M} \to \mathcal{M}^C$ that sends M to $(M \otimes C, (a_{M,C,C})^{-1} \circ (M \otimes \Delta))$ and $f\colon M \to N$ to $f \otimes C\colon M \otimes C \to N \otimes C$ defines a functor. Moreover, T is right adjoint to $U\colon \mathcal{M}^C \to \mathcal{M}$, the underlying functor. The unit and counit of the adjunction are given, for every $(M, \rho_M) \in \mathcal{M}^C$ and $N \in \mathcal{M}$, by:*

$$\eta_M := \rho_M\colon M^\bullet \to M \otimes C^\bullet \quad \text{and} \quad \epsilon_N := r_N \circ (N \otimes \varepsilon)\colon N \otimes C \to N \tag{3.29}$$

(the upper full dots denote the given C-coaction).

Proof. First of all, we should prove that $(M \otimes C, (a_{M,C,C})^{-1} \circ (M \otimes \Delta))$ is a C-comodule.

- Consider the following diagram:

$$\begin{tikzcd}
M \otimes C \arrow[r, "M\otimes\Delta"] \arrow[dd, "M\otimes\Delta"'] & M \otimes (C \otimes C) \arrow[r, "a^{-1}_{M,C,C}"] \arrow[d, dashed, "M\otimes(\Delta\otimes C)"'] & (M \otimes C) \otimes C \arrow[d, "(M\otimes\Delta)\otimes C"] \\
 & M \otimes ((C \otimes C) \otimes C) \arrow[r, dashed, "a^{-1}_{M,C\otimes C,C}"] \arrow[d, dashed, "M\otimes a_{C,C,C}"] & (M \otimes (C \otimes C)) \otimes C \arrow[dd, "a^{-1}_{M,C,C}\otimes C"] \\
M \otimes (C \otimes C) \arrow[r, dashed, "M\otimes(C\otimes\Delta)"] \arrow[d, "a^{-1}_{M,C,C}"'] & M \otimes (C \otimes (C \otimes C)) \arrow[d, dashed, "a^{-1}_{M,C,C\otimes C}"] & \\
(M \otimes C) \otimes C \arrow[r, "(M\otimes C)\otimes\Delta"'] & (M \otimes C) \otimes (C \otimes C) \arrow[r, "a^{-1}_{M\otimes C,C,C}"'] & ((M \otimes C) \otimes C) \otimes C
\end{tikzcd}$$

  The upper left square commutes since C is a coalgebra and $M \otimes -$ is a functor. The upper right square and the lower left square commute since a is natural. The lower right square commutes in view of the Pentagon Axiom (1.9). The external square is the first compatibility condition that a coaction should satisfy (cfr. (1.31)).

- Consider the following diagram:

$$\begin{tikzcd}
M \otimes C \arrow[rr, equal] \arrow[d, "M\otimes\Delta"'] & & M \otimes C \\
M \otimes (C \otimes C) \arrow[r, dashed, "M\otimes(C\otimes\varepsilon)"] \arrow[d, "a^{-1}_{M,C,C}"'] & M \otimes (C \otimes \mathbb{I}) \arrow[ur, dashed, "M\otimes r_C"] \arrow[dr, dashed, "a^{-1}_{M,C,\mathbb{I}}"'] & \\
(M \otimes C) \otimes C \arrow[rr, "(M\otimes C)\otimes\varepsilon"'] & & (M \otimes C) \otimes \mathbb{I} \arrow[uu, "r_{M\otimes C}"']
\end{tikzcd}$$

The upper left square commutes because C is a coalgebra. The lower left square commutes since a is natural. The right triangle commutes in view of (1.12). The external path is the second compatibility condition that a coaction should satisfy (cfr. (1.31)).

Thus $(M \otimes C, (a_{M,C,C})^{-1} \circ (M \otimes \Delta))$ is a C-comodule. If $f\colon M \to N$ is a linear map, then $f \otimes C\colon M \otimes C \to N \otimes C$ is trivially a right C-comodule map. Therefore $T\colon \mathcal{M} \to \mathcal{M}^C$ is a functor. Let us prove that (U, T) is an adjoint pair.

- Define the unit and counit of the adjunction:

$$\eta_M\colon M^\bullet \to TU(M) = M \otimes C^\bullet$$

$$\epsilon_M\colon UT(M) = M \otimes C \to M$$

  to be:

$$\eta_M = \rho_M \quad \left(\forall\, M \in \mathcal{M}^C\right) \qquad \text{and} \qquad \epsilon_M = r_M \circ (M \otimes \varepsilon) \quad (\forall\, M \in \mathcal{M})\,.$$

- Let us verify that η_M is a right C-comodule morphism:

$$\rho_{M\otimes C} \circ \eta_M = a^{-1}_{M,C,C} \circ (M \otimes \Delta) \circ \rho_M \overset{(1.31)}{=} (\rho_M \otimes C) \circ \rho_M = (\eta_M \otimes C) \circ \rho_M$$

  since (M, ρ_M) is a C-comodule.

- We have that $\epsilon_M = r_M \circ (M \otimes \varepsilon)$ and so it is a morphism in $\mathcal{M}$.

- Naturality of η. For each $f\colon M \to N$, right C-comodule morphism, we have that $(f \otimes C) \circ \rho_M = \rho_N \circ f$. Since $\eta_M = \rho_M$ we deduce that η is natural.

- Naturality of ϵ. For every $f\colon M \to N$, linear map,

$$\begin{tikzcd}
M \otimes C \arrow[r, "M\otimes\varepsilon"] \arrow[d, "f\otimes C"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes \Bbbk \arrow[r, "r_M"] \arrow[d, dotted, "f\otimes\Bbbk"] & M \arrow[d, "f"] \\
N \otimes C \arrow[r, "N\otimes\varepsilon"'] & N \otimes \Bbbk \arrow[r, "r_N"'] & N
\end{tikzcd}$$

  commutes by naturality of r_M.

- Triangular identities:

$$\begin{tikzcd}
UTU(M) \arrow[r, "\epsilon_{U(M)}"] & U(M) \\
U(M) \arrow[u, "U(\eta_M)"] \arrow[ur, "\mathrm{Id}_{U(M)}"'] &
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
TUT(M) \arrow[r, "T(\epsilon_M)"] & T(M) \\
T(M) \arrow[u, "\eta_{T(M)}"] \arrow[ur, "\mathrm{Id}_{T(M)}"'] &
\end{tikzcd}$$

  If $M^\bullet \in \mathcal{M}^C$, then $UTU(M) = M \otimes C$. Thus

$$\epsilon_{U(M)} \circ U(\eta_M) = r_M \circ (M \otimes \varepsilon) \circ \rho_{M^\bullet} = \mathrm{Id}_M,$$

since (M, ρ_M) is a right C-comodule.
If $M \in \mathcal{M}$, then $TUT(M) = (M \otimes C) \otimes C^{\bullet}$. Consider the following diagram:

$$
\begin{tikzcd}
M \otimes C \arrow[rr, equal] \arrow[d, "M\otimes\Delta"'] & & M \otimes C \\
M \otimes (C \otimes C) \arrow[r, dashed, "M\otimes(\varepsilon\otimes C)"] \arrow[d, "a^{-1}_{M,C,C}"'] & M \otimes (\mathbb{I} \otimes C) \arrow[ur, dashed, "M\otimes l_C"] \arrow[dr, dashed, "a^{-1}_{M,\mathbb{I},C}"'] & \\
(M \otimes C) \otimes C \arrow[rr, "(M\otimes\varepsilon)\otimes C"'] & & (M \otimes \mathbb{I}) \otimes C \arrow[uu, "r_M\otimes C"']
\end{tikzcd}
$$

The upper left square commutes because C is a coalgebra. The lower left square commutes since a is natural. The right triangle commutes in view of the Triangle Axiom (1.10). Thus:

$$
\begin{aligned}
T(\epsilon_M) \circ \eta_{T(M)} &= (\epsilon_M \otimes C) \circ \rho_{M\otimes C} = \\
&= ((r_M \circ (M \otimes \varepsilon)) \otimes C) \circ a^{-1}_{M,C,C} \circ (M \otimes \Delta) = \\
&= \mathrm{Id}_M.
\end{aligned}
$$

The thesis now follows. $\square$

In view of Proposition 3.2.3 and Lemma 3.2.7, we have and adjunction:

$$(U, T, \eta, \epsilon) \colon {}_A\mathcal{M}_A^A \rightharpoonup {}_A\mathcal{M}_A.$$

Explicitly:

$$
\begin{array}{cccc}
T: & {}_A\mathcal{M}_A & \longrightarrow & {}_A\mathcal{M}_A^A \\
 & {}_\bullet M_\bullet & \longmapsto & {}_\bullet M_\bullet \otimes {}_\bullet A_\bullet^\bullet
\end{array}
$$

where

$$
\begin{array}{cccc}
\rho_{M\otimes A}: & M \otimes A & \longrightarrow & (M \otimes A) \otimes A \\
 & m \otimes a & \longmapsto & \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2) \cdot \Phi
\end{array}. \tag{3.30}
$$

The unit and counit of the adjunction are given by:

$$
\begin{array}{cccc}
\eta_M: & {}_\bullet M_\bullet^\bullet & \longrightarrow & {}_\bullet M_\bullet \otimes {}_\bullet A_\bullet^\bullet \\
 & m & \longmapsto & m_0 \otimes m_1
\end{array} \qquad \forall\, M \in {}_A\mathcal{M}_A^A,
$$

$$
\begin{array}{cccc}
\epsilon_M: & {}_\bullet M_\bullet \otimes {}_\bullet A_\bullet & \longrightarrow & {}_\bullet M_\bullet \\
 & m \otimes a & \longmapsto & m\,\varepsilon(a)
\end{array} \qquad \forall\, M \in {}_A\mathcal{M}_A.
$$

Now, recall that the structure theorem for ordinary Hopf modules involves the concept of coinvariants as the equalizer in $\mathcal{M}$ of the maps:

$$
\begin{tikzcd}
0 \arrow[r] & M^{\mathrm{Co}H} \arrow[r] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "i_1"'] & M \otimes H
\end{tikzcd}
$$

where

$$\begin{array}{rccc} i_1: & M & \longrightarrow & M \otimes H \\ & m & \longmapsto & m \otimes 1 \end{array}$$

The dual concept leads us to define what will be shown to be an appropriate generalization of the coinvariants in the case of quasi-bialgebras.
Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra and (M, μ^l_M, μ^r_M) be a (A, A)-bimodule. Consider the coequalizer in the category of $\Bbbk$-vector spaces:

$$\begin{tikzcd} M \otimes A \arrow[r, shift left, "\mu^r_M"] \arrow[r, shift right, "\xi_M"'] & M \arrow[r, "\pi"] & \frac{M}{\mathrm{Im}(\mu^r_M - \xi_M)} \arrow[r] & 0 \end{tikzcd}$$

where ξ_M is the trivial right A-action: $\xi(m \otimes a) = m\,\varepsilon(a)$.

Remark 3.2.8. Pay attention: μ^r_M and ξ_M are maps in $\mathcal{M}$. This means that the quotient (as a coequalizer) comes with the structure of $\Bbbk$-vector space. However, if we take $n \in \mathrm{Im}(\mu - \xi)$, there exists an element $\sum_i m_i \otimes a_i \in M \otimes A$ such that $n = (\mu - \xi)(\sum_i m_i \otimes a_i) = \sum_i m_i \cdot (a_i - \varepsilon(a_i)1)$. Note that:

$$b \cdot n = b \cdot \sum_i m_i \cdot (a_i - \varepsilon(a_i)1) = \sum_i b \cdot m_i \cdot (a_i - \varepsilon(a_i)1) = (\mu - \xi)(\sum_i b \cdot m_i \otimes a_i) \in \mathrm{Im}(\mu - \xi).$$

Thus we have that $\mathrm{Im}(\mu - \xi)$ is a left A-submodule of M and $\frac{M}{\mathrm{Im}(\mu-\xi)}$ is a left A-module.

Consider the short exact sequence (SES):

$$\begin{tikzcd} 0 \arrow[r] & A^+ \arrow[r, hook] & A \arrow[r, "\varepsilon"] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

where we set $A^+ := \ker(\varepsilon)$. Note that A^+ is a two-sides ideal of A, the so called *augmentation ideal* of A (if A is an associative algebra over a ring R, then it is called *augmented* or *supplemented* if it is equipped with a ring homomorphism $\varepsilon\colon A \to R$. The kernel $\ker(\varepsilon)$ is called the *augmentation ideal* of A).

Lemma 3.2.9. $\mathrm{Im}(\mu - \xi) = MA^+$.

Proof. Observe that $a - \varepsilon(a)1 \in A^+$, so that $\mathrm{Im}(\mu - \xi) \subseteq MA^+$. On the other hand, if $a \in A^+$, then $a = a - \varepsilon(a)1$. That implies that, for all $\sum_i m_i \cdot a_i \in MA^+$, $\sum_i m_i \cdot a_i = \sum_i m_i \cdot (a_i - \varepsilon(a_i)1) \in \mathrm{Im}(\mu - \xi)$. □

Remark 3.2.10. Note that $\mathrm{Im}(\mu - \xi)$ is not a right A-submodule of M, in general. However,

$$\overline{m \cdot a} = \overline{m\,\varepsilon(a)} \tag{3.31}$$

holds in $\frac{M}{MA^+}$.

Concluding, what we got is that

$$\frac{M}{\mathrm{Im}(\mu - \xi)} = \frac{M}{MA^+},$$

that is exactly what we need. Indeed, consider the assignment $R\colon {}_A\mathcal{M} \to {}_A\mathcal{M}_A$ that sends ${}_\bullet M$ to ${}_\bullet M_\circ$ and $f\colon {}_\bullet M \to {}_\bullet N$ to $f\colon {}_\bullet M_\circ \to {}_\bullet N_\circ$, where the empty dot denotes the

trivial right A-action.
R is a functor and it is right adjoint to the functor $L\colon {}_A\mathcal{M}_A \to {}_A\mathcal{M}$ that sends ${}_\bullet M_\bullet$ to ${}_\bullet \frac{M}{MA^+}$.
To see how L operates on morphism, let $f\colon M \to N$ be a morphism of (A,A)-bimodules. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & MA^+ \arrow[r, "i"] \arrow[ddr, "\pi_N \circ f \circ i"'] & M \arrow[r, "\pi_M"] \arrow[d, "f"] & \frac{M}{MA^+} \arrow[r] \arrow[ddl, dotted, "\overline{f}"] & 0 \\
 & & N \arrow[d, "\pi_N"] & & \\
 & & \frac{N}{NA^+} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

If $\sum_i m_i a_i \in MA^+$, then $f(\sum_i m_i a_i) = \sum_i f(m_i)a_i \in NA^+$ since f is in particular a morphism of right A-modules. Thus f passes to the quotient and we can set $L(f) = \overline{f}$, morphism of left A-modules. Note that on elements we have: $\overline{f}(\overline{m}) = \overline{f(m)}$.

Proposition 3.2.11. *(L,R) forms an adjunction between ${}_A\mathcal{M}_A$ and ${}_A\mathcal{M}$. The unit and counit of the adjunction are given by:*

$$\tilde{\eta}_M = \pi_M : {}_\bullet M_\bullet \longrightarrow {}_\bullet \frac{M}{MA^+}{}_\circ : m \longmapsto \overline{m} \tag{3.32}$$

$$\tilde{\epsilon}_M = Id_M : {}_\bullet M \longrightarrow {}_\bullet M : m \longmapsto m \tag{3.33}$$

Proof. Note that $LR({}_\bullet M) = L({}_\bullet M_\circ) = {}_\bullet \frac{M}{MA^+}$, but if $m \in M$ and $a \in A^+$, $m \cdot a = m\,\varepsilon(a) = 0$. Thus $MA^+ = 0$ and we can identify $LR({}_\bullet M)$ with ${}_\bullet M$. Moreover, $RL({}_\bullet M_\bullet) = R\left({}_\bullet \frac{M}{MA^+}\right) = {}_\bullet \frac{M}{MA^+}{}_\circ$. These observations suggest the definitions of the unit and the counit:

$$\tilde{\eta}_M = \pi_M : \quad \begin{array}{ccc} {}_\bullet M_\bullet & \longrightarrow & {}_\bullet \frac{M}{MA^+}{}_\circ \\ m & \longmapsto & \overline{m} \end{array} ,$$

$$\tilde{\epsilon}_M = \mathrm{Id}_M : \quad \begin{array}{ccc} {}_\bullet M & \longrightarrow & {}_\bullet M \\ m & \longmapsto & m \end{array} .$$

Clearly $\tilde{\epsilon}$ is a natural isomorphism of left A-modules. Let us prove that also $\tilde{\eta}$ is a natural morphism:

- $\tilde{\eta}_M$ is a morphism of (A,A)-bimodules, for all $M \in {}_A\mathcal{M}_A$. Indeed, for every $m \in M$ and $a \in A$:
$$a \cdot \tilde{\eta}_M(m) = a \cdot \overline{m} = \overline{a \cdot m} = \tilde{\eta}_M(a \cdot m),$$
$$\tilde{\eta}_M(m) \cdot a = \tilde{\eta}_M(m)\,\varepsilon(a) = \overline{m\,\varepsilon(a)} \overset{(3.31)}{=} \overline{m \cdot a} = \tilde{\eta}_M(m \cdot a).$$
- $\tilde{\eta}$ is natural. Let $f\colon M \to N$ be a morphism of (A,A)-bimodules. $RL(f) = R(\overline{f}) = \overline{f}$, thus:
$$\overline{f}(\tilde{\eta}_M(m)) = \overline{f}(\overline{m}) = \overline{f(m)} = \tilde{\eta}_N(f(m))$$
for all $m \in M$.

Hence the only things left are the triangular identities. Now, $\tilde{\eta}_{R(M)}$ is again the identity, since $({}_\bullet M_\circ)A^+ = 0$ and so the composition

$$R(M) \xrightarrow{\tilde{\eta}_{R(M)}} RLR(M) \xrightarrow{R(\tilde{\epsilon}_M)} R(M)$$

is trivially the identity. Moreover, since $\tilde{\eta}_M$ is just the projection, $L(\tilde{\eta}_M) = \overline{\tilde{\eta}_M}$ is the identity of the quotient. Therefore, even

$$L(M) \xrightarrow{L(\tilde{\eta}_M)} LRL(M) \xrightarrow{\tilde{\epsilon}_{L(M)}} L(M)$$

is the identity map. $\square$

In view of Theorem 1.1.15 we can compose the adjunctions that we constructed and obtain an adjunction between ${}_A\mathcal{M}_A^A$ and ${}_A\mathcal{M}$, as desired. Define:

$$F := LU \colon {}_A\mathcal{M}_A^A \to {}_A\mathcal{M} \qquad \text{and} \qquad G := TR \colon {}_A\mathcal{M} \to {}_A\mathcal{M}_A^A$$

so that $F({}_\bullet M_\bullet^\bullet) = {}_\bullet \frac{M}{MA^+}$ and $G({}_\bullet M) = {}_\bullet M_\circ \otimes {}_\bullet A_\bullet^\bullet$.
Explicitly, the structures on $G({}_\bullet M)$ are given by:

$$x \cdot (m \otimes a) = x_1 \cdot m \otimes x_2 a, \tag{3.34}$$
$$(m \otimes a) \cdot x = m \otimes ax, \tag{3.35}$$
$$\rho(m \otimes a) = \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2) \tag{3.36}$$

for every $x, a \in A$ and $m \in M$, as:

$$\begin{aligned}\rho(m \otimes a) &\overset{(3.30)}{=} \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2) \cdot \Phi = \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2) \cdot (\varepsilon \otimes A \otimes A)(\Phi) = \\ &= \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2)\end{aligned}$$

The unit and counit are given by the compositions:

$$\hat{\eta}_M := T(\tilde{\eta}_{U(M)}) \circ \eta_M = (\tilde{\eta}_{U(M)} \otimes A) \circ \eta_M,$$

$$\hat{\epsilon}_M := \tilde{\epsilon}_M \circ L(\epsilon_{R(M)}) = \overline{\epsilon_{R(M)}},$$

so that:

$$\hat{\eta}_M : \begin{array}{ccc} M & \longrightarrow & \frac{M}{MA^+} \otimes A \\ m & \longmapsto & \overline{m_0} \otimes m_1 \end{array} \qquad \forall\, M \in {}_A\mathcal{M}_A^A$$

and

$$\hat{\epsilon}_M : \begin{array}{ccc} \frac{M \otimes A}{(M \otimes A)A^+} & \longrightarrow & M \\ \overline{m \otimes a} & \longmapsto & m\,\varepsilon(a) \end{array} \qquad \forall\, M \in {}_A\mathcal{M}.$$

Remark 3.2.12. Note that, since the right A-module structure on $G({}_\bullet M) = {}_\bullet M_\circ \otimes {}_\bullet A_\bullet^\bullet$ is given by the multiplication on the second factor, we have that $(M \otimes A)A^+ = M \otimes A^+$. Moreover, if we tensor the SES

$$0 \longrightarrow A^+ \hookrightarrow A \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

by M on the left we get the SES:

$$0 \longrightarrow M \otimes A^+ \hookrightarrow M \otimes A \xrightarrow{M\otimes\varepsilon} M \otimes \Bbbk \cong M \longrightarrow 0$$

from which we deduce that $\frac{M\otimes A}{M\otimes A^+} \cong M$ via the isomorphism

$$\overline{M \otimes \varepsilon}: \quad \begin{array}{ccc} \dfrac{M \otimes A}{M \otimes A^+} & \longrightarrow & M \\ \overline{m \otimes a} & \longmapsto & m\,\varepsilon(a) \end{array}$$

Now, it is evident that $\overline{M \otimes \varepsilon} = \hat{\epsilon}_M$ for every $M \in {}_A\mathcal{M}$, i.e., $\hat{\epsilon}$ is a natural isomorphism. Another way to see this is to observe that for all $m \otimes a$ in $M \otimes A$, $m \otimes (a - \varepsilon(a)1)$ is in $M \otimes A^+$, so that $\overline{m \otimes a} = \overline{m\,\varepsilon(a) \otimes 1}$.

Summing up, we have just proved the following result (cfr. [Sc1, proof of Theorem 3.1]) where we use a different notation for the (co)unit for the sake of simplicity.

Theorem 3.2.13. *The functor $F\colon {}_A\mathcal{M}_A^A \to {}_A\mathcal{M}$, $F(M) = \frac{M}{MA^+} =: \overline{M}$, is left adjoint to the functor $G\colon {}_A\mathcal{M} \to {}_A\mathcal{M}_A^A$, $G(M) = M \otimes A$. Moreover, the unit and the counit of this adjunction are given by:*

$$\eta_N : N \longrightarrow \overline{N} \otimes A : n \longmapsto \overline{n_0} \otimes n_1 \tag{3.37}$$
$$\epsilon_M : \overline{M \otimes A} \longrightarrow M : \overline{m \otimes a} \longmapsto m\,\varepsilon(a) \tag{3.38}$$

for all $M \in {}_A\mathcal{M}$ and for all $N \in {}_A\mathcal{M}_A^A$, and ϵ is a natural isomorphism.

Note that we dropped the ˆ decoration on the unit and counit in order to lighten the notation, since from now on we will be interested in this last adjunction only.

Theorem 3.2.14. (Dual to [AP1, Proposition 3.3]) *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. The following assertions are equivalent:*

(1) The adjunction (F, G, η, ϵ) is an equivalence of categories.

(2) For each $M \in {}_A\mathcal{M}_A^A$, there exists a $\Bbbk$-linear map $\tilde{\tau}\colon \overline{M} \to M$ such that:

$$(i)\ \ \tilde{\tau}(\overline{m})_0 \otimes \tilde{\tau}(\overline{m})_1 = \tilde{\tau}(\phi^1 \cdot \overline{m}) \cdot \phi^2 \otimes \phi^3, \text{ where } \Phi^{-1} = \phi^1 \otimes \phi^2 \otimes \phi^3; \tag{3.39}$$
$$(ii)\ \ a \cdot \tilde{\tau}(\overline{m}) = \tilde{\tau}(\overline{a_1 \cdot m}) \cdot a_2; \tag{3.40}$$
$$(iii)\ \ \tilde{\tau}(\overline{m_0}) \cdot m_1 = m; \tag{3.41}$$

for all $m \in M$, $a \in A$.

(3) For each $M \in {}_A\mathcal{M}_A^A$, there exists a $\Bbbk$-linear map $\tilde{\tau}\colon \overline{M} \to M$ such that (iii) holds and:

$$(iv)\ \ \overline{\tilde{\tau}(\overline{m})_0} \otimes \tilde{\tau}(\overline{m})_1 = \overline{m} \otimes 1; \tag{3.42}$$

for all $m \in M$.

Proof. (1) $\Rightarrow$ (2). By hypothesis (F, G, η, ϵ) is an equivalence, so that η is a natural isomorphism. For each $M \in {}_A\mathcal{M}_A^A$ define:

$$\tilde{\tau}_M = \left(\frac{M}{MA^+} \xrightarrow{i_1} \frac{M}{MA^+} \otimes A \xrightarrow{\eta_M^{-1}} M \right)$$

where:

$$i_1 \colon \frac{M}{MA^+} \longrightarrow \frac{M}{MA^+} \otimes A \colon \overline{m} \longmapsto \overline{m} \otimes 1,$$

i.e., $\tilde{\tau}_M(\overline{m}) = \eta_M^{-1}(\overline{m} \otimes 1)$.

- Since η_M is right A-linear, also η_M^{-1} is:

$$\begin{tikzcd}
\left(\frac{M}{MA^+} \otimes A\right) \otimes A \arrow[r, "\eta_M^{-1} \otimes A"] \arrow[d, "\mu^r_{\overline{M} \otimes A}"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes A \arrow[d, "\mu^r_M"] \\
\frac{M}{MA^+} \otimes A \arrow[r, "\eta_M^{-1}"'] & M
\end{tikzcd}$$

  so that $\eta_M^{-1}(\overline{m} \otimes a) \cdot b = \eta_M^{-1}(\overline{m} \otimes ab)$. Evaluating at $a = 1$ we get that:

$$\eta_M^{-1}(\overline{m} \otimes a) = \tilde{\tau}(\overline{m}) \cdot a. \tag{3.43}$$

- Since η_M is right A-colinear, also η_M^{-1} is:

$$\begin{tikzcd}
\frac{M}{MA^+} \otimes A \arrow[r, "\eta_M^{-1}"] \arrow[d, "\rho_{\overline{M} \otimes A}"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "\rho_M"] \\
\left(\frac{M}{MA^+} \otimes A\right) \otimes A \arrow[r, "\eta_M^{-1} \otimes A"'] & M \otimes A
\end{tikzcd}$$

  so that

$$\rho(\tilde{\tau}(\overline{m}) \cdot a) \overset{(3.43)}{=} \rho\left(\eta_M^{-1}(\overline{m} \otimes a)\right) = \eta_M^{-1}\left(\phi^1 \cdot \overline{m} \otimes \phi^2 \cdot a_1\right) \otimes \phi^3 \cdot a_2 =$$
$$\overset{(3.43)}{=} \tilde{\tau}(\phi^1 \cdot \overline{m}) \cdot \phi^2 \cdot a_1 \otimes \phi^3 \cdot a_2.$$

  Evaluating at $a = 1$ we get (3.39):

$$\tilde{\tau}(\overline{m})_0 \otimes \tilde{\tau}(\overline{m})_1 = \tilde{\tau}\left(\phi^1 \cdot \overline{m}\right) \cdot \phi^2 \otimes \phi^3.$$

- Again, η_M is left A-linear, and so is η_M^{-1}:

$$b \cdot \eta_M^{-1}\left(\overline{m} \otimes 1\right) = \eta_M^{-1}\left(b_1 \cdot \overline{m} \otimes b_2\right),$$

  i.e., by (3.43), (3.40) holds:

$$b \cdot \tilde{\tau}(\overline{m}) = \tilde{\tau}\left(b_1 \cdot \overline{m}\right) \cdot b_2.$$

- η_M^{-1} is the inverse of η_M:

$$m = \eta_M^{-1}\left(\eta_M(m)\right) = \eta_M^{-1}\left(\overline{m_0} \otimes m_1\right) \overset{(3.43)}{=} \tilde{\tau}\left(\overline{m_0}\right) \cdot m_1,$$

  i.e., (3.41).

(2) $\Rightarrow$ (3). Applying the canonical projection on the quotient, π, to (3.41) and recalling (3.31), we get that:

$$\overline{m} = \overline{\tilde{\tau}(\overline{m_0}) \cdot m_1} = \overline{\tilde{\tau}(\overline{m_0})\, \varepsilon(m_1)} = \overline{\tilde{\tau}(\overline{m})},$$

i.e.:

$$\overline{\tilde{\tau}(\overline{m})} = \overline{m}. \tag{3.44}$$

In the same way, applying $\pi \otimes A$ to (3.39) we find:

$$\begin{aligned}\overline{\tilde{\tau}(\overline{m})_0} \otimes \tilde{\tau}(\overline{m})_1 &= \overline{\tilde{\tau}\left(\phi^1 \cdot \overline{m}\right) \cdot \phi^2} \otimes \phi^3 = \overline{\tilde{\tau}\left(\phi^1 \cdot \overline{m}\right)\, \varepsilon\left(\phi^2\right)} \otimes \phi^3 \overset{(3.24)}{=} \\ &= \overline{\tilde{\tau}(\overline{m})} \otimes 1 \overset{(3.44)}{=} \overline{m} \otimes 1\end{aligned}$$

so that (3.42) holds.
(3) $\Rightarrow$ (1). It's enough to give an inverse for η_M, for all $M \in {}_A\mathcal{M}_A^A$. As ((1) $\Rightarrow$ (2)) suggests, we can consider:

$$\psi_M = \left(\frac{M}{MA^+} \otimes A \xrightarrow{\tilde{\tau}\otimes A} M \otimes A \xrightarrow{\mu_M} M\right)$$

so that $\psi_M(\overline{m} \otimes a) = \tilde{\tau}(\overline{m}) \cdot a$. Thus:

$$m \overset{\eta_M}{\longmapsto} \overline{m_0} \otimes m_1 \overset{\psi_M}{\longmapsto} \tilde{\tau}(\overline{m_0}) \cdot m_1 \overset{(3.41)}{=} m$$

and:

$$\overline{m} \otimes a \overset{\psi_M}{\longmapsto} \tilde{\tau}(\overline{m}) \cdot a \overset{\eta_M}{\longmapsto} \overline{\tilde{\tau}(\overline{m})_0 \cdot a_1} \otimes \tilde{\tau}(\overline{m})_1 \cdot a_2$$

but:

$$\begin{aligned}\overline{\tilde{\tau}(\overline{m})_0 \cdot a_1} \otimes \tilde{\tau}(\overline{m})_1 \cdot a_2 &\overset{(3.31)}{=} \overline{\tilde{\tau}(\overline{m})_0} \otimes \tilde{\tau}(\overline{m})_1 \cdot a = \\ &\overset{(3.35)}{=} \left(\overline{\tilde{\tau}(\overline{m})_0} \otimes \tilde{\tau}(\overline{m})_1\right) \cdot a \overset{(3.42)}{=} (\overline{m} \otimes 1) \cdot a \overset{(3.35)}{=} \overline{m} \otimes a\end{aligned}$$

□

Remark 3.2.15. It follows from the previous theorem that, if $\tilde{\tau} \colon \overline{M} \to M$ is a $\Bbbk$-linear map that satisfies (3.41), then it satisfies (3.39) and (3.40) if and only if it satisfies (3.42).

Definition 3.2.16. (Preantipode)
A *preantipode* for a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a $\Bbbk$-linear map $S \colon A \to A$ such that:

(P1) For every $a, b \in A$,

$$b_1 S(ab_2) = S(a)\varepsilon(b). \tag{3.45}$$

(P2) For every $a, b \in A$,

$$S(a_1 b)a_2 = \varepsilon(a)S(b). \tag{3.46}$$

(P3) If $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$, then:

$$\Phi^1 S(\Phi^2)\Phi^3 = 1 \tag{3.47}$$

Remark 3.2.17. Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode S. Then the following equalities hold, taking $a = 1$ in (P1) and $b = 1$ in (P2):

$$a_1 S(a_2) = S(1)\varepsilon(a) = S(a_1)a_2.$$

Let us take a little bit of time to investigate some elementary properties of quasi-bialgebras with preantipode.

Lemma 3.2.18. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode S. The following identities hold:*

$$S(\phi^1)\phi^2 S(\phi^3) = S(1) \tag{3.48}$$

$$\varepsilon(S(1)) = 1 \tag{3.49}$$

$$\varepsilon \circ S = \varepsilon \tag{3.50}$$

Proof. In what follows, the use of two copies of Φ^{-1} will be required, so that we are going to indicate with $\Psi^{-1} = \psi^1 \otimes \psi^2 \otimes \psi^3$ another copy of Φ^{-1}, in order to avoid confusion.

In view of (3.19):

$$(\Delta \otimes A \otimes A)(\Phi^{-1})(A \otimes A \otimes \Delta)(\Psi^{-1})(1 \otimes \Phi) = (\Phi^{-1} \otimes 1)(A \otimes \Delta \otimes A)(\Psi^{-1}),$$

i.e.:

$$(\phi^1)_1\psi^1 \otimes (\phi^1)_2\psi^2\Phi^1 \otimes \phi^2(\psi^3)_1\Phi^2 \otimes \phi^3(\psi^3)_2\Phi^3 = \phi^1\psi^1 \otimes \phi^2(\psi^2)_1 \otimes \phi^3(\psi^2)_2 \otimes \psi^3.$$

Applying $S \otimes A \otimes S \otimes A$ to both sides and then multiplying:

$$\underbrace{S((\phi^1)_1\psi^1)(\phi^1)_2}_{\overset{(3.46)}{=}\varepsilon(\phi^1)S(\psi^1)}\psi^2\Phi^1 S(\phi^2(\psi^3)_1\Phi^2)\phi^3(\psi^3)_2\Phi^3 = S(\phi^1\psi^1)\phi^2\underbrace{(\psi^2)_1 S(\phi^3(\psi^2)_2)}_{\overset{(3.45)}{=}\varepsilon(\psi^2)S(\phi^3)}\psi^3,$$

simplifying using (3.24):

$$S(\psi^1)\psi^2\Phi^1\underbrace{S((\psi^3)_1\Phi^2)(\psi^3)_2}_{\overset{(3.46)}{=}\varepsilon(\psi^3)S(\Phi^2)}\Phi^3 = S(\phi^1)\phi^2 S(\phi^3)$$

and, in view of (3.47) and (3.24) again:

$$S(1) = S(\phi^1)\phi^2 S(\phi^3).$$

To prove (3.49) it's enough to apply ε to both sides of (3.47):

$$1 = \varepsilon(1) \overset{(3.47)}{=} \varepsilon(\Phi^1 S(\Phi^2)\Phi^3) = \varepsilon(\varepsilon(\Phi^1)S(\Phi^2)\varepsilon(\Phi^3)) \overset{(3.24)}{=} \varepsilon(S(1)).$$

Now, (3.50) follows directly from (3.49) by applying ε to both sides of (3.46) and evaluating at $b = 1$:

$$\varepsilon(a) \overset{(3.49)}{=} \varepsilon(a)\varepsilon(S(1)) = \varepsilon(\varepsilon(a)S(1)) \overset{(3.46)}{=} \varepsilon(a_1 S(a_2)) = \varepsilon(a_1)\varepsilon(S(a_2)) = \varepsilon(S(a)),$$

for all $a \in A$. $\square$

Note that, unless $S(1)$ is invertible, (3.48) and (3.47) are not equivalent. Indeed, in the last proof we showed that:

$$S(1)\Phi^1 S(\Phi^2)\Phi^3 = S(\phi^1)\phi^2 S(\phi^3).$$

The following Proposition is the natural generalization of [HN, Proposition 3.4], to quasi-bialgebras with preantipode.

Proposition 3.2.19. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode S and $M \in {}_A\mathcal{M}_A^A$. Define*

$$\tau : M \longrightarrow M : m \longmapsto \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3 \tag{3.51}$$

and, for all $a \in A$, $m \in M$,

$$a \blacktriangleright m := \tau(a \cdot m). \tag{3.52}$$

Then they satisfy:

(1) $\tau(m \cdot a) = \tau(m)\,\varepsilon(a)$,

(2) $\tau^2 = \tau$,

(3) $a \blacktriangleright \tau(m) = \tau(a \cdot m)$,

(4) $a \blacktriangleright (b \blacktriangleright m) = (ab) \blacktriangleright m$,

(5) $a \cdot \tau(m) = \tau(a_1 \cdot m) \cdot a_2 = (a_1 \blacktriangleright \tau(m)) \cdot a_2$,

(6) $\tau(m_0) \cdot m_1 = m$,

(7) $\tau(\tau(m)_0) \otimes \tau(m)_1 = \tau(m) \otimes 1$,

for all $a, b \in A$, $m \in M$.

Proof. Property (1) is quite easy to prove, indeed:

$$\tau(m \cdot a) = \Phi^1 \cdot m_0 \cdot a_1 S(\Phi^2 m_1 a_2)\Phi^3 \overset{(3.45)}{=} \Phi^1 \cdot m_0 \cdot \varepsilon(a)\, S(\Phi^2 m_1)\Phi^3 = \tau(m)\,\varepsilon(a).$$

To prove (3) one uses (1) to compute:

$$\begin{aligned}
\tau(a \cdot \tau(m)) &= \tau(a\Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3) = \\
&= \tau(a\Phi^1 \cdot m_0)\,\varepsilon(S(\Phi^2 m_1))\varepsilon(\Phi^3) \overset{(3.50)}{=} \\
&= \tau(a\Phi^1 \cdot m_0)\,\varepsilon(\Phi^2 m_1)\varepsilon(\Phi^3) = \\
&= \tau(a \cdot m_0)\varepsilon(m_1) = \tau(a \cdot m),
\end{aligned}$$

for all $a \in A$ and $m \in M$. Now, (2) is just (3) with $a = 1$ and (4) follows directly from (3) since, for every $a, b \in A$ and $m \in M$:

$$a \blacktriangleright (b \blacktriangleright m) = a \blacktriangleright \tau(b \cdot m) = \tau(ab \cdot m) = (ab) \blacktriangleright m,$$

Statement (5) is a consequence of the quasi-coassociativity of Δ:

$$\begin{aligned}\tau(a_1\cdot m)\cdot a_2 &= \Phi^1(a_1)_1\cdot m_0\cdot S(\Phi^2(a_1)_2 m_1)\Phi^3 a_2 \overset{(3.21)}{=}\\ &= a_1\Phi^1\cdot m_0\cdot S((a_2)_1\Phi^2 m_1)(a_2)_2\Phi^3 \overset{(3.46)}{=}\\ &= a_1\Phi^1\cdot m_0\cdot S(\Phi^2 m_1)\Phi^3\varepsilon(a_2) = a\cdot\tau(m).\end{aligned}$$

for all $a\in A$ and $m\in M$. Furthermore, (6) follows from the fact that, for all $m\in M$:

$$\begin{aligned}\tau(m_0)\cdot m_1 &= \Phi^1\cdot(m_0)_0\cdot S(\Phi^2(m_0)_1)\Phi^3 m_1 \overset{(3.28)}{=}\\ &= m_0\cdot\Phi^1 S((m_1)_1\Phi^2)(m_1)_2\Phi^3 \overset{(3.46)}{=}\\ &= m_0\cdot\Phi^1 S(\Phi^2)\varepsilon(m_1)\Phi^3 \overset{(3.47)}{=} m\end{aligned}$$

Now, let us prove (7). *Again, in the calculations that follows, two copies of Φ will be required, so that we are going to indicate with $\Psi=\Psi^1\otimes\Psi^2\otimes\Psi^3$ another copy of Φ.*

For all $m\in M$

$$\begin{aligned}\tau(\tau(m)_0)\otimes\tau(m)_1 &= \tau\left(\left(\Phi^1\cdot m_0\cdot S(\Phi^2 m_1)\Phi^3\right)_0\right)\otimes\left(\Phi^1\cdot m_0\cdot S(\Phi^2 m_1)\Phi^3\right)_1 \overset{(1)}{=}\\ &= \tau\left(\left(\Phi^1\cdot m_0\right)_0\right)\varepsilon\left(\left(S(\Phi^2 m_1)\Phi^3\right)_1\right)\otimes\left(\Phi^1\cdot m_0\right)_1\left(S(\Phi^2 m_1)\Phi^3\right)_2 =\\ &= \tau\left((\Phi^1)_1\cdot(m_0)_0\right)\otimes(\Phi^1)_2(m_0)_1 S(\Phi^2 m_1)\Phi^3 \overset{(3.28)}{=}\\ &= \tau\left((\Phi^1)_1\phi^1\cdot m_0\cdot\Psi^1\right)\otimes(\Phi^1)_2\phi^2(m_1)_1\Psi^2 S(\Phi^2\phi^3(m_1)_2\Psi^3)\Phi^3 \overset{(1)}{=}\\ &= \tau\left((\Phi^1)_1\phi^1\cdot m_0\right)\varepsilon\left(\Psi^1\right)\otimes(\Phi^1)_2\phi^2(m_1)_1\Psi^2 S(\Phi^2\phi^3(m_1)_2\Psi^3)\Phi^3 \overset{(3.24)}{=}\\ &= \tau\left((\Phi^1)_1\phi^1\cdot m_0\right)\otimes(\Phi^1)_2\phi^2(m_1)_1 S(\Phi^2\phi^3(m_1)_2)\Phi^3 \overset{(3.45)}{=}\\ &= \tau\left((\Phi^1)_1\phi^1\cdot m_0\right)\otimes(\Phi^1)_2\phi^2\varepsilon(m_1)S(\Phi^2\phi^3)\Phi^3 \overset{(*)}{=}\\ &= \tau\left(\phi^1\Psi^1\cdot m\right)\otimes\phi^2\Phi^1\left(\Psi^2\right)_1 S\left(\left(\phi^3\right)_1\Phi^2\left(\Psi^2\right)_2\right)\left(\phi^3\right)_2\Phi^3\Psi^3 \overset{(3.45)}{=}\\ &= \tau\left(\phi^1\Psi^1\cdot m\right)\otimes\phi^2\Phi^1\varepsilon\left(\Psi^2\right)S\left(\left(\phi^3\right)_1\Phi^2\right)\left(\phi^3\right)_2\Phi^3\Psi^3 \overset{(3.46)}{=}\\ &= \tau\left(\phi^1\cdot m\right)\otimes\phi^2\Phi^1 S\left(\Phi^2\right)\varepsilon\left(\phi^3\right)\Phi^3 \overset{(3.24)}{=}\\ &= \tau(m)\otimes\Phi^1 S\left(\Phi^2\right)\Phi^3 \overset{(3.47)}{=} \tau(m)\otimes 1.\end{aligned}$$

where in $(*)$ we used the identity (3.19):

$$(\Delta\otimes A\otimes A)(\Phi)(\Phi^{-1}\otimes 1) = (A\otimes A\otimes\Delta)(\Phi^{-1})(1\otimes\Phi)(A\otimes\Delta\otimes A)(\Psi).$$

$\square$

The following theorem shows how important are in this treatment the preantipode and the map τ.

Theorem 3.2.20. *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. Let $S\colon A \to A$ be a preantipode for A. Then the map*

$$\tilde{\tau} : \frac{M}{MA^+} \longrightarrow M : \overline{m} \longmapsto \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3 \tag{3.53}$$

is $\Bbbk$-linear and satisfies (iii) and (iv) of Theorem 3.2.14.

Proof. Consider the $\Bbbk$-linear map:

$$\tau : M \longrightarrow M : m \longmapsto \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3$$

defined in (3.51). It satisfies $\tau(m \cdot a) = \tau(m)\,\varepsilon(a)$ ((1) of Proposition 3.2.19). Thus τ factors through the quotient:

$$\tilde{\tau}\colon \frac{M}{MA^+} \to M,$$

since for all $\sum_i m_i \cdot a_i \in MA^+$

$$\tau\left(\sum_i m_i \cdot a_i\right) = \sum_i \tau(m_i)\,\varepsilon(a_i) = 0.$$

Now, keeping in mind Proposition 3.2.19, let us show that $\tilde{\tau}$ satisfies (*iii*) and (*iv*):

(iii) For all $m \in M$

$$\tilde{\tau}(\overline{m_0}) \cdot m_1 = \tau(m_0) \cdot m_1 \overset{(6)}{=} m.$$

(iv) From (6) we deduce also that, for all $m \in M$,

$$\overline{m} = \overline{\tilde{\tau}(\overline{m_0}) \cdot m_1} = \overline{\tilde{\tau}(\overline{m})} = \overline{\tau(m)}, \tag{3.54}$$

Putting together this last equality, (7) and the fact that $\tilde{\tau}(\overline{m}) = \tau(m)$ we find out that:

$$\begin{aligned}\overline{\tilde{\tau}(\overline{m})_0} \otimes \tilde{\tau}(\overline{m})_1 = \overline{\tau(m)_0} \otimes \tau(m)_1 &\overset{(3.54)}{=} \\ = \overline{\tau(\tau(m)_0)} \otimes \tau(m)_1 &\overset{(7)}{=} \\ = \overline{\tau(m)} \otimes 1 &\overset{(3.54)}{=} \overline{m} \otimes 1.\end{aligned}$$

The proof is now complete. ∎

Remark 3.2.21. Note that if $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra and there exists a map τ that satisfies (1)-(7) then $\tilde{\tau}$ satisfies properties (*iii*) and (*iv*) of Theorem 3.2.14. Indeed, nowhere in the previous proof we used the actual definition of τ, but only its properties.

What we are going to do now is to show that if the adjunction (F, G, η, ϵ) is an equivalence, then we can construct a map S from η^{-1} that satisfies condition (P1), (P2) and (P3) of Definition 3.2.16, i.e., a preantipode for A.

Indeed, consider the tensor product $A \otimes A$ with the following structures:

$$A\hat{\otimes}A := T({}_\circ A_\bullet) = {}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet$$

where the tensor product is taken in ${}_A\mathcal{M}_A$. Explicitly:

$$x \cdot (a \otimes b) = a \otimes xb \tag{3.55a}$$
$$(a \otimes b) \cdot x = ax_1 \otimes bx_2 \tag{3.55b}$$
$$\rho(a \otimes b) = ((a \otimes b_1) \otimes b_2) \cdot \Phi \tag{3.55c}$$

for all $a, b, x \in A$ (recall relation (3.30)). Set

$$\begin{array}{rccc} \hat{\eta}_A := \eta_{A\hat{\otimes}A} : & A\hat{\otimes}A & \longrightarrow & \frac{A\hat{\otimes}A}{(A\hat{\otimes}A)A^+} \otimes A \\ & a \otimes b & \longmapsto & \overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3 \end{array} \tag{3.56}$$

The structures on $\frac{A\hat{\otimes}A}{(A\hat{\otimes}A)A^+} \otimes A$ are given by:

$$x \cdot (\overline{a \otimes b} \otimes c) = \overline{a \otimes x_1 b} \otimes x_2 c$$
$$(\overline{a \otimes b} \otimes c) \cdot x = \overline{a \otimes b} \otimes cx$$
$$\rho(\overline{a \otimes b} \otimes c) = (\overline{a \otimes \phi^1 b} \otimes \phi^2 c_1) \otimes \phi^3 c_2$$

for all $a, b, c, x \in A$. If we assume that (F, G) is an equivalence, then $\hat{\eta}_A$ is an isomorphism in ${}_A\mathcal{M}_A^A$. This means that $\hat{\eta}_A^{-1}$ exists and it is an isomorphism too.

- $\hat{\eta}_A^{-1}$ is right A-linear. In particular:

$$\hat{\eta}_A^{-1}(\overline{a \otimes b} \otimes c) = \hat{\eta}_A^{-1}(\overline{a \otimes b} \otimes 1) \cdot c, \tag{3.57}$$

  so that it suffices to see how it works on elements of the form $\overline{a \otimes b} \otimes 1$. Define

$$a^1 \otimes a^2 := \hat{\eta}_A^{-1}(\overline{1 \otimes a} \otimes 1). \tag{3.58}$$

- Define a new map:

$$\beta : \frac{A\hat{\otimes}A}{(A\hat{\otimes}A)A^+} \longrightarrow A : \overline{a \otimes b} \longmapsto (A \otimes \varepsilon)\hat{\eta}_A^{-1}(\overline{a \otimes b} \otimes 1). \tag{3.59}$$

  Consider the left A-action on $A \otimes A$ given by the multiplication on the first factor. It is trivially an action and $(A \otimes A)A^+$ is left A-submodule of $A \otimes A$ with this new action. Consider the same A-action on the first factor on $\frac{A\otimes A}{(A\otimes A)A^+}$. Then $\hat{\eta}_A$ in left A-linear with respect to this A-action:

$$\begin{tikzcd} x \otimes (a \otimes b) \arrow[r, mapsto, "A\otimes\hat{\eta}_A"] \arrow[d, mapsto, "m\otimes A"'] \arrow[dr, phantom, "\circlearrowleft"] & x \otimes (\overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3) \arrow[d, mapsto, "m\otimes A\otimes A"] \\ xa \otimes b \arrow[r, mapsto, "\hat{\eta}_A"'] & \overline{xa\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3 \end{tikzcd} \tag{3.60}$$

This implies that also $\hat{\eta}_A^{-1}$ is. In particular, if we write down this left A-linearity of $\hat{\eta}_A^{-1}$ and we apply it to $\overline{1 \otimes a} \otimes 1$ we get that:

$$\begin{tikzcd}
x \otimes (a^1 \otimes a^2) \arrow[d, mapsto, "m\otimes A"'] \arrow[dr, phantom, "\circlearrowleft"] & x \otimes (\overline{1 \otimes a} \otimes 1) \arrow[l, mapsto, "A\otimes\hat{\eta}_A^{-1}"'] \arrow[d, mapsto, "m\otimes A\otimes A"] \\
xa^1 \otimes a^2 & \overline{x \otimes a} \otimes 1 \arrow[l, mapsto, "\hat{\eta}_A^{-1}"]
\end{tikzcd}$$

i.e.:

$$\hat{\eta}_A^{-1}(\overline{a \otimes b} \otimes 1) = ab^1 \otimes b^2 \tag{3.61}$$

$$\beta(\overline{a \otimes b}) = ab^1\, \varepsilon(b^2) \tag{3.62}$$

- Define
$$S(a) := a^1\, \varepsilon(a^2)$$
for all $a \in A$. Then $S\colon A \to A$ is clearly $\Bbbk$-linear. Moreover
$$\beta(\overline{a \otimes b}) = aS(b). \tag{3.63}$$

- Since $\overline{a \otimes b} \in \frac{A\otimes A}{(A\otimes A)A^+}$, we have that:
$$\overline{ax_1 \otimes bx_2} = \overline{(a \otimes b) \cdot x} = \overline{a \otimes b}\, \varepsilon(x).$$
This implies that:
$$ax_1 S(bx_2) = \beta(\overline{ax_1 \otimes bx_2}) = \beta(\overline{a \otimes b})\, \varepsilon(x) = aS(b)\, \varepsilon(x).$$
Evaluating at $a = 1$, we get (P1): $x_1 S(bx_2) = \varepsilon(x) S(b)$.

- $\hat{\eta}_A^{-1}$ is left A-linear with respect to the original left A-action too. Hence:
$$\begin{aligned} a^1 \otimes xa^2 = x \cdot (a^1 \otimes a^2) = x \cdot \hat{\eta}_A^{-1}(\overline{1 \otimes a} \otimes 1) = \hat{\eta}_A^{-1}(x \cdot (\overline{1 \otimes a} \otimes 1)) = \\ = \hat{\eta}_A^{-1}(\overline{1 \otimes x_1 a} \otimes x_2) = \Big((x_1 a)^1 \otimes (x_1 a)^2\Big) \cdot x_2 = \\ = (x_1 a)^1 (x_2)_1 \otimes (x_1 a)^2 (x_2)_2. \end{aligned}$$
Applying $A \otimes \varepsilon$ on both sides we find (P2):
$$\varepsilon(x)\, S(a) = S(x_1 a) x_2.$$

- $\hat{\eta}_A^{-1}$ is the inverse of $\hat{\eta}_A$, so that:
$$\begin{aligned} a \otimes b = \hat{\eta}_A^{-1}(\hat{\eta}_A(a \otimes b)) = \hat{\eta}_A^{-1}(\overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3) = \\ = \hat{\eta}_A^{-1}(\overline{a\Phi^1 \otimes b_1\Phi^2} \otimes 1) \cdot b_2\Phi^3 \overset{(3.61)}{=} (a\Phi^1 (b_1\Phi^2)^1 \otimes (b_1\Phi^2)^2) \cdot b_2\Phi^3. \end{aligned}$$
Applying $A \otimes \varepsilon$ to both sides we get:
$$a\, \varepsilon(b) = a\Phi^1 S\left(b_1 \Phi^2\right) b_2 \Phi^3.$$
For $a = b = 1$ we find (P3):
$$1 = \Phi^1 S(\Phi^2) \Phi^3.$$

Remark 3.2.22. We can use the right A-colinearity of $\hat{\eta}_A^{-1}$ to express it explicitly as a function of S. Indeed, consider the following commutative diagram, that shows the A-colinearity of $\hat{\eta}_A^{-1}$ when applied to $\overline{1 \otimes b} \otimes 1$:

$$\begin{tikzcd}
\overline{1 \otimes b} \otimes 1 \arrow[r, mapsto, "\hat{\eta}_A^{-1}"] \arrow[d, mapsto, "\rho_{\overline{A \hat{\otimes} A} \otimes A}"'] \arrow[dr, phantom, "\circlearrowleft"] & b^1 \otimes b^2 \arrow[d, mapsto, "\rho_{A \hat{\otimes} A}"] \\
\left(\overline{1 \otimes \phi^1 b} \otimes \phi^2\right) \otimes \phi^3 \arrow[r, mapsto, "\hat{\eta}_A^{-1} \otimes A"'] & (*)
\end{tikzcd}$$

where in $(*)$ should take place the following equality:

$$\begin{aligned}
\left(b^1\Phi^1 \otimes (b^2)_1\Phi^2\right) \otimes (b^2)_2\Phi^3 &= \hat{\eta}_A^{-1}\left(\overline{1 \otimes \phi^1 b} \otimes \phi^2\right) \otimes \phi^3 \overset{(3.57)}{=} \\
&= \hat{\eta}_A^{-1}\left(\overline{1 \otimes \phi^1 b} \otimes 1\right) \cdot \phi^2 \otimes \phi^3 \overset{\text{Def.}(3.58)}{=} \\
&= \left((\phi^1 b)^1 \otimes (\phi^1 b)^2\right) \cdot \phi^2 \otimes \phi^3 = \\
&= (\phi^1 b)^1 (\phi^2)_1 \otimes (\phi^1 b)^2 (\phi^2)_2 \otimes \phi^3
\end{aligned}$$

Applying $A \otimes \varepsilon \otimes A$ to both sides and in view of (3.20):

$$\hat{\eta}_A^{-1}(\overline{1 \otimes b} \otimes 1) = b^1 \otimes b^2 = S(\phi^1 b)\phi^2 \otimes \phi^3. \tag{3.64}$$

Recalling (3.61) we can conclude that:

$$\hat{\eta}_A^{-1}(\overline{a \otimes b} \otimes c) = aS(\phi^1 b)\phi^2 c_1 \otimes \phi^3 c_2. \tag{3.65}$$

Theorem 3.2.23. (Fundamental Structure Theorem for quasi-Hopf bimodules) *For a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ the following assertions are equivalent:*

(1) The adjunction (F, G, η, ϵ) is an equivalence of categories.

(2) $\hat{\eta}_A$ is bijective.

(3) There exists a preantipode.

Proof. (1) $\Rightarrow$ (2). It follows from the fact that $\hat{\eta}_A = \eta_{A \hat{\otimes} A}$.
(2) $\Rightarrow$ (3). It is the preceding discussion.
(3) $\Rightarrow$ (1). It follows from Theorem 3.2.14 and Theorem 3.2.20. □

A careful observer can object that the Structure Theorem for an ordinary Hopf module M involves $M^{\mathrm{Co}H}$ and not this unexpected quotient $\frac{M}{MA^+}$. Actually, they are the same object, but to see this we need a different generalization of the concept of coinvariants for a quasi-bialgebra.

The results that follow have been proven for quasi-Hopf algebras (that we will introduce later) by Hausser and Nill in [HN]. Here we generalize these results to quasi-bialgebras with preantipode and in the next section we will show how the classical ones can be recovered from this new ones. The references that can be found near the statements refer to the analogue result for ordinary quasi-Hopf algebras.

Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode S, let M be a quasi-Hopf bimodule and consider again the map $\tau\colon M \to M$ defined in (3.51),

$$\tau(m) = \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3.$$

Definition 3.2.24. ([HN, Definition 3.5]) (Coinvariants)
Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. The *space of coinvariants* of a quasi-Hopf A-bimodule M is defined to be

$$M^{\mathrm{Co}A} := \tau(M),$$

where τ is the map defined in (3.51).

Proposition 3.2.25. *If A is a quasi-bialgebra with preantipode S and M is a quasi-Hopf A-bimodule, then the following descriptions of $M^{\mathrm{Co}A}$ hold:*

$$M^{\mathrm{Co}A} = \{n \in M \mid \tau(n) = n\} = \{n \in M \mid \tau(n_0) \otimes n_1 = \tau(n) \otimes 1\}\,. \tag{3.66}$$

Proof. Let $n \in M$ be such that $\tau(n) = n$. Trivially $n \in \tau(M)$, so that

$$M^{\mathrm{Co}A} \supseteq \{n \in M \mid \tau(n) = n\}\,.$$

Moreover, applying ρ_M to both sides, we find that:

$$n_0 \otimes n_1 = \tau(n)_0 \otimes \tau(n)_1.$$

Thus:

$$\tau(n_0) \otimes n_1 = \tau(\tau(n)_0) \otimes \tau(n)_1 = \tau(n) \otimes 1,$$

in view of (7) of Proposition 3.2.19. Hence:

$$\{n \in M \mid \tau(n) = n\} \subseteq \{n \in M \mid \tau(n_0) \otimes n_1 = \tau(n) \otimes 1\}\,.$$

Now, let $n \in M$ be such that $\tau(n_0) \otimes n_1 = \tau(n) \otimes 1$ and apply μ^r_M to both sides:

$$n = \tau(n_0) \cdot n_1 = \tau(n),$$

by (6), Proposition 3.2.19. This implies that:

$$\{n \in M \mid \tau(n) = n\} \supseteq \{n \in M \mid \tau(n_0) \otimes n_1 = \tau(n) \otimes 1\}\,.$$

For the remaining inclusion, let $n \in M$ be such that $n \in \tau(M)$. Thus there exists $m \in M$ such that $n = \tau(m)$. Applying τ to both sides and recalling (2) of Proposition 3.2.19 we find that:

$$\tau(n) = \tau(\tau(m)) = \tau(m) = n.$$

Therefore $M^{\mathrm{Co}A} \subseteq \{n \in M \mid \tau(n) = n\}$. $\square$

Corollary 3.2.26. ([HN, Corollary 3.9]) *For any quasi-Hopf A-bimodule M we have*

$$M^{\mathrm{Co}A} = \left\{n \in M \mid \rho_M(n) = \tau(\phi^1 \cdot n) \cdot \phi^2 \otimes \phi^3\right\}. \tag{3.67}$$

Proof. Let $n \in M$ be such that $n_0 \otimes n_1 = \rho_M(n) = \tau(\phi^1 \cdot n) \cdot \phi^2 \otimes \phi^3$. Applying $A \otimes \varepsilon$ to both sides we get:

$$n = n_0\, \varepsilon(n_1) = \tau(\phi^1 \cdot n) \cdot \phi^2\, \varepsilon(\phi^3) = \tau(n).$$

Conversely, let us recall the A-colinearity of η_M (as defined in (3.37)):

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "\rho_M"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{M} \otimes A \arrow[d, "\rho_{\overline{M} \otimes A}"] \\ M \otimes A \arrow[r, "\eta_{M \otimes A}"'] & (\overline{M} \otimes A) \otimes A \end{tikzcd}$$

Hence:

$$\rho_M = (\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A} \circ \eta_M.$$

Now, if $n \in M^{\mathrm{Co}A}$ then:

$$\begin{aligned} \rho_M(n) &= ((\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A} \circ \eta_M)(n) \overset{(3.37)}{=} \\ &= ((\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A})(\overline{n_0} \otimes n_1) \overset{n \in M^{\mathrm{Co}A}}{=} \\ &= ((\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A})(\overline{\tau(n)_0} \otimes \tau(n)_1) = \\ &= ((\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A})(\overline{\tilde{\tau}(\overline{n})_0} \otimes \tilde{\tau}(\overline{n})_1) \overset{(3.42)}{=} \\ &= ((\eta_M^{-1} \otimes A) \circ \rho_{\overline{M} \otimes A})(\overline{n} \otimes 1) \overset{(3.43)}{=} \\ &= \tilde{\tau}(\overline{\phi^1 \cdot n}) \cdot \phi^2 \otimes \phi^3 = \\ &= \tau(\phi^1 \cdot n) \cdot \phi^2 \otimes \phi^3 \end{aligned}$$

and even the other inclusion holds. □

Remark 3.2.27. Note that if $n \in M$ is such that $\rho_M(n) = n \otimes 1$, then $n \in M^{\mathrm{Co}A}$. However, up to this moment we found no evidence that the converse is true or not, in general.

Lemma 3.2.28. ([HN, Lemma 3.6]) *Let M be a left A-module. Then the coinvariants of the quasi-Hopf bimodule*

$$G(M) = {}_\bullet M_\circ \otimes {}_\bullet A^\bullet_\bullet$$

are given by

$$(M \otimes A)^{\mathrm{Co}A} = M \otimes 1,$$

and for $m \in M$ and $a \in A$ we have that $\tau(m \otimes a) = m \otimes \varepsilon(a)$.

Proof. In view of (3.35) and (1) of Proposition 3.2.19, we have that

$$\tau(m \otimes a) = \tau((m \otimes 1) \cdot a) = \tau(m \otimes 1)\, \varepsilon(a). \tag{3.68}$$

Moreover:

$$
\begin{aligned}
\tau(m\otimes 1) &= \Phi^1\cdot(m\otimes 1)_0\cdot S(\Phi^2(m\otimes 1)_1)\Phi^3 = \\
&= \Phi^1\cdot(\phi^1 m\otimes\phi^2)\cdot S(\Phi^2\phi^3)\Phi^3 \overset{(*)}{=} \\
&= \phi^1\Phi^1\cdot m\otimes\phi^2\Psi^1(\Phi^2)_1 S\left((\phi^3)_1\Psi^2(\Phi^2)_2\right)(\phi^3)_2\Psi^3\Phi^3 \overset{(3.46)}{=} \\
&= \phi^1\Phi^1\cdot m\otimes\phi^2\Psi^1\,\varepsilon(\Phi^2)\,S\left((\phi^3)_1\Psi^2\right)(\phi^3)_2\Psi^3\Phi^3 \overset{(3.45)}{=} \\
&= \phi^1\Phi^1\cdot m\otimes\phi^2\Psi^1\,\varepsilon(\Phi^2)\,S\left(\Psi^2\right)\,\varepsilon(\phi^3)\,\Psi^3\Phi^3 \overset{(3.24)}{=} \\
&= m\otimes\Psi^1 S(\Psi^2)\Psi^3 = \\
&= m\otimes 1
\end{aligned}
\tag{3.69}
$$

where, again, we put $\Psi=\Phi$ in order to avoid confusion and in $(*)$ we used (3.19) in the form:

$$(\Delta\otimes A\otimes A)(\Phi)(\Phi^{-1}\otimes 1) = (A\otimes A\otimes\Delta)(\Phi^{-1})(1\otimes\Phi)(A\otimes\Delta\otimes A)(\Phi).$$

Identity (3.69) implies that $M\otimes 1\subseteq (M\otimes A)^{\mathrm{Co}A}$. Furthermore, combined with (3.68), shows that if $m\otimes a\in(M\otimes A)^{\mathrm{Co}A}$, then

$$m\otimes a = \tau(m\otimes a) = m\otimes\varepsilon(a) = m\,\varepsilon(a)\otimes 1.$$

Hence $M\otimes 1\supseteq (M\otimes A)^{\mathrm{Co}A}$. □

Note that this last result is in accordance with the classical one about Hopf algebras: if B is an ordinary bialgebra and V is a $\Bbbk$-vector space, then the unit of the adjunction (L,R) of Theorem 2.1.7 is always a natural isomorphism and we saw in Proposition 2.1.8 that $(V\otimes B)^{\mathrm{Co}B} = V\otimes 1\cong V$.

Proposition 3.2.29. *Let $(A,m,u,\Delta,\varepsilon,\Phi,S)$ be a quasi-bialgebra with preantipode S and consider $M^{\mathrm{Co}A}$ as a left A-module with A-action given by (3.52). Then*

$$\tilde{\tau}\colon \frac{M}{MA^+}\to M^{\mathrm{Co}A}$$

as defined in (3.53) is an isomorphism of left A-modules with inverse given by

$$
\begin{array}{cccc}
\sigma: & M^{\mathrm{Co}A} & \longrightarrow & \dfrac{M}{MA^+} \\
 & m & \longmapsto & \overline{m}
\end{array}
$$

Proof. Let us start by showing that $\tilde{\tau}$ is bijective:

$$\overline{m}\xmapsto{\tilde{\tau}}\tilde{\tau}(\overline{m})\xmapsto{\sigma}\overline{\tilde{\tau}(\overline{m})}\overset{(3.44)}{=}\overline{m}$$

$$n\xmapsto{\sigma}\overline{n}\xmapsto{\tilde{\tau}}\tilde{\tau}(\overline{n}) = \tau(n)\overset{(*)}{=} n$$

where in $(*)$ we used the fact that $n \in M^{\mathrm{Co}A}$ by hypothesis and (3.66).
Now, consider the left A-action on $M^{\mathrm{Co}A}$ given by (3.52):

$$a \blacktriangleright m := \tau(a \cdot m),$$

for all $a \in A$ and $m \in M^{\mathrm{Co}A}$. By definition of $M^{\mathrm{Co}A} = \tau(M)$ and in view of (4) of Proposition 3.2.19, in order to prove that $\blacktriangleright$ is an action it's enough to prove that

$$u(k) \blacktriangleright m = km,$$

but

$$u(k) \blacktriangleright m = \tau(ku(1) \cdot m) = k\tau(m) \overset{(3.66)}{=} km$$

for all $k \in \Bbbk$ and $m \in M^{\mathrm{Co}A}$. Moreover, (3) of Proposition 3.2.19 guarantees that τ is A-linear with respect to this left A-action. $\square$

Corollary 3.2.30. ([HN, Theorem 3.8]) *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. Let M be a quasi-Hopf A-bimodule. Consider $N := M^{\mathrm{Co}A}$ as a left A-module with A-action $\blacktriangleright$ as in (3.52), and ${}_\bullet N_\circ \otimes {}_\bullet A^\bullet_\bullet$ as a quasi-Hopf A-bimodule with structures indicated by the dots. Then:*

$$\begin{array}{cccc} \nu: & N \otimes A & \longrightarrow & M \\ & n \otimes a & \longmapsto & n \cdot a \end{array}$$

provides an isomorphism of quasi-Hopf A-bimodules with inverse given by

$$\nu^{-1}(m) = \tau(m_0) \otimes m_1.$$

Proof. In view of Proposition 3.2.29,

$$\begin{array}{cccc} \sigma: & {}_\bullet M^{\mathrm{Co}A}{}_\circ & \longrightarrow & {}_\bullet \dfrac{M}{MA^+}{}_\circ \\ & m & \longmapsto & \overline{m} \end{array}$$

is an isomorphism of (A, A)-bimodules. Thus $G(\sigma) = \sigma \otimes A \colon M^{\mathrm{Co}A} \otimes A \to \frac{M}{MA^+} \otimes A$ is an isomorphism. Hence:

$$\begin{tikzcd} M^{\mathrm{Co}A} \otimes A \arrow[r, "\sigma \otimes A"] \arrow[dr, "\eta_M^{-1} \circ (\sigma \otimes A)"'] & \frac{M}{MA^+} \otimes A \arrow[d, "\eta_M^{-1}"] \\ & M \end{tikzcd}$$

is an isomorphism and

$$\left(\eta_M^{-1} \circ (\sigma \otimes A)\right)(n \otimes a) \overset{(3.43)}{=} \tilde{\tau}(\overline{n}) \cdot a = \tau(n) \cdot a = n \cdot a$$

for all $n \in M^{\mathrm{Co}A}$ and $a \in A$. The inverse is given by

$$((\tilde{\tau} \otimes A) \circ \eta_M)(m) = \tilde{\tau}(\overline{m_0}) \otimes m_1 = \tau(m_0) \otimes m_1,$$

for every $m \in M$. $\square$

3.3 Quasi-Hopf algebras and some classical results.

Once the Structure Theorem is proven, it is a good thing to verify if it is in accordance with the classical results. This is why we open here a digression on quasi-Hopf algebras.

By the way, we also develop some new important results about quasi-bialgebras with preantipode, as the uniqueness of the preantipode and the fact that quasi-bialgebras with preantipode are closed under gauge twisting. Thereby we can highlight some links and some differences that intervene between the ordinary quasi-antipode and our preantipode.

Definition 3.3.1. ([Dr1, page 1424]) (Quasi-Hopf algebra)
A quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a *quasi-Hopf algebra* if there exist elements α and β in A and an antiendomorphism s of A such that:

$$s(a_1)\alpha a_2 = \varepsilon(a)\,\alpha \tag{3.70}$$
$$a_1\beta s(a_2) = \varepsilon(a)\,\beta \tag{3.71}$$
$$\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = 1 \tag{3.72}$$
$$s(\phi^1)\alpha\phi^2\beta s(\phi^3) = 1 \tag{3.73}$$

where, as usual, $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$ and $\Phi^{-1} = \phi^1 \otimes \phi^2 \otimes \phi^3$. The triple (s, α, β) is usually called *antipode* ([HN]) or *quasi-antipode* ([Sc1]). We will use the second terminology, in order to distinguish this one from the ordinary antipode of a Hopf algebra.

Actually, condition (3.72) is superfluous (as the subsequent proposition shows) since it can be derived by the other three axioms. Nevertheless, expressing both (3.72) and (3.73) as part of the definition of a quasi-Hopf algebra makes things more symmetric and we will see that it does not make proofs dull.

Proposition 3.3.2. *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra, and suppose given an antiendomorphism s of A and elements α and β in A that satisfy (3.70) and (3.71), then $g := \Phi^1\beta s(\Phi^2)\alpha\Phi^3$ is in the center of A and $h := s(\phi^1)\alpha\phi^2\beta s(\phi^3)$ commutes with $s(A)$. Moreover, if h is equal to 1, so is g.*

Proof. Let us proceed step by step:

- First of all, we prove that $\Phi^1\beta s(\Phi^2)\alpha\Phi^3$ belongs to the center of A. Let us apply the map from $A \otimes A \otimes A$ to A that sends $a \otimes b \otimes c$ into $a\beta s(b)\alpha c$ to both sides of (3.21):
$$\Phi^1(a_1)_1\beta s(\Phi^2(a_1)_2)\alpha\Phi^3 a_2 = a_1\Phi^1\beta s((a_2)_1\Phi^2)\alpha(a_2)_2\Phi^3.$$
Simplifying the left hand side using (3.71) we get:
$$\Phi^1(a_1)_1\beta s(\Phi^2(a_1)_2)\alpha\Phi^3 a_2 = \Phi^1(a_1)_1\beta s((a_1)_2)s(\Phi^2)\alpha\Phi^3 a_2 = ga$$
and simplifying the right hand side using (3.70) we get:
$$a_1\Phi^1\beta s((a_2)_1\Phi^2)\alpha(a_2)_2\Phi^3 = a_1\Phi^1\beta s(\Phi^2)s((a_2)_1)\alpha(a_2)_2\Phi^3 = ag.$$
Hence $ga = ag$ for all $a \in A$ and g belongs to the center of A.

- Secondly, let us show that $s(\phi^1)\alpha\phi^2\beta s(\phi^3)$ commutes with $s(A)$. In view of (3.21), apply the $\Bbbk$-linear map $A \otimes A \otimes A \longrightarrow A$ that takes $a \otimes b \otimes c$ into $s(a)\alpha b\beta s(c)$ to both sides of
$$(\Delta \otimes A)(\Delta(a)) \cdot \Phi^{-1} = \Phi^{-1} \cdot (A \otimes \Delta)(\Delta(a)),$$
so that
$$s((a_1)_1\phi^1)\alpha(a_1)_2\phi^2\beta s(a_2\phi^3) = s(\phi^1 a_1)\alpha\phi^2(a_2)_1\beta s(\phi^3(a_2)_2).$$
On one hand, the left hand side simplifies to:
$$\begin{aligned} s((a_1)_1\phi^1)\alpha(a_1)_2\phi^2\beta s(a_2\phi^3) &= s(\phi^1)s((a_1)_1)\alpha(a_1)_2\phi^2\beta s(\phi^3)s(a_2) \overset{(3.70)}{=} \\ &= s(\phi^1)\alpha\phi^2\beta s(\phi^3)s(a). \end{aligned}$$
On the other hand, the right hand side simplifies to:
$$\begin{aligned} s(\phi^1 a_1)\alpha\phi^2(a_2)_1\beta s(\phi^3(a_2)_2) &= s(a_1)s(\phi^1)\alpha\phi^2(a_2)_1\beta s((a_2)_2)s(\phi^3) \overset{(3.71)}{=} \\ &= s(a)s(\phi^1)\alpha\phi^2\beta s(\phi^3). \end{aligned}$$
Hence $hs(a) = s(a)h$ for all $a \in A$.

- Now, consider (3.19) in the form:
$$(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes A \otimes A)(\Phi)(\Phi^{-1} \otimes 1) = (1 \otimes \Phi)(A \otimes \Delta \otimes A)(\Phi)$$
and apply the map
$$A \otimes A \otimes A \otimes A \longrightarrow A : a \otimes b \otimes c \otimes d \longmapsto s(a)\alpha b\beta s(c)\alpha d$$
to both sides. If we write $\Psi = \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \Phi$, for clearness sake, we get:
$$\begin{aligned} &s(\Phi^1(\Psi^1)_1\phi^1)\alpha\Phi^2(\Psi^1)_2\phi^2\beta s((\Phi^3)_1\Psi^2\phi^3)\alpha(\Phi^3)_2\Psi^3 = \\ &\qquad\qquad = s(\Psi^1)\alpha\Phi^1(\Psi^2)_1\beta s(\Phi^2(\Psi^2)_2)\alpha\Phi^3\Psi^3. \end{aligned}$$
For the left hand side:
$$\begin{aligned} &s(\Phi^1(\Psi^1)_1\phi^1)\alpha\Phi^2(\Psi^1)_2\phi^2\beta s((\Phi^3)_1\Psi^2\phi^3)\alpha(\Phi^3)_2\Psi^3 = \\ &\quad = s(\Phi^1(\Psi^1)_1\phi^1)\alpha\Phi^2(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)s((\Phi^3)_1)\alpha(\Phi^3)_2\Psi^3 \overset{(3.70)}{=} \\ &\quad = s(\Phi^1(\Psi^1)_1\phi^1)\alpha\Phi^2(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)\,\varepsilon(\Phi^3)\,\alpha\Psi^3 = \\ &\quad = s((\Psi^1)_1\phi^1)\alpha(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)\alpha\Psi^3 = \\ &\quad = s(\phi^1)s((\Psi^1)_1)\alpha(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)\alpha\Psi^3 \overset{(3.70)}{=} \\ &\quad = s(\phi^1)\,\varepsilon(\Psi^1)\alpha\phi^2\beta s(\Psi^2\phi^3)\alpha\Psi^3 = \\ &\quad = h\alpha \end{aligned}$$
while, for the right hand side:
$$\begin{aligned} s(\Psi^1)\alpha\Phi^1(\Psi^2)_1\beta s(\Phi^2(\Psi^2)_2)\alpha\Phi^3\Psi^3 &= s(\Psi^1)\alpha\Phi^1(\Psi^2)_1\beta s((\Psi^2)_2)s(\Phi^2)\alpha\Phi^3\Psi^3 \overset{(3.71)}{=} \\ &= s(\Psi^1)\alpha\Phi^1\,\varepsilon(\Psi^2)\,\beta s(\Phi^2)\alpha\Phi^3\Psi^3 = \\ &= \alpha\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = \\ &= \alpha g \end{aligned}$$
so that we have $\alpha g = h\alpha$.

- Pick an element $f \in s(A)\alpha A$ of the form $f = s(a)\alpha b$:

$$fg = s(a)\alpha bg = s(a)\alpha gb = s(a)h\alpha b = hs(a)\alpha b = hf. \tag{3.74}$$

  If we choose $f = h$ we get $hg = h^2$. Hence, if $h = 1$, then $g = 1$.

$\square$

Remark 3.3.3. ([Dr1, Proposition 1.3]) Note that if s is bijective (as it was in the original definition of quasi-Hopf algebras given by Drinfel'd), then also $s(\phi^1)\alpha\phi^2\beta s(\phi^3)$ is in the center of A (since s is surjective) and $h = 1$ if and only if $g = 1$. Indeed, it's enough to consider $f = g$ in (3.74) to get that also $g^2 = hg$ holds. Observe further that if α is cancellable, then from $g\alpha = \alpha g = h\alpha$ we get $g = h$.

Theorem 3.3.4. *Let $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a quasi-Hopf algebra. The application $S(a) = \beta s(a)\alpha$ is a preantipode.*

Proof. It's just a question of verifying that the axioms are satisfied:

(P1) $b_1 S(ab_2) = b_1\beta s(ab_2)\alpha = b_1\beta s(b_2)s(a)\alpha \overset{(3.71)}{=} \varepsilon(b)\beta s(a)\alpha = \varepsilon(b)S(a)$,

(P2) $S(a_1 b)a_2 = \beta s(b)s(a_1)\alpha a_2 \overset{(3.70)}{=} \beta s(b)\alpha\varepsilon(a) = S(b)\varepsilon(a)$,

(P3) $\Phi^1 S(\Phi^2)\Phi^3 = \Phi^1\beta s(\Phi^2)\alpha\Phi^3 \overset{(3.72)}{=} 1$.

$\square$

Therefore we have that every quasi-Hopf algebra is a quasi-bialgebra with preantipode. It is more than likely that the converse does not hold, even if we are not able to provide an example at the moment. Actually, there exists an example of a dual quasi-bialgebra with preantipode that is not a dual quasi-Hopf algebra. The interested reader may refer to [Sc3, Example 4.5.1], where Schauenburg exhibits a dual quasi-bialgebra H that does not admit a quasi-antipode but such that the category ${}^{H}\mathcal{M}_f$ of finite dimensional left H-comodules is left and right rigid. By the left-handed version of [Sc1, Theorem 2.6], this is equivalent to say that the adjunction (F, G) of [AP1, Theorem 2.7] is an equivalence of categories and hence, by [AP1, Theorem 3.9], H admits a preantipode (cfr. also [AP1, Remark 3.12] and [AP2, Remark 2.17]).

Corollary 3.3.5. *Let $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a quasi-Hopf algebra. Then $\varepsilon \circ s = \varepsilon$.*

Proof. We proved in Lemma 3.2.18 that if $(A, m, u, \Delta, \varepsilon, \Phi, S)$ is a quasi-bialgebra with preantipode, then $\varepsilon \circ S = \varepsilon$. In view of Theorem 3.3.4, $S(\cdot) = \beta s(\cdot)\alpha$ is a preantipode and

$$1 = \varepsilon(S(1)) = \varepsilon(\beta s(1)\alpha) = \varepsilon(\beta)\varepsilon(\alpha). \tag{3.75}$$

Hence

$$\varepsilon(a) = \varepsilon(S(a)) = \varepsilon(\beta)\varepsilon(s(a))\varepsilon(\alpha) = \varepsilon(s(a))$$

for every $a \in A$. $\square$

Corollary 3.3.6. *Let $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a quasi-Hopf algebra. Then the adjunction (F, G, η, ϵ) is an equivalence of categories. In particular, for each $M \in {}_A\mathcal{M}_A^A$,*

$$M \cong \frac{M}{MA^+} \otimes A,$$

where the structures on $\frac{M}{MA^+} \otimes A$ are given by:

$$\begin{aligned} x \cdot (\overline{m} \otimes a) = x_1 \cdot \overline{m} \otimes x_2 a \\ (\overline{m} \otimes a) \cdot x = \overline{m} \otimes ax \\ \rho(\overline{m} \otimes a) = \Phi^{-1} \cdot ((\overline{m} \otimes a_1) \otimes a_2) \end{aligned}$$

for every $a, x \in A$ and $m \in M$, and the isomorphism is given by the unit of the adjunction:

$$\begin{array}{cccc} \eta_M : & M & \longrightarrow & \dfrac{M}{MA^+} \otimes A \\ & m & \longmapsto & \overline{m_0} \otimes m_1 \end{array}$$

(cfr. [Sc1, proof of Theorem 3.1]).

Remark 3.3.7. Let $(H, m, u, \Delta, \varepsilon, s)$ be an ordinary Hopf algebra. Set $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$. Thus $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a quasi-Hopf algebra with quasi-antipode $(s, 1, 1)$. By Theorem 3.3.4, s is a preantipode.

Now, note that if $n \in M$ is such that $\tau(n) = n$, then

$$\rho_M(n) = n_0 \otimes n_1 \overset{(3.67)}{=} \tau(\phi^1 \cdot n) \cdot \phi^2 \otimes \phi^3 = \tau(n) \otimes 1 = n \otimes 1,$$

so that the ordinary definition of coinvariants and Definition 3.2.24 coincides. As a consequence, we can apply Corollary 3.2.30 to get back the ordinary Structure Theorem for Hopf modules. Indeed, for every Hopf module M,

$$M \cong M^{\mathrm{Co}H} \otimes H$$

via the isomorphisms:

$$\begin{aligned} \nu : M^{\mathrm{Co}H} \otimes H \longrightarrow M : m \otimes h \longmapsto m \cdot h \\ \nu^{-1} : M \longrightarrow M^{\mathrm{Co}H} \otimes H : m \longmapsto \tau(m_0) \otimes m_1 \end{aligned}$$

Moreover, note that $\tau(m) = \Phi^1 \cdot m_0 \cdot s(\Phi^2 m_1)\Phi^3 = m_0 \cdot s(m_1)$, so that $\nu^{-1}(m) = m_0 \cdot s(m_1) \otimes m_2$ as we found in Theorem 2.3.1.

Remark 3.3.8. Let $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a quasi-Hopf algebra with quasi-antipode (s, α, β) and assume that s is bijective. By Theorem 3.3.4, $S(\cdot) = \beta s(\cdot)\alpha$ is a preantipode. Thus the map τ has the form:

$$\tau(m) = \Phi^1 \cdot m_0 \cdot \beta s(\Phi^2 m_1)\alpha\Phi^3 = \Phi^1 \cdot m_0 \cdot \beta s(s^{-1}(\alpha\Phi^3)\Phi^2 m_1).$$

Note that this τ is exactly the projection E of Hausser and Nill ([HN]) and $M^{\mathrm{Co}H}$, obtained as image of τ, is the same $M^{\mathrm{Co}H}$ that appears in [HN, Definition 3.5] and [HN, Corollary 3.9]. Moreover, the A-action $\blacktriangleright$ coincides with the action they indicate with $\triangleright$ and Corollary 3.2.30 is precisely [HN, Theorem 3.8].

This last remark and the previous one show how the theory we developed here latch on to the traditional results about Hopf and quasi-Hopf bimodules. Now, let us spend some time to show that the preantipode presents some advantages with respect to the quasi-antipode.

First of all, we prove that the quasi-antipode is not unique, but it is just uniquely determined up to an invertible element u.

Proposition 3.3.9. [Dr1, Proposition 1.1] *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. If (s, α, β) is a quasi-antipode for A, then for any invertible element $u \in A$, also $(\overline{s}, \overline{\alpha}, \overline{\beta})$ is still a quasi-antipode, where*

$$\overline{s}(a) = us(a)u^{-1}, \quad \overline{\alpha} = u\alpha, \quad \overline{\beta} = \beta u^{-1}. \tag{3.76}$$

Furthermore, if A admits two quasi-antipodes (s, α, β) and $(\overline{s}, \overline{\alpha}, \overline{\beta})$, then they are connected by a transformation (3.76), where u is uniquely determined.

Proof. The first statement is a trivial observation. Indeed:

$$\overline{s}(a_1)\,\overline{\alpha}\,a_2 = us(a_1)u^{-1}u\alpha a_2 = us(a_1)\alpha a_2 \overset{(3.70)}{=} \varepsilon(a)u\alpha = \varepsilon(a)\,\overline{\alpha},$$

$$a_1\,\overline{\beta}\,\overline{s}(a_2) = a_1\beta u^{-1}us(a_2)u^{-1} = a_1\beta s(a_2)u^{-1} \overset{(3.71)}{=} \varepsilon(a)\beta u^{-1} = \varepsilon(a)\,\overline{\beta}$$

$$\Phi^1\,\overline{\beta}\,\overline{s}(\Phi^2)\,\overline{\alpha}\,\Phi^3 = \Phi^1\beta s(\Phi^2)\alpha\Phi^3 \overset{(3.72)}{=} 1$$

$$\overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\,\overline{\beta}\,\overline{s}(\phi^3) = us(\phi^1)\alpha\phi^2\beta s(\phi^3)u^{-1} \overset{(3.73)}{=} uu^{-1} = 1$$

For the second claim, define:

$$u = \overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\beta s(\phi^3)$$

and let us verify that it fulfils (3.76). To prove that $us(a) = \overline{s}(a)u$ for all $a \in A$ it is enough to recall (3.21):

$$(\Delta \otimes A)(\Delta(a))\Phi^{-1} = \Phi^{-1}(A \otimes \Delta)(\Delta(a)).$$

Indeed:

$$\begin{aligned}
us(a) &= \overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\beta s(\phi^3)s(a) = \\
&= \overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\beta s(\phi^3)s(\varepsilon(a_1)a_2) = \\
&= \overline{s}(\phi^1)\,\varepsilon(a_1)\,\overline{\alpha}\,\phi^2\beta s(\phi^3)s(a_2) \overset{(3.70)}{=} \\
&= \overline{s}(\phi^1)\,\overline{s}((a_1)_1)\,\overline{\alpha}\,(a_1)_2\,\phi^2\beta s(\phi^3)s(a_2) = \\
&= \overline{s}((a_1)_1\phi^1)\,\overline{\alpha}\,(a_1)_2\,\phi^2\beta s(a_2\phi^3) \overset{(3.21)}{=} \\
&= \overline{s}(\phi^1 a_1)\,\overline{\alpha}\,\phi^2(a_2)_1\beta s(\phi^3(a_2)_2) = \\
&= \overline{s}(a_1)\overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2(a_2)_1\beta s((a_2)_2)s(\phi^3) \overset{(3.71)}{=} \\
&= \overline{s}(a_1)\overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\,\varepsilon(a_2)\beta s(\phi^3) = \\
&= \overline{s}(a)u
\end{aligned}$$

To prove that $u\alpha = \overline{\alpha}$ apply

$$\xi \colon A \otimes A \otimes A \otimes A \longrightarrow A \colon a \otimes b \otimes c \otimes d \longmapsto \overline{s}(a)\,\overline{\alpha}\,b\beta s(c)\alpha d$$

to both sides of (3.19) in the form:

$$(A\otimes A\otimes \Delta)(\Phi)(\Delta\otimes A\otimes A)(\Psi)(\Phi^{-1}\otimes 1)=(1\otimes\Phi)(A\otimes\Delta\otimes A)(\Psi)$$

where, again, $\Psi=\Psi^1\otimes\Psi^2\otimes\Psi^3=\Phi$. What we get is:

$$\begin{aligned}\overline{s}(\Phi^1(\Psi^1)_1\phi^1)\,\overline{\alpha}\,\Phi^2(\Psi^1)_2\phi^2\beta s((\Phi^3)_1\Psi^2\phi^3)\alpha(\Phi^3)_2\Psi^3 &=\\ &=\overline{s}(\Psi^1)\,\overline{\alpha}\,\Phi^1(\Psi^2)_1\beta s(\Phi^2(\Psi^2)_2)\alpha\Phi^3\Psi^3\end{aligned}$$

Consider the left hand member:

$$\begin{aligned}&\overline{s}(\Phi^1(\Psi^1)_1\phi^1)\,\overline{\alpha}\,\Phi^2(\Psi^1)_2\phi^2\beta s((\Phi^3)_1\Psi^2\phi^3)\alpha(\Phi^3)_2\Psi^3=\\ &\qquad=\overline{s}(\Phi^1(\Psi^1)_1\phi^1)\,\overline{\alpha}\,\Phi^2(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)s((\Phi^3)_1)\alpha(\Phi^3)_2\Psi^3\overset{(3.70)}{=}\\ &\qquad=\overline{s}(\Phi^1(\Psi^1)_1\phi^1)\,\overline{\alpha}\,\Phi^2(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)\alpha\,\varepsilon(\Phi^3)\Psi^3\overset{(3.24)}{=}\\ &\qquad=\overline{s}((\Psi^1)_1\phi^1)\,\overline{\alpha}(\Psi^1)_2\phi^2\beta s(\Psi^2\phi^3)\alpha\,\Psi^3\overset{(3.70)}{=}\\ &\qquad=\overline{s}(\phi^1)\,\overline{\alpha}\phi^2\beta s(\phi^3)\alpha=\\ &\qquad=u\alpha.\end{aligned}$$

On the other hand, the right hand member simplifies to:

$$\overline{s}(\Psi^1)\,\overline{\alpha}\,\Phi^1(\Psi^2)_1\beta s(\Phi^2(\Psi^2)_2)\alpha\Phi^3\Psi^3\overset{(3.71)}{=}\overline{\alpha}\,\Phi^1\beta s(\Phi^2)\alpha\Phi^3\overset{(3.72)}{=}\overline{\alpha}.$$

Analogously, applying

$$\xi'\colon A\otimes A\otimes A\otimes A\longrightarrow A\colon a\otimes b\otimes c\otimes d\longmapsto a\,\overline{\beta}\,\overline{s}(b)\,\overline{\alpha}\,c\beta s(d)$$

to both sides of (3.19) in the form:

$$(1\otimes\Phi^{-1})(A\otimes A\otimes\Delta)(\Phi)(\Delta\otimes A\otimes A)(\Psi)=(A\otimes\Delta\otimes A)(\Phi)(\Psi\otimes 1)$$

leads us to $\overline{\beta}u=\beta$. Finally, let us prove that u is invertible. We claim that

$$v:=s(\phi^1)\alpha\phi^2\,\overline{\beta}\,\overline{s}(\phi^3)$$

is the inverse element to u. Indeed:

$$uv=us(\phi^1)\alpha\phi^2\,\overline{\beta}\,\overline{s}(\phi^3)\overset{(3.76)}{=}\overline{s}(\phi^1)u\alpha\phi^2\,\overline{\beta}\,\overline{s}(\phi^3)\overset{(3.76)}{=}\overline{s}(\phi^1)\,\overline{\alpha}\,\phi^2\,\overline{\beta}\,\overline{s}(\phi^3)\overset{(3.73)}{=}1$$

and

$$vu=s(\phi^1)\alpha\phi^2\,\overline{\beta}\,\overline{s}(\phi^3)u=s(\phi^1)\alpha\phi^2\,\overline{\beta}us(\phi^3)=s(\phi^1)\alpha\phi^2\beta s(\phi^3)\overset{(3.73)}{=}1.$$

□

Remark 3.3.10. Note that (3.75) implies that $\varepsilon(\alpha)=\varepsilon(\beta)^{-1}$, so that we can always assume, without loss of generality, that $\varepsilon(\alpha)=1=\varepsilon(\beta)$. In fact, $(s,\varepsilon(\alpha)^{-1}\alpha,\beta\,\varepsilon(\beta)^{-1})$ is still a quasi-antipode (for example, by Proposition 3.3.9).

In spite of what happens with quasi-antipodes, it turns out that the preantipode is unique.

Theorem 3.3.11. *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. If there exists a preantipode S for A, then it is unique.*

Proof. Assume that S and T are both preantipodes for A. Then we know that the adjunction (F, G, η, ϵ) defined in Theorem 3.2.13 is an equivalence and the unit η is a natural isomorphism. Moreover, in view of Theorem 3.2.20, the maps

$$\tilde{\sigma}_M : \frac{M}{MA^+} \longrightarrow M : \overline{m} \longmapsto \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3$$
$$\tilde{\tau}_M : \frac{M}{MA^+} \longrightarrow M : \overline{m} \longmapsto \Phi^1 \cdot m_0 \cdot T(\Phi^2 m_1)\Phi^3$$

satisfies (iii) and (iv) of Theorem 3.2.14, so that, by $(3) \Rightarrow (1)$ in the proof of the same theorem:

$$\mu_M \circ (\tilde{\sigma}_M \otimes A) = \eta_M^{-1} = \mu_M \circ (\tilde{\tau}_M \otimes A). \tag{3.77}$$

Now, recall that we can construct a preantipode R for A from $\hat{\eta}_A^{-1}$. In particular, in view of (3.59) and (3.63), for all $a \in A$:

$$R(a) = \beta(\overline{1 \otimes a}) = (A \otimes \varepsilon)(\hat{\eta}_A^{-1}(\overline{1 \otimes a} \otimes 1)).$$

If we use as $\hat{\eta}_A^{-1}$ the one given by the left hand equality in (3.77):

$$\begin{aligned} R(a) &= (A \otimes \varepsilon)(\hat{\eta}_A^{-1}(\overline{1 \otimes a} \otimes 1)) \overset{(3.65)}{=} \\ &= (A \otimes \varepsilon)(S(\phi^1 a)\phi^2 \otimes \phi^3) = \\ &= S(a) \end{aligned}$$

for all $a \in A$. Analogously, using the right hand side of (3.77), one finds out that $R(a) = T(a)$ for every $a \in A$, so that $S = T$. $\square$

Next, recall that we know that twisting a bialgebra A by an ordinary invertible element $F \in A \otimes A$ takes us out of the class of ordinary bialgebras. Even if we consider a gauge transformation F as defined in Definition 3.1.13, we do not get back a bialgebra, since we introduce an associativity deficit

$$\Phi = (1 \otimes F)(A \otimes \Delta)(F)(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1),$$

as we observed in Remark 3.1.14. This implies that, unless we strengthen the hypothesis on F, we cannot hope to find a Hopf algebra by twisting another Hopf algebra. On the other hand, this associativity deficit is not a problem if we start from a quasi-bialgebra instead of an ordinary bialgebra, as we saw in Theorem 3.1.6. The following propositions show that both quasi-Hopf algebras and quasi-bialgebras with preantipode behave well with respect to gauge twisting.

Proposition 3.3.12. ([Dr1, Remark 5 on page 1425]) *Let $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a quasi-Hopf algebra and $F = F^1 \otimes F^2$ be a gauge transformation on H with inverse $F^{-1} = f^1 \otimes f^2$. Set*

$$\alpha_F := s(f^1)\alpha f^2 \qquad \text{and} \qquad \beta_F := F^1 \beta s(F^2).$$

Then $(H_F, m, u, \Delta_F, \varepsilon, \Phi_F, s, \alpha_F, \beta_F)$ is a quasi-Hopf algebra.

Proof. We already know that $(H_F, m, u, \Delta_F, \varepsilon, \Phi_F)$ is a quasi-bialgebra, in view of Theorem 3.1.6 and Remark 3.1.14. Thus let us show that (s, α_F, β_F) is a quasi-antipode.

In what follows three copies of F and F^{-1} are required, let us denote with $E = E^1 \otimes E^2$ and $G = G^1 \otimes G^2$ other two copies of F and with $E^{-1} = e^1 \otimes e^2$ and $G^{-1} = g^1 \otimes g^2$ other two copies of F^{-1}.

By definition of Δ_F we have that:

$$\Delta_F(h) = F\Delta(h)F^{-1} = F^1 h_1 f^1 \otimes F^2 h_2 f^2$$

for all $h \in H$. Hence:

$$\begin{aligned} s(F^1 h_1 f^1)\alpha_F F^2 h_2 f^2 &= s(F^1 h_1 f^1)s(g^1)\alpha g^2 F^2 h_2 f^2 = \\ &= s(g^1 F^1 h_1 f^1)\alpha g^2 F^2 h_2 f^2 = \\ &= s(h_1 f^1)\alpha h_2 f^2 = s(f^1)s(h_1)\alpha h_2 f^2 \overset{(3.70)}{=} \\ &= s(f^1)\alpha f^2 \varepsilon(h) = \alpha_F \varepsilon(h) \end{aligned}$$

moreover:

$$\begin{aligned} F^1 h_1 f^1 \beta_F s(F^2 h_2 f^2) &= F^1 h_1 f^1 G^1 \beta s(G^2) s(F^2 h_2 f^2) = \\ &= F^1 h_1 f^1 G^1 \beta s(F^2 h_2 f^2 G^2) = \\ &= F^1 h_1 \beta s(F^2 h_2) = F^1 h_1 \beta s(h_2) s(F^2) \overset{(3.71)}{=} \\ &= F^1 \beta s(F^2)\varepsilon(h) = \beta_F \varepsilon(h) \end{aligned}$$

Furthermore, by definition of Φ_F we have that:

$$\begin{aligned} \Phi_F &= (1 \otimes G)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(G^{-1} \otimes 1) = \\ &= F^1 \Phi^1 (f^1)_1 g^1 \otimes G^1 (F^2)_1 \Phi^2 (f^1)_2 g^2 \otimes G^2 (F^2)_2 \Phi^3 f^2. \end{aligned}$$

Therefore

$$\begin{aligned} &\Phi_F^1 \beta_F s(\Phi_F^2)\alpha_F \Phi_F^3 = \\ &\quad = F^1 \Phi^1 (f^1)_1 g^1 E^1 \beta s(E^2) s(G^1 (F^2)_1 \Phi^2 (f^1)_2 g^2) s(e^1)\alpha e^2 G^2 (F^2)_2 \Phi^3 f^2 = \\ &\quad = F^1 \Phi^1 (f^1)_1 g^1 E^1 \beta s(e^1 G^1 (F^2)_1 \Phi^2 (f^1)_2 g^2 E^2)\alpha e^2 G^2 (F^2)_2 \Phi^3 f^2 = \\ &\quad = F^1 \Phi^1 (f^1)_1 \beta s((F^2)_1 \Phi^2 (f^1)_2)\alpha (F^2)_2 \Phi^3 f^2 \overset{(3.71)}{=} \\ &\quad = F^1 \Phi^1 \varepsilon(f^1)\beta s((F^2)_1 \Phi^2)\alpha (F^2)_2 \Phi^3 f^2 \overset{(3.70)}{=} \\ &\quad = F^1 \Phi^1 \varepsilon(f^1)\beta s(\Phi^2)\alpha \varepsilon(F^2)\Phi^3 f^2 \overset{(3.18)}{=} \\ &\quad = \Phi^1 \beta s(\Phi^2)\alpha \Phi^3 \overset{(3.72)}{=} 1 \end{aligned}$$

Finally, from:

$$\begin{aligned} \Phi_F^{-1} &= (G \otimes 1)(\Delta \otimes A)(F)\Phi^{-1}(A \otimes \Delta)(F^{-1})(1 \otimes G^{-1}) = \\ &= G^1 (F^1)_2 \phi^1 f^1 \otimes G^2 (F^1)_2 \phi^2 (f^2)_1 g^1 \otimes F^2 \phi^3 (f^2)_2 g^2 \end{aligned}$$

it follows that:

$$
\begin{aligned}
&s(\phi_F^1)\alpha_F\phi_F^2\beta_F s(\phi_F^3) = \\
&\quad = s(G^1(F^1)_2\phi^1 f^1)s(e^1)\alpha e^2G^2(F^1)_2\phi^2(f^2)_1g^1E^1\beta s(E^2)s(F^2\phi^3(f^2)_2g^2) = \\
&\quad = s(e^1G^1(F^1)_2\phi^1 f^1)\alpha e^2G^2(F^1)_2\phi^2(f^2)_1g^1E^1\beta s(F^2\phi^3(f^2)_2g^2E^2) = \\
&\quad = s((F^1)_2\phi^1 f^1)\alpha(F^1)_2\phi^2(f^2)_1\beta s(F^2\phi^3(f^2)_2) \overset{(3.70)}{=} \\
&\quad = s(\phi^1 f^1)\alpha\varepsilon(F^1)\phi^2(f^2)_1\beta s(F^2\phi^3(f^2)_2) \overset{(3.71)}{=} \\
&\quad = s(\phi^1 f^1)\alpha\varepsilon(F^1)\phi^2\varepsilon(f^2)\beta s(F^2\phi^3) \overset{(3.18)}{=} \\
&\quad = s(\phi^1)\alpha\phi^2\beta s(\phi^3) \overset{(3.73)}{=} 1
\end{aligned}
$$

□

Quasi-Hopf algebras then are a larger class than ordinary Hopf algebras, but one that is closed under gauge twisting. In the same way, quasi-bialgebras with preantipode are an even larger class, but still closed under gauge twisting.

Proposition 3.3.13. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and $F \in A \otimes A$ be a gauge transformation on A. Define, for $a \in A$,*

$$S_F(a) := F^1S(f^1aF^2)f^2.$$

Then $(A_F, m, u, \Delta_F, \varepsilon, \Phi_F, S_F)$ is a quasi-bialgebra with preantipode.

Proof. As above, we already know that $(A_F, m, u, \Delta_F, \varepsilon, \Phi_F)$ is a quasi-bialgebra, so that it is enough to show that S_F satisfies (3.45), (3.46) and (3.47) of Definition 3.2.16.

With the same notation used for proving Proposition 3.3.12, we have that

$$
\begin{aligned}
F^1b_1f^1S_F(aF^2b_2f^2) &= F^1b_1f^1G^1S(g^1aF^2b_2f^2G^2)g^2 = \\
&= F^1b_1S(g^1aF^2b_2)g^2 \overset{(3.45)}{=} \\
&= F^1S(g^1aF^2)g^2\varepsilon(b) = S_F(a)\varepsilon(b), \\
S_F(F^1a_1f^1b)F^2a_2f^2 &= G^1S(g^1F^1a_1f^1bG^2)g^2F^2a_2f^2 = \\
&= G^1S(a_1f^1bG^2)a_2f^2 \overset{(3.46)}{=} \\
&= \varepsilon(a)G^1S(f^1bG^2)f^2 = \\
&= \varepsilon(a)S_F(b)
\end{aligned}
$$

and, finally, that

$$
\begin{aligned}
\Phi_F^1S_F(\Phi_F^2)\Phi_F^3 &= F^1\Phi^1(f^1)_1g^1E^1S\left(e^1G^1(F^2)_1\Phi^2(f^1)_2g^2E^2\right)e^2G^2(F^2)_2\Phi^3f^2 = \\
&= F^1\Phi^1(f^1)_1S\left((F^2)_1\Phi^2(f^1)_2\right)(F^2)_2\Phi^3f^2 \overset{(3.45)}{=} \\
&= F^1\Phi^1\varepsilon(f^1)S\left((F^2)_1\Phi^2\right)(F^2)_2\Phi^3f^2 \overset{(3.46)}{=} \\
&= F^1\Phi^1\varepsilon(f^1)S\left(\Phi^2\right)\varepsilon(F^2)\Phi^3f^2 \overset{(3.18)}{=} \\
&= \Phi^1S\left(\Phi^2\right)\Phi^3 \overset{(3.47)}{=} 1
\end{aligned}
$$

□

Remark 3.3.14. Assume that $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a quasi-Hopf algebra and that F is a gauge transformation on H. By Theorem 3.3.4, we can twist H via F, and then consider it as a quasi-bialgebra with preantipode, obtaining

$$(H_F, m, u, \Delta_F, \varepsilon, \Phi_F, \widetilde{S}),$$

where

$$\widetilde{S}(\cdot) = \beta_F s(\cdot)\alpha_F.$$

Or, we can first consider it as a quasi-bialgebra with preantipode $S(\cdot) = \beta s(\cdot)\alpha$ and then twist it via F. What we get is exactly the same quasi-bialgebra with preantipode, since for all $h \in H$:

$$\begin{aligned}\widetilde{S}(h) = \beta_F s(h)\alpha_F &= F^1\beta s(F^2)s(h)s(f^1)\alpha f^2 = \\ &= F^1\beta s(f^1hF^2)\alpha f^2 = F^1 S(f^1hF^2)f^2 = \\ &= S_F(h)\end{aligned}$$

The following lemma comes from [Ma, Example 2.4.1] and it is retrieved here because it has been the inspiration of the result in Remark 3.3.17.

Lemma 3.3.15. *Let $(H, m, u, \Delta, \varepsilon, s)$ be an ordinary Hopf algebra. Let $\Phi \in H \otimes H \otimes H$ be an invertible element that satisfies (3.19), (3.20) and (3.21), that is:*

- $(H \otimes H \otimes \Delta)(\Phi)(\Delta \otimes H \otimes H)(\Phi) = (1 \otimes \Phi)(H \otimes \Delta \otimes H)(\Phi)(\Phi \otimes 1)$
- $(H \otimes \varepsilon \otimes H)(\Phi) = 1 \otimes 1$
- $(H \otimes \Delta)(\Delta(h))\Phi = \Phi(\Delta \otimes H)(\Delta(h))$*, for all* $h \in H$*.*

Assume that $c := \Phi^1 s(\Phi^2)\Phi^3 \in H$ is invertible. Let $\beta = \left(\Phi^1 s(\Phi^2)\Phi^3\right)^{-1}$ and $\alpha = 1$. Then $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a quasi-Hopf algebra and $\beta \in \mathcal{Z}(H)$ where $\mathcal{Z}(H)$ is the center of H, i.e., $\mathcal{Z}(H) := \{h \in H \mid hl = lh, \forall\, l \in H\}$. Furthermore, $(H, m, u, \Delta, \varepsilon, \Phi, S)$ is a quasi-bialgebra with preantipode defined by $S(h) = \beta s(h)$, for each $h \in H$.

Proof. Obviously, if Φ satisfies (3.19), (3.20) and (3.21), then $(H, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra.

Let us show that $c \in Z(H)$. Consider (3.21) and apply

$$m \circ (m \otimes H) \circ (H \otimes s \otimes H)$$

to both sides:

$$h_1\Phi^1 s(\Phi^2)\underbrace{s(h_2)h_3}_{\varepsilon(h_2)}\Phi^3 = \Phi^1\underbrace{h_1 s(h_2)}_{\varepsilon(h_1)}s(\Phi^2)\Phi^3 h_3,$$

from which we conclude that $hc = ch$ for all $h \in H$.

Consider (3.19) in the form:

$$(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes A \otimes A)(\Psi)(\Phi^{-1} \otimes 1) = (1 \otimes \Phi)(A \otimes \Delta \otimes A)(\Psi),$$

where again $\Psi = \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \Phi$, and apply $m \circ (m \otimes m) \circ (s \otimes A \otimes s \otimes A)$ to both sides:

$$s(\phi^1)s((\Psi^1)_1)s(\Phi^1)\Phi^2(\Psi^1)_2\phi^2 s(\phi^3)s(\Psi^2)\overbrace{s((\Phi^3)_1)(\Phi^3)_2}^{\varepsilon(\Phi^3)}\Psi^3 = \\ = s(\Psi^1)\Phi^1\underbrace{(\Psi^2)_1 s((\Psi^2)_2)}_{\varepsilon(\Psi^2)}s(\Phi^2)\Phi^3\Psi^3.$$

Simplifying in view of (3.24):

$$\Phi^1 s(\Phi^2)\Phi^3 = s(\phi^1)\underbrace{s((\Psi^1)_1)(\Psi^1)_2}_{\varepsilon(\Psi^1)}\phi^2 s(\phi^3)s(\Psi^2)\Psi^3 = s(\phi^1)\phi^2 s(\phi^3). \tag{3.78}$$

Now, let us use these two properties to verify that the axioms of a quasi-Hopf algebra are satisfied:

$$s(a_1)\alpha a_2 = s(a_1)a_2 = \varepsilon(a)1 = \varepsilon(a)\alpha$$
$$a_1\beta s(a_2) = a_1 s(a_2)\beta = \varepsilon(a)\beta$$
$$\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = \beta\Phi^1 s(\Phi^2)\Phi^3 = 1$$

What is left is (3.73) but, in view of (3.78):

$$s(\phi^1)\alpha\phi^2\beta s(\phi^3) = \beta s(\phi^1)\phi^2 s(\phi^3) = \beta\left(\Phi^1 s(\Phi^2)\Phi^3\right) = 1$$

and (3.73) follows. □

Remark 3.3.16. The previous example allows us to observe that S is not, in general, an antiendomorphism of algebras, since:

$$S(ab) = \beta s(ab) = \beta s(b)s(a) = \beta s(b)\beta^{-1}\beta s(a) = \beta^{-1}S(b)S(a),$$

nor an antiendomorphism of coalgebras, since:

$$S(a_2)\otimes S(a_1) = \beta s(a_2)\otimes\beta s(a_1) \overset{(2.9c)}{=} (\beta\otimes\beta)(\Delta(s(a))) \neq \beta_1 s(a)_1\otimes\beta_2 s(a)_2 = \Delta(\beta s(a)).$$

Actually, in this particular situation, it depends on β.

Remark 3.3.17. Inspired by the following observations (that come out from Lemma 3.3.15):

$$\beta = S(1) \qquad s(\cdot) = \beta^{-1}S(\cdot),$$

we tried to prove the converse of that lemma and we came to formulate the following result:

Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. If S satisfies:

- *$S(1)$ is invertible in A,*
- *$S(ab) = S(b)S(1)^{-1}S(a)$ for all $a, b \in A$,*

then $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ *is a quasi-Hopf algebra with* $\alpha = 1$*,* $\beta = S(1)$ *and* $s(a) = S(1)^{-1}S(a)$*, for all* $a \in A$*.*
Indeed, we have that s is an antiendomorphism of A:

$$s(ab) = \beta^{-1}S(ab) = \beta^{-1}S(b)\beta^{-1}S(a) = s(b)s(a)$$
$$s(u(k)) = s(k1) = \beta^{-1}S(k1) = \beta^{-1}\beta k = k1 = u(k)$$

and the axioms of quasi-Hopf algebra hold:

- $s(a_1)\alpha a_2 = \beta^{-1}S(a_1)a_2 = \beta^{-1}\varepsilon(a)S(1) = \varepsilon(a)1 = \varepsilon(a)\alpha$
- $a_1\beta s(a_2) = a_1S(a_2) = \varepsilon(a)S(1) = \varepsilon(a)\beta$
- $\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = \Phi^1S(\Phi^2)\Phi^3 = 1$
- $s(\phi^1)\alpha\phi^2\beta s(\phi^3) = \beta^{-1}S(\phi^1)\phi^2S(\phi^3) = 1$

3.4 The other way round: from preantipodes to quasi-antipodes

Even though we claimed that a quasi-bialgebra with preantipode is not, in general, a quasi-Hopf algebra, there exist partial converses to Theorem 3.3.4. The subsequent proposition retrieves an easy one. We will conclude this last section with a result, due to Schauenburg, that proves that in the finite dimensional case these two concepts are equivalent and with some examples in which this equivalence is explicit.

Proposition 3.4.1. *Let* $(A, m, u, \Delta, \varepsilon, \Phi, S)$ *be a quasi-bialgebra with preantipode. If* Φ *is in the center of* $A \otimes A \otimes A$*, then* $(A, m, u, \Delta, \varepsilon, s)$ *is an ordinary Hopf algebra where*

$$s(a) = \Phi^1S(a\Phi^2)\Phi^3, \tag{3.79}$$

for all $a \in A$*. Furthermore* $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ *is a quasi-Hopf algebra with* $\alpha = 1$ *and* $\beta = S(1)$*. Moreover one has*

$$S(a) = \beta s(a) \tag{3.80}$$

for all $a \in A$*.*

Proof. In view of (3.22) and (3.23), we know that ε is a counit for Δ. Moreover, commutativity of Φ ensures that Δ is coassociative. Indeed, by (3.21):

$$(\Delta \otimes A)(\Delta(a)) = \Phi^{-1}((A \otimes \Delta)(\Delta(a)))\Phi = ((A \otimes \Delta)(\Delta(a)))\Phi^{-1}\Phi = (A \otimes \Delta)(\Delta(a))$$

for every $a \in A$, so that $(A, m, u, \Delta, \varepsilon)$ is an ordinary bialgebra. Let us show that s is an antipode:

$$(s * \mathrm{Id})(a) = s(a_1)a_2 = \Phi^1S(a_1\Phi^2)\Phi^3a_2 \overset{(*)}{=} \Phi^1S(a_1\Phi^2)a_2\Phi^3 \overset{(3.46)}{=}$$
$$= \Phi^1S(\Phi^2)\Phi^3\,\varepsilon(a) \overset{(3.47)}{=} (u \circ \varepsilon)(a)$$

where in $(*)$ we used

$$(\Phi^1 \otimes a_1 \otimes \Phi^2 \otimes \Phi^3 a_2) = (\Phi^1 \otimes a_1 \otimes \Phi^2 \otimes a_2 \Phi^3). \tag{3.81}$$

Analogously:

$$\begin{aligned}(\mathrm{Id} * s)(a) = a_1 s(a_2) = a_1 \Phi^1 S(a_2 \Phi^2)\Phi^3 &\overset{(**)}{=} \Phi^1 a_1 S(\Phi^2 a_2)\Phi^3 \overset{(3.45)}{=} \\ &= \Phi^1 S(\Phi^2)\Phi^3 \, \varepsilon(a) \overset{(3.47)}{=} (u \circ \varepsilon)(a)\end{aligned}$$

where $(**)$ follows from

$$(\Delta(a) \otimes 1)\Phi = \Phi(\Delta(a) \otimes 1).$$

Hence $(A, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra. Moreover:

$$S(a) = S(a_1 \varepsilon(a_2)) = S(a_1)\varepsilon(a_2) \overset{(\circ)}{=} S(a_1) a_2 s(a_3) \overset{(3.46)}{=} S(1)\varepsilon(a_1) s(a_2) = \beta s(a).$$

where in $(\circ)$ we used that $\mathrm{Id} * s = u \circ \varepsilon$ and coassociativity of Δ to renumber. Now, let us show that $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a quasi-Hopf algebra:

- We know that s is an antiendomorphism of A, since it is an ordinary antipode.
- Since $\alpha = 1$, we have $s(a_1)\alpha a_2 = s(a_1) a_2 = \varepsilon(a)1 = \varepsilon(a)\alpha$
- In view of (3.80), $a_1 \beta s(a_2) = a_1 S(a_2) \overset{(3.46)}{=} \varepsilon(a) S(1) = \varepsilon(a)\beta$.
- Again, by (3.80) $\Phi^1 \beta s(\Phi^2)\alpha \Phi^3 = \Phi^1 S(\Phi^2)\Phi^3 = 1$.
- In order to prove the remaining identity, first of all apply

  $$m \circ (m \otimes A) \circ (S \otimes A \otimes S)$$

  to both sides of:

  $$(\Delta \otimes A)(\Delta(a))\Phi^{-1} \overset{(3.21)}{=} \Phi^{-1}(A \otimes \Delta)(\Delta(a))$$

  and simplify using (3.45) and (3.46) to get that

  $$S(\phi^1)\phi^2 S(a\phi^3) = S(\phi^1 a)\phi^2 S(\phi^3) \tag{3.82}$$

  for all $a \in A$. Now:

  $$\begin{aligned} s(\phi^1)\alpha\phi^2\beta s(\phi^3) = s(\phi^1)\phi^2 S(\phi^3) &\overset{(3.79)}{=} \Phi^1 S(\phi^1 \Phi^2)\Phi^3 \phi^2 S(\phi^3) \overset{(\blacktriangle)}{=} \\ &= \Phi^1 S(\phi^1 \Phi^2)\phi^2 S(\phi^3)\Phi^3 \overset{(3.82)}{=} \\ &= \Phi^1 S(\phi^1)\phi^2 S(\Phi^2 \phi^3)\Phi^3 \overset{(\triangle)}{=} \\ &= \Phi^1 S(\phi^1)\phi^2 S(\phi^3) s(\Phi^2)\Phi^3 \overset{(3.48)}{=} \\ &= \Phi^1 S(1) s(\Phi^2)\Phi^3 \overset{(3.80)}{=} \Phi^1 S(\Phi^2)\Phi^3 \overset{(3.47)}{=} 1 \end{aligned}$$

where in ($\triangle$) we used: $S(ab) = \beta s(ab) = \beta s(b)s(a) = S(b)s(a)$ and in ($\blacktriangle$) we used (3.81) again, with $a = \phi^2 S(\phi^3)$. $\square$

Corollary 3.4.2. (Dual to [AP2, Theorem 2.16]) *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. If A is commutative, then all the conclusions of Proposition 3.4.1 hold for A. In particular, it is an ordinary Hopf algebra.*

Remark 3.4.3. We can deduce from the proof of Proposition 3.4.1 that if $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra and $\Phi \in \mathcal{Z}(A \otimes A \otimes A)$, then A is an ordinary bialgebra. Observe also that, since $S(1)$ is not 1 in general, we do not have that

$$(A, m, u, \Delta, \varepsilon, 1 \otimes 1 \otimes 1, s, 1, S(1))$$

is a quasi-Hopf algebra, as one can expect. Otherwise, in light of Proposition 3.3.9, we should have that $(s, 1, 1)$ and $(s, 1, S(1))$ are connected by an invertible element (necessarily 1) and thus should coincide.

Let us retrieve now a less trivial result, due to Schauenburg, that states that, at least in the finite dimensional case, the existence of a preantipode is equivalent to the existence of a quasi-antipode.

Theorem 3.4.4. ([Sc1, Theorem 3.1]) *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a finite dimensional quasi-bialgebra. The following are equivalent:*

(1) A is a quasi-Hopf algebra,

(2) the adjunction (F, G, η, ϵ) is a category equivalence.

Proof. (1) $\Rightarrow$ (2) is Corollary 3.3.6. Hence, let us prove (2) $\Rightarrow$ (1). Recall that the map

$$\begin{array}{rccc} \hat{\eta}_A := \eta_{A\hat{\otimes}A} : & A\hat{\otimes}A & \longrightarrow & \dfrac{A\hat{\otimes}A}{(A\hat{\otimes}A)A^+} \otimes A \\ & a \otimes b & \longmapsto & \overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3 \end{array}$$

defined in (3.56) is an isomorphism of quasi-Hopf A-bimodules and it is also A-linear with respect to the left A-module structures given by multiplication on the first factor (recall (3.60)). Hence we have an isomorphism:

$$\hat{\eta} : \ ({}_\bullet A_\bullet \otimes {}_\circ A_\bullet^\bullet) \longrightarrow {}_\bullet \frac{(A \otimes A)}{(A \otimes A)A^+}{}_\circ \otimes {}_\bullet A_\bullet^\bullet.$$

Since A is finite dimensional as $\Bbbk$-vector space, we have that $A \cong \Bbbk^{\dim(A)}$. If we define $\overline{A \otimes A} := \frac{(A\otimes A)}{(A\otimes A)A^+}$, we get (recalling that the tensor product distributes over the direct sum) ${}_\bullet\overline{A \otimes A}^{\dim(A)} \cong A^{\dim(A)}$, so that we have ${}_\bullet\overline{A \otimes A} \cong A$ as left A-modules by Krull-Schmidt.

Indeed, note that both A and $\overline{A \otimes A}$ are finite dimensional $\Bbbk$-vector spaces. Moreover, every A-submodule is, in particular, a $\Bbbk$-vector subspace, so that they are both Artinian and Noetherian as A-modules and we can apply Corollary 4.1.13.

Now, our target is to show that, at least in the finite dimensional case and theoretically, a quasi-antipode could be constructed from $\hat{\eta}$.

In order to do this, pick an isomorphism $\widetilde{\gamma}\colon \overline{A\otimes A}\to A$ of left A-modules and define $\gamma(a)=\widetilde{\gamma}(\overline{1\otimes a})$, for every $a\in A$. Observe that

$$a_1\gamma(a_2)=\widetilde{\gamma}\left(\overline{a_1\otimes a_2}\right)=\widetilde{\gamma}\left(\overline{(1\otimes 1)\cdot a}\right)=\widetilde{\gamma}\left(\overline{(1\otimes 1)\,\varepsilon(a)}\right)=\varepsilon(a)\,\gamma(1). \tag{3.83}$$

Furthermore, since $\widetilde{\gamma}$ is A-linear with respect to the left A-action given by multiplication on the first factor, we have that, for all $a,b\in A$:

$$\widetilde{\gamma}(\overline{a\otimes b})=\widetilde{\gamma}(a\cdot\overline{1\otimes b})=a\gamma(b). \tag{3.84}$$

Next, $A\otimes A$ comes with a natural structure of left $A\otimes A$-module given by its algebra structure:

$$\begin{array}{rccc} \mu_{A\otimes A}: & (A\otimes A)\otimes(A\otimes A) & \longrightarrow & A\otimes A \\ & (a\otimes b)\otimes(x\otimes y) & \longmapsto & ax\otimes by \end{array}$$

which induces a left $A\otimes A$-module structure on $\overline{A\otimes A}$ (since $(A\otimes A)A^+$ is a left $A\otimes A$-submodule of $A\otimes A$ itself, w.r.t this left action). Let us indicate with $\mu_{\overline{A\otimes A}}$ this last one:

$$\begin{array}{rccc} \mu_{\overline{A\otimes A}}: & (A\otimes A)\otimes\left(\overline{A\otimes A}\right) & \longrightarrow & \overline{A\otimes A} \\ & (a\otimes b)\otimes(\overline{x\otimes y}) & \longmapsto & \overline{ax\otimes by} \end{array} \tag{3.85}$$

Since $\widetilde{\gamma}$ is bijective, we can fill in the following diagram in such a way that it becomes commutative and we get an $A\otimes A$-module structure on A:

$$\begin{tikzcd} A\otimes A\otimes\overline{A\otimes A} \arrow[r, "A\otimes A\otimes\widetilde{\gamma}"] \arrow[d, "\mu_{\overline{A\otimes A}}"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A\otimes A \arrow[d, dashed, "\mu_A"] \\ \overline{A\otimes A} \arrow[r, "\widetilde{\gamma}"'] & A \end{tikzcd} \tag{3.86}$$

If we indicate with $\overline{a^1\otimes a^2}$ the preimage $\widetilde{\gamma}^{-1}(a)$, we have that

$$a=\widetilde{\gamma}(\widetilde{\gamma}^{-1}(a))=\widetilde{\gamma}(\overline{a^1\otimes a^2})\overset{(3.84)}{=}a^1\gamma(a^2)\qquad(\forall\, a\in A) \tag{3.87}$$

and, if we indicate with $(x\otimes y)\triangleright a:=\mu_A((x\otimes y)\otimes a)$ the left $A\otimes A$-action we get from (3.86), we can write, for all $x,y,a\in A$

$$\begin{aligned} (x\otimes y)\triangleright a=\mu_A((x\otimes y)\otimes a)&\overset{(3.86)}{=}\widetilde{\gamma}\left(\mu_{\overline{A\otimes A}}\left((x\otimes y)\otimes\overline{a^1\otimes a^2}\right)\right)\overset{(3.85)}{=}\\ &=\widetilde{\gamma}(\overline{xa^1\otimes ya^2})\overset{(3.84)}{=}xa^1\gamma(ya^2). \end{aligned} \tag{3.88}$$

Note that A gains also two left A-module structures via the algebra maps

$$\begin{aligned} i_1: \ A\longrightarrow A\otimes A: \ a\longmapsto a\otimes 1 \\ i_2: \ A\longrightarrow A\otimes A: \ a\longmapsto 1\otimes a \end{aligned}$$

The action of the left tensor factor is given by:

$$x\cdot a:=(x\otimes 1)\triangleright a\overset{(3.88)}{=}xa^1\gamma(a^2)\overset{(3.87)}{=}xa$$

for all $a, x \in A$, i.e. the regular module structure of A. Now, observe that as $\widetilde{\gamma}$ is A-linear with respect to the left A-action given by the multiplication on the first factor, even $\widetilde{\gamma}^{-1}$ is. Therefore, we have:

$$\begin{tikzcd}
a \otimes \overline{b^1 \otimes b^2} \arrow[d, mapsto, "\cdot"'] & & a \otimes b \arrow[ll, mapsto, "A \otimes \widetilde{\gamma}^{-1}"'] \arrow[d, mapsto, "\cdot"] \\
\overline{ab^1 \otimes b^2} \arrow[r, phantom, "\circlearrowleft" near end, shift left=4ex] & & ab \arrow[ll, mapsto, "\widetilde{\gamma}^{-1}"]
\end{tikzcd}$$

for every $a, b \in A$. This implies that, evaluating at $b = 1$:

$$a \cdot \overline{1^1 \otimes 1^2} = \overline{a^1 \otimes a^2} \qquad (\forall\, a \in A) \tag{3.89}$$

and that (3.87) becomes:

$$a = a1^1\gamma(1^2) \tag{3.90}$$

Define, for each $y \in A$,

$$s(y) := y \bullet 1, \tag{3.91}$$

where $y \bullet a := (1 \otimes y) \triangleright a$ for all $a, y \in A$. We have that

$$s(y) \overset{(3.88)}{=} 1^1\gamma(y1^2) \tag{3.92}$$

and

$$\begin{aligned} y \bullet a = (1 \otimes y) \triangleright a = \widetilde{\gamma}((1 \otimes y) \cdot (\overline{a^1 \otimes a^2})) &\overset{(3.89)}{=} \\ = \widetilde{\gamma}((1 \otimes y) \cdot (\overline{a1^1 \otimes 1^2})) \overset{(3.85)}{=} \widetilde{\gamma}((\overline{a1^1 \otimes y1^2})) &\overset{(3.84)}{=} \\ = a1^1\gamma(y1^2) \overset{(3.92)}{=} as(y). & \end{aligned} \tag{3.93}$$

Thus, by definition of s, we can deduce also that, for all $a, y \in A$,

$$y \bullet a = a(y \bullet 1) \tag{3.94}$$

Let us show that s defined above is an algebra antiendomorphism on A:

$$s(ab) = ab \bullet 1 = a \bullet (b \bullet 1) \overset{(3.94)}{=} (b \bullet 1)(a \bullet 1) = s(b)s(a),$$
$$s(1) \overset{(3.92)}{=} 1^1\gamma(1^2) \overset{(3.87)}{=} 1.$$

Hence we have the candidate quasi-antipode that we were looking for: s. What is left is to find α and β and to verify the axioms of a quasi-Hopf algebra. Until now we proved only that there exists an antiendomorphism s such that the action of the right tensor factor has the form:

$$y \bullet a = as(y).$$

Observe that, for all $a, b, x \in A$

$$\begin{aligned} a\gamma(xb) \overset{(3.84)}{=} \widetilde{\gamma}(\overline{a \otimes xb}) \overset{(3.85)}{=} \widetilde{\gamma}((1 \otimes x) \cdot (\overline{a \otimes b})) = \\ \overset{(3.86)}{=} (1 \otimes x) \triangleright \widetilde{\gamma}(\overline{a \otimes b}) \overset{(3.93)}{=} \widetilde{\gamma}(\overline{a \otimes b})s(x) = \\ \overset{(3.84)}{=} a\gamma(b)s(x) \end{aligned}$$

In particular, evaluating in $a = b = 1$, we have

$$\gamma(x) = \beta s(x) \tag{3.95}$$

where $\beta := \gamma(1)$. Define

$$\theta := (\widetilde{\gamma} \otimes A) \circ \hat{\eta} : \ A \otimes A \longrightarrow A \otimes A. \tag{3.96}$$

We find that:

$$\begin{aligned}\theta(a \otimes b) = (\widetilde{\gamma} \otimes A)(\hat{\eta}(a \otimes b)) &\overset{(3.56)}{=} (\widetilde{\gamma} \otimes A)(\overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3) \overset{(3.84)}{=} \\ = a\Phi^1\gamma(b_1\Phi^2) \otimes b_2\Phi^3 &\overset{(3.95)}{=} a\Phi^1\beta s(b_1\Phi^2) \otimes b_2\Phi^3.\end{aligned}$$

Note that

$$\theta \colon {}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet \longrightarrow {}_\bullet({}_sA)_\circ \otimes {}_\bullet A_\bullet^\bullet \colon a \otimes b \longmapsto a\Phi^1\beta s(b_1\Phi^2) \otimes b_2\Phi^3 \tag{3.97}$$

is a morphism in ${}_A\mathcal{M}_A^A$ with the indicated structures, where ${}_sA$ denotes the left A-module structure on A given by s, that is:

$$x \bullet a = as(x)$$

for all $a, x \in A$. Indeed, if we let $a \otimes b$ vary in $A \otimes A$:

$$\begin{aligned}\theta(x \cdot (a \otimes b)) = \theta(a \otimes xb) &\overset{(3.97)}{=} a\Phi^1\beta s(x_1b_1\Phi^2) \otimes x_2b_2\Phi^3 = \\ &= a\Phi^1\beta s(b_1\Phi^2)s(x_1) \otimes x_2b_2\Phi^3 = \\ &= x_1 \bullet (a\Phi^1\beta s(b_1\Phi^2)) \otimes x_2b_2\Phi^3 = \\ &= x \cdot (a\Phi^1\beta s(b_1\Phi^2)) \otimes b_2\Phi^3) \overset{(3.97)}{=} x \cdot \theta(a \otimes b),\end{aligned} \tag{3.98}$$

$$\begin{aligned}\theta((a \otimes b) \cdot x) = \theta(ax_1 \otimes bx_2) &\overset{(3.97)}{=} ax_1\Phi^1\beta s(b_1(x_2)_1\Phi^2)) \otimes b_2(x_2)_2\Phi^3 \overset{(3.21)}{=} \\ &= a\Phi^1(x_1)_1\beta s(b_1\Phi^2(x_1)_2)) \otimes b_2\Phi^3x_2 = \\ &= a\Phi^1(x_1)_1\beta s((x_1)_2)s(b_1\Phi^2) \otimes b_2\Phi^3x_2 \overset{(3.83)}{=} \\ &= a\Phi^1\,\varepsilon(x_1)\,\beta s(b_1\Phi^2) \otimes b_2\Phi^3x_2 = \\ &= a\Phi^1\beta s(b_1\Phi^2) \otimes b_2\Phi^3x \overset{(3.97)}{=} \theta(a \otimes b) \cdot x\end{aligned} \tag{3.99}$$

for all $a, b, x \in A$ and the right colinearity follows from the fact that $\hat{\eta}$ is right colinear and $\widetilde{\gamma} \otimes A$ does not affect the two rightmost factors. In addition, θ is an A-module map with respect to the left A-module structures given by the regular action of A on the left tensor factor. Indeed, for all $a, b, x \in A$:

$$\theta(x \star (a \otimes b)) = \theta(xa \otimes b) \overset{(3.97)}{=} xa\Phi^1\beta s(b_1\Phi^2) \otimes b_2\Phi^3 \overset{(3.97)}{=} x \star \theta(a \otimes b). \tag{3.100}$$

We may summarize the three variants of A-linearity in the formula:

$$\begin{aligned}\theta((x \otimes y)\xi(z_1 \otimes z_2)) &\overset{(3.100)}{=} (x \otimes 1)\theta((1 \otimes y)\xi(z_1 \otimes z_2)) \overset{(3.99)}{=} \\ &= (x \otimes 1)\theta((1 \otimes y)\xi)(1 \otimes z) \overset{(3.98)}{=} (x \otimes y_2)\theta(\xi)(s(y_1) \otimes z)\end{aligned} \tag{3.101}$$

for $x, y, z \in A$ and $\xi \in A \otimes A$. Now we are ready to show that the axioms of quasi-Hopf algebra are satisfied. Let us begin with (3.71): by (3.83) we know that $a_1\gamma(a_2) = \varepsilon(a)\,\beta$ and by (3.95) $\gamma(x) = \beta s(x)$, so that

$$a_1\beta s(a_2) = \varepsilon(a)\,\beta \qquad (\forall\, a \in A)\,.$$

Next, observe that since θ is A-linear with respect to the three A-actions, the same holds true for θ^{-1}, so that:

$$\theta^{-1}(a \otimes b) = \theta^{-1}((a \otimes 1)(1 \otimes 1)(1 \otimes b)) = (a \otimes 1)\theta^{-1}(1 \otimes 1)(b_1 \otimes b_2), \tag{3.102}$$

then we set $\alpha := (A \otimes \varepsilon)(\theta^{-1}(1 \otimes 1))$, and find:

$$(A \otimes \varepsilon)(\theta^{-1}(a \otimes b)) = (A \otimes \varepsilon)((a \otimes 1)\theta^{-1}(1 \otimes 1)(b_1 \otimes b_2)) = a\alpha b \tag{3.103}$$

for all $a, b \in A$. This implies further

$$\begin{aligned} s(a_1)\alpha a_2 &\overset{(3.103)}{=} (A \otimes \varepsilon)(\theta^{-1}(s(a_1) \otimes a_2)) = \\ &= (A \otimes \varepsilon)(\theta^{-1}((1 \otimes a_2)(1 \otimes 1)(s(a_1) \otimes 1))) = \\ &= (A \otimes \varepsilon)((1 \otimes a)\theta^{-1}(1 \otimes 1)) = \varepsilon(a)\alpha \end{aligned}$$

for all $a \in A$, and

$$\Phi^1\beta s(\Phi^2)\alpha\Phi^3 \overset{(3.103)}{=} (A \otimes \varepsilon)(\theta^{-1}(\Phi^1\beta s(\Phi^2) \otimes \Phi^3)) \overset{(3.97)}{=} (A \otimes \varepsilon)(\theta^{-1}(\theta(1 \otimes 1))) = 1.$$

In order to prove the remaining axiom observe that:

$$\begin{tikzcd} {}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet \arrow[r, equal] \arrow[d, "\rho_{{}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet}"'] & A \otimes A \\ ({}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet) \otimes A \arrow[r, "(A\otimes\varepsilon\otimes A)"'] & A \otimes \Bbbk \otimes A \arrow[u, "A\otimes l_A"'] \end{tikzcd} \qquad \begin{tikzcd} a \otimes b \arrow[r, equal] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & a \otimes b \\ ((a \otimes b_1) \otimes b_2)\Phi \arrow[r, mapsto] & a \otimes 1 \otimes b \arrow[u, mapsto] \end{tikzcd}$$

For the sake of clearness, let us define $\rho := \rho_{{}_\circ A_\bullet \otimes {}_\bullet A_\bullet^\bullet}$ and $\rho' := \rho_{\bullet({}_sA)_\circ \otimes {}_\bullet A_\bullet^\bullet}$. In view of (3.30):

$$\rho'(a \otimes b) = \phi^1 \bullet a \otimes \phi^2 b_1 \otimes \phi^3 b_2 = as(\phi^1) \otimes \phi^2 b_1 \otimes \phi^3 b_2. \tag{3.104}$$

Since θ^{-1} is A-colinear with respect to these two coactions, we can determine explicitly its form:

$$\begin{aligned} \theta^{-1}(a \otimes b) &= (A \otimes \varepsilon \otimes A)(\rho(\theta^{-1}(a \otimes b))) = \\ &= (A \otimes \varepsilon \otimes A)(\theta^{-1} \otimes A)(\rho'(a \otimes b)) \overset{(3.104)}{=} \\ &= (A \otimes \varepsilon \otimes A)(\theta^{-1} \otimes A)(as(\phi^1) \otimes \phi^2 b_1 \otimes \phi^3 b_2) = \\ &= (A \otimes \varepsilon)(\theta^{-1})(as(\phi^1) \otimes \phi^2 b_1) \otimes \phi^3 b_2 \overset{(3.103)}{=} \\ &= as(\phi^1)\alpha\phi^2 b_1 \otimes \phi^3 b_2. \end{aligned}$$

This implies that:

$$s(\phi^1)\alpha\phi^2\beta s(\phi^3) \overset{(*)}{=} (A\otimes\varepsilon)(\theta(s(\phi^1)\alpha\phi^2\otimes\phi^3)) = (A\otimes\varepsilon)(\theta(\theta^{-1}(1\otimes 1))) = 1$$

where in $(*)$ we used:

$$(A\otimes\varepsilon)(\theta(a\otimes b)) \overset{(3.97)}{=} (A\otimes\varepsilon)(a\Phi^1\beta s(b_1\Phi^2)\otimes b_2\Phi^3) = a\beta s(b).$$

This concludes the proof. □

Note that the maps $\widetilde{\gamma}$ and γ that appear in the previous proof remind us of the map β and the preantipode S (compare relations (3.45) and (3.63) with (3.83) and (3.84)). The problem is that β needs not to be bijective, and so cannot be used in lieu of $\widetilde{\gamma}$. Nevertheless, the subsequent result holds. In order to avoid confusion, let us denote with ξ the map β, i.e.,

$$\xi\colon \overline{A\otimes A} \longrightarrow A\colon \overline{a\otimes b} \longmapsto aS(b). \tag{3.105}$$

Corollary 3.4.5. *Let $(A,m,u,\Delta,\varepsilon,\Phi,S)$ be a quasi-bialgebra with preantipode. If ξ as defined in (3.105) is bijective, then A is a quasi-Hopf algebra with quasi-antipode given by $\alpha = 1$, $\beta = S(1)$ and, for all $a\in A$,*

$$s(a) = \xi\left((1\otimes a)\cdot\xi^{-1}(1)\right) = 1^1S(a1^2)$$

where $\overline{1^1\otimes 1^2} = \xi^{-1}(1)$.

Proof. In proving Schauenburg's result 3.4.4 we used the finiteness condition on A just to find an isomorphism $\widetilde{\gamma}\colon \overline{A\otimes A}\to A$. By hypothesis, we already have such an isomorphism:

$$\xi\colon \overline{A\otimes A} \longrightarrow A\colon \overline{a\otimes b} \longmapsto aS(b).$$

Hence, let us substitute this ξ to $\widetilde{\gamma}$ in the previous proof. We get that $\gamma = S$ and $\beta = S(1)$. Moreover, with the same conventions as above (see also (3.88), (3.91) and (3.92)),

$$s(a) = (1\otimes a)\triangleright 1 = \xi\left((1\otimes a)\cdot\xi^{-1}(1)\right) = \xi\left(\overline{1^1\otimes a1^2}\right) = 1^1S(a1^2)$$

and, recalling (3.96),

$$\alpha = (A\otimes\varepsilon)\left(\theta^{-1}(1\otimes 1)\right) = (A\otimes\varepsilon)\left(\hat{\eta}_A^{-1}(\xi^{-1}(1)\otimes 1)\right) \overset{(3.59)}{=} \xi\left(\xi^{-1}(1)\right) = 1$$

as claimed. □

Example 3.4.6. ([EG, Preliminaries 2.3], [BCT, Example 2.5]) Let $C_2 = \langle g\rangle$ be the cyclic group of order 2 with generator g and let $\Bbbk$ be a field of characteristic different from 2. Consider the group algebra $H(2) := \Bbbk C_2$ with bialgebra structure given as in Example 1.3.22, i.e.,

$$\Delta(g) = g\otimes g \qquad \text{and} \qquad \varepsilon(g) = 1.$$

Observe that $H(2)$ is a two dimensional commutative algebra. Now, let us denote by

$$p := \frac{1}{2}(1-g) \tag{3.106}$$

and note that the following relations hold:

$$pg = \frac{1}{2}(1-g)g = \frac{1}{2}(g-1) = -p \tag{3.107a}$$

$$p^2 = \frac{1}{4}(1-2g+g^2) = \frac{1}{2}(1-g) = p \tag{3.107b}$$

$$\Delta(p) = \frac{1}{2}((1\otimes 1) \pm (1\otimes g) - (g\otimes g)) = (1\otimes p) + (p\otimes g) \tag{3.107c}$$

$$\Delta(p) = \frac{1}{2}((1\otimes 1) \pm (g\otimes 1) - (g\otimes g)) = (p\otimes 1) + (g\otimes p) \tag{3.107d}$$

$$\varepsilon(p) = \frac{1}{2}(1-1) = 0 \tag{3.107e}$$

$$(p\otimes p)\Delta(p) = \frac{1}{2}(p\otimes p)((1\otimes 1) - (g\otimes g)) \overset{(3.107a)}{=} \frac{1}{2}((p\otimes p) - (p\otimes p)) = 0 \tag{3.107f}$$

Next, let us introduce the non trivial reassociator

$$\Phi := (1\otimes 1\otimes 1) - 2(p\otimes p\otimes p).$$

Since $H(2)$ is commutative, (3.21) is fulfilled, and since ε is a counit for Δ also (3.22) and (3.23) are satisfied. Moreover, (3.107e) implies that

$$(H(2)\otimes\varepsilon\otimes H(2))(\Phi) = 1\otimes 1.$$

Hence, it remains to verify (3.19). Therefore consider:

$$\begin{aligned}
&(1\otimes\Phi)(H(2)\otimes\Delta\otimes H(2))(\Phi)(\Phi\otimes 1) = \\
&= \begin{bmatrix} ((1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p))\cdot \\ \cdot((1\otimes 1\otimes 1\otimes 1) - 2(p\otimes\Delta(p)\otimes p))\cdot \\ \cdot((1\otimes 1\otimes 1\otimes 1) - 2(p\otimes p\otimes p\otimes 1)) \end{bmatrix} = \\
&= \begin{bmatrix} ((1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p) - 2(p\otimes\Delta(p)\otimes p) + \\ + 4(p\otimes(p\otimes p)\Delta(p)\otimes p))\cdot((1\otimes 1\otimes 1\otimes 1) - 2(p\otimes p\otimes p\otimes 1)) \end{bmatrix} \overset{(3.107f)}{=} \\
&= \begin{bmatrix} (1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p) - 2(p\otimes\Delta(p)\otimes p) + \\ - 2(p\otimes p\otimes p\otimes 1) + 4(p\otimes p\otimes p\otimes p) + 4(p\otimes\Delta(p)(p\otimes p)\otimes p) \end{bmatrix} \overset{\substack{(3.107f)\\(3.107d)}}{=} \\
&= \begin{bmatrix} (1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p) - 2(p\otimes g\otimes p\otimes p) + \\ - 2(p\otimes p\otimes 1\otimes p) - 2(p\otimes p\otimes p\otimes 1) + 4(p\otimes p\otimes p\otimes p) \end{bmatrix} \overset{(3.106)}{=} \\
&= \begin{bmatrix} (1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p) - 2(p\otimes g\otimes p\otimes p) + \\ - 2(p\otimes p\otimes 1\otimes p) - 2(p\otimes p\otimes p\otimes 1) + 2(p\otimes p\otimes 1\otimes p) + \\ - 2(p\otimes p\otimes g\otimes p) \end{bmatrix} = \\
&= \begin{bmatrix} (1\otimes 1\otimes 1\otimes 1) - 2(1\otimes p\otimes p\otimes p) - 2(p\otimes g\otimes p\otimes p) + \\ - 2(p\otimes p\otimes p\otimes 1) - 2(p\otimes p\otimes g\otimes p) \end{bmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
(H(2)\otimes &H(2)\otimes\Delta)(\Phi)(\Delta\otimes H(2)\otimes H(2))(\Phi)= \\
&= ((1\otimes 1\otimes 1\otimes 1)-2(p\otimes p\otimes \Delta(p)))((1\otimes 1\otimes 1\otimes 1)-2(\Delta(p)\otimes p\otimes p))= \\
&= \begin{bmatrix}(1\otimes 1\otimes 1\otimes 1)-2(\Delta(p)\otimes p\otimes p)-2(p\otimes p\otimes\Delta(p))+ \\ +4(\Delta(p)(p\otimes p)\otimes \Delta(p)(p\otimes p))\end{bmatrix} \overset{(3.107f)}{=} \\
&= (1\otimes 1\otimes 1\otimes 1)-2(\Delta(p)\otimes p\otimes p)-2(p\otimes p\otimes\Delta(p)) \overset{(3.107c)}{=} \\
&= \begin{bmatrix}(1\otimes 1\otimes 1\otimes 1)-2(1\otimes p\otimes p\otimes p)+ \\ -2(p\otimes g\otimes p\otimes p)-2(p\otimes p\otimes \Delta(p))\end{bmatrix} \overset{(3.107d)}{=} \\
&= \begin{bmatrix}(1\otimes 1\otimes 1\otimes 1)-2(1\otimes p\otimes p\otimes p)-2(p\otimes g\otimes p\otimes p)+ \\ -2(p\otimes p\otimes p\otimes 1)-2(p\otimes p\otimes g\otimes p)\end{bmatrix}
\end{aligned}
$$

then even (3.19) is fulfilled. Furthermore, note that

$$\Phi^2 = (1\otimes 1\otimes 1)-4(p\otimes p\otimes p)+4(p\otimes p\otimes p)=1\otimes 1\otimes 1$$

and so Φ is invertible. By [EG], $H(2)$ is a quasi-Hopf algebra with quasi-antipode given by $\beta=1$, $\alpha=g$ and $s=\mathrm{Id}_{H(2)}$. However, for the moment we don't care about this. By the contrary, we consider the linear map:

$$S\colon H(2)\longrightarrow H(2)\colon x\longmapsto xg.$$

Write $x=a+bg$ and $y=c+dg$ in $H(2)$, for a, b, c, d in $\Bbbk$. Since

$$\Delta(x)=\Delta(a+bg)=a(1\otimes 1)+b(g\otimes g),$$

we have that

$$
\begin{gathered}
S(x_1y)x_2 = x_1ygx_2 = ayg+bgygg=(a+b)yg=\varepsilon(x)S(y) \\
y_1S(xy_2)=y_1xy_2g=cxg+dgxgg=(c+d)xg=\varepsilon(y)S(x) \\
\Phi^1S(\Phi^2)\Phi^3 = g-2ppgp \overset{(3.107a)}{=} g+2p=1
\end{gathered}
$$

and then S is a preantipode. As above consider the map ξ of (3.105):

$$\xi\colon \overline{H(2)\otimes H(2)}\longrightarrow H(2)\colon \overline{x\otimes y}\longmapsto xyg.$$

We have that ξ is surjective, because $x=\xi(\overline{x\otimes g})$ for each $x\in H(2)$. Hence it is bijective and so we can construct a quasi-antipode by virtue of Corollary 3.4.5. What we find is $\alpha=1$, $\beta=S(1)=g$ and

$$s(x)=1^1S(x1^2)$$

where $\overline{1^1\otimes 1^2}=\xi^{-1}(1)$. However, we can give an explicit inverse for ξ. Indeed, consider the function

$$\psi\colon H(2)\longrightarrow \overline{H(2)\otimes H(2)}\colon x\longmapsto \overline{x\otimes g}.$$

By composing with ξ we find:

$$x\xrightarrow{\psi}\overline{x\otimes g}\xrightarrow{\xi} xgg=x$$

and since we know that ξ is invertible, we have that $\psi = \xi^{-1}$. Therefore,

$$\overline{1^1 \otimes 1^2} = \xi^{-1}(1) = \psi(1) = \overline{1 \otimes g}$$

and $s(x) = S(xg) = xgg = x$. Finally, by recalling that g is trivially invertible, it is easy to see that g itself plays the role of u of Proposition 3.3.9 and thus we recovered the structure given previously.

Remark 3.4.7. Observe that, once we have proven that $H(2)$ is a quasi-bialgebra with preantipode S, we can come to the same conclusions of Example 3.4.6 by simply applying Corollary 3.4.2.

In [EG] is claimed that $H(2)$ is not twist equivalent to an ordinary Hopf algebra. We can give an alternative proof to this claim, based on the theory we developed and the following lemma.

Lemma 3.4.8. *Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \Phi_A)$ and $(B, m_B, u_B, \Delta_B, \varepsilon_B, \Phi_B)$ be isomorphic quasi-bialgebra, via the isomorphism $\varphi \colon A \to B$, and assume that A admits a preantipode S. Then B admits a preantipode.*

Proof. Since φ is invertible, we can define a preantipode for B by setting $T := \varphi \circ S \circ \varphi^{-1}$. Indeed:

$$\begin{aligned} T\left(x_1 y\right) x_2 &= \varphi\left(S\left(\varphi^{-1}\left(x_1 y\right)\right)\right) x_2 = \varphi\left(S\left(\varphi^{-1}\left(x_1 y\right)\right) \varphi^{-1}\left(x_2\right)\right) \overset{(*)}{=} \\ &= \varphi\left(S\left(\varphi^{-1}\left(x_1\right) \varphi^{-1}\left(y\right)\right) \varphi^{-1}\left(x_2\right)\right) \overset{(*)}{=} \\ &= \varphi\left(S\left(\left(\varphi^{-1}(x)\right)_1 \varphi^{-1}(y)\right) \left(\varphi^{-1}(x)\right)_2\right) \overset{(3.46)}{=} \\ &= \varphi\left(S\left(\varphi^{-1}(y)\right) \varepsilon_A\left(\varphi^{-1}(x)\right)\right) \overset{(*)}{=} T(y)\, \varepsilon_B(x), \end{aligned}$$

$$\begin{aligned} y_1 T(x y_2) &= y_1 \varphi\left(S\left(\varphi^{-1}(x y_2)\right)\right) \overset{(*)}{=} \varphi\left(\varphi^{-1}(y_1) S\left(\varphi^{-1}(x) \varphi^{-1}(y_2)\right)\right) \overset{(*)}{=} \\ &= \varphi\left(\left(\varphi^{-1}(y)\right)_1 S\left(\varphi^{-1}(x) \left(\varphi^{-1}(y)\right)_2\right)\right) \overset{(3.45)}{=} T(x)\, \varepsilon_B(x) \end{aligned}$$

and

$$\begin{aligned} \Phi_B^1 T(\Phi_B^2) \Phi_B^3 &= \varphi\left(\varphi^{-1}\left(\Phi_B^1\right) S\left(\varphi^{-1}(\Phi_B^2)\right) \varphi^{-1}(\Phi_B^3)\right) \overset{(*)}{=} \\ &= \varphi\left(\Phi_A^1 S(\Phi_A^2) \Phi_A^3\right) \overset{(3.47)}{=} \varphi(1) \overset{(*)}{=} 1 \end{aligned}$$

where in $(*)$ we used the fact that φ and φ^{-1} are morphisms of quasi-bialgebras. □

Now, assume by contradiction that there exists a Hopf algebra $(H, m_H, u_H, \Delta_H, \varepsilon_H, s)$ and a gauge transformation F on $H(2)$ such that $H \cong H(2)_F$ via an isomorphism of quasi-bialgebras that we can denote again by $\varphi \colon H(2)_F \to H$. In view of Proposition 3.3.13, the twisted S_F is a preantipode for $H(2)_F$ and by Lemma 3.4.8 we have that $T := \varphi \circ S_F \circ \varphi^{-1}$ is a preantipode for H. However, a preantipode for an ordinary Hopf algebra with trivial associator $1 \otimes 1 \otimes 1$ should be an ordinary antipode, by Proposition 3.4.1, and so T should coincide with s, but

$$T(1) = \varphi\left(S_F\left(\varphi^{-1}(1)\right)\right) = \varphi\left(F^1 f^1 F^2 g f^2\right) = \varphi(g) \neq 1 = s(1).$$

Contradiction.

Going back to Theorem 3.4.4, we should observe that it states that, at least in the finite dimensional case, there is a close connection between the preantipode and the quasi-antipode, but this connection is not explicit because of the Krull-Schmidt Theorem. Indeed, we have no informations about the isomorphism $\widetilde{\gamma}$ and we don't know how to relate it with the preantipode, in general. Corollary 3.4.5 retrieves, in particular, what it seems to be a limited family of quasi-bialgebras with preantipode for which it is possible to recover an explicit relation with the quasi-Hopf algebra structure (as the one of Example 3.4.6). Let us show briefly that it is actually a large class of quasi-Hopf algebras.

Let $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a finite dimensional quasi-Hopf algebra. Then we know, by Theorem 3.3.4, that H admits a preantipode $S(\cdot) := \beta s(\cdot)\alpha$ and so the Structure Theorem holds for the quasi-Hopf H-bimodules. Applying Schauenburg's result 3.4.4 we get, a posteriori, a quasi-antipode (s', α', β') for H such that the morphism $\widetilde{\gamma}(\overline{x \otimes y}) = x\beta' s'(y)$ is invertible. By Proposition 3.3.9 there exists an invertible element $u \in H$ such that (s, α, β) and (s', α', β') are connected by relations (3.76). In particular, if α is invertible, then also α' is invertible. By the way, note that s', α' and β' are not known to us, since they are obtained by $\widetilde{\gamma}$.

Next, assume that α is invertible in H. Hence

$$\xi \colon \overline{H \otimes H} \longrightarrow H : \overline{x \otimes y} \longmapsto xS(y) = x\beta' s'(y)\alpha' = \widetilde{\gamma}(\overline{x \otimes y})\alpha' \tag{3.108}$$

is invertible with 'explicit' inverse given by

$$\xi^{-1}(h) := \widetilde{\gamma}^{-1}\left(h(\alpha')^{-1}\right) = \widetilde{\gamma}^{-1}\left(h\alpha^{-1}u^{-1}\right). \tag{3.109}$$

Thus we can apply Corollary 3.4.5. This implies that, if α is invertible, it is always possible to recover explicitly the quasi-antipode from the preantipode, at least theoretically. It is just a question of finding an explicit inverse to the map ξ, that we know it is invertible.

There is even something more that we can say in this situation. Indeed, recall relations (3.88) and (3.93). These together implies that for all $x, y, h \in H$

$$\widetilde{\gamma}\left((x \otimes y) \cdot \widetilde{\gamma}^{-1}(h)\right) = xhs'(y). \tag{3.110}$$

Now, denote with $(\widehat{s}, \widehat{\alpha}, \widehat{\beta})$ the quasi-antipode that we get from the last corollary. If we write it down *a posteriori*, we find

$$\widehat{\alpha} = 1, \quad \widehat{\beta} = S(1) = \beta\alpha \qquad \text{and} \tag{3.111a}$$

$$\begin{aligned} \widehat{s}(h) &= \xi\left((1 \otimes h) \cdot \xi^{-1}(1)\right) \stackrel{(3.109)}{=} \xi\left((1 \otimes h) \cdot \widetilde{\gamma}^{-1}\left(\alpha^{-1}u^{-1}\right)\right) \stackrel{(3.108)}{=} \\ &= \widetilde{\gamma}\left((1 \otimes h) \cdot \widetilde{\gamma}^{-1}\left(\alpha^{-1}u^{-1}\right)\right) u\alpha \stackrel{(3.110)}{=} \alpha^{-1}u^{-1}s'(h)u\alpha \stackrel{(3.76)}{=} \\ &= \alpha^{-1}s(h)\alpha \end{aligned} \tag{3.111b}$$

for each $h \in H$, i.e., the same quasi-antipode that we would get from relations (3.76) with $u = \alpha^{-1}$.

Observe that in this setting fall all finite dimensional Hopf algebras, the quasi-Hopf algebras $H(2)$, $H_{\pm}(8)$ and $H(32)$ of [EG], the twisted quantum doubles $D^{\omega}(G)$ introduced by Dijkgraaf, Pasquier and Roche (cfr. [DPR], [Ka, Section XV.5], [CP, Chapter 16]), the basic quasi-Hopf algebras $A(q)$ of [Ge].

In order to find interesting examples of the relation that intervenes between quasi-antipodes and preantipodes, one should look for a quasi-Hopf algebra that is finite dimensional and such that α is not invertible. Unfortunately, it will not be enough to twist a quasi-Hopf algebra with trivial α (let us call *trivial* α when it is invertible) via a gauge transformation F, as the following remark shows.

Remark 3.4.9. Let $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ be a (finite dimensional) quasi-Hopf algebra with α invertible. We have the preantipode $S(\cdot) = \beta s(\cdot)\alpha$ and the quasi-antipode $(\widehat{s}, \widehat{\alpha}, \widehat{\beta})$ obtained from S with the same steps as above. Let $F \in H \otimes H$ be a gauge transformation on H and consider the quasi-antipode (s, α_F, β_F) as defined in Proposition 3.3.12. In general, $\alpha_F = s(f^1)\alpha f^2$ needs not to be invertible.

Nevertheless, consider the preantipode $S_F(\cdot) = \beta_F s(\cdot)\alpha_F$ (it is effectively the twisting of S, as we observed in Remark 3.3.14) and denote by $E = E^1 \otimes E^2$ and $G = G^1 \otimes G^2$ other two copies of F. We have that, for all $h \in H$:

$$\begin{aligned} \widehat{s}(h) &= 1^1 S(h1^2) = 1^1 \left(S_F\right)_{F^{-1}}(h1^2) = 1^1 f^1 S_F(F^1 h1^2 f^2)F^2, \\ \widehat{\alpha}_F &= \widehat{s}(g^1)\widehat{\alpha} g^2 = 1^1 f^1 S_F(F^1 g^1 1^2 f^2)F^2 g^2 = 1^1 f^1 S_F(1^2 f^2), \\ \widehat{\beta}_F &= G^1\, \widehat{\beta}\, \widehat{s}(G^2) = G^1 S(1)\widehat{s}(G^2) = G^1 e^1 S_F(E^1 e^2)E^2 1^1 f^1 S_F(F^1 G^2 1^2 f^2)F^2 \end{aligned}$$

and this quasi-antipode $(\widehat{s}, \widehat{\alpha}_F, \widehat{\beta}_F)$ on H_F is written 'explicitly' using just F, S_F and $\xi^{-1}(1)$. Furthermore, it is connected to (s, α_F, β_F) by relations (3.76) where α^{-1} plays the role of u. Indeed, recalling relations (3.111) we have that:

$$\begin{aligned} \widehat{s}(\cdot) &= \alpha^{-1} s(\cdot)\alpha, \\ \widehat{\alpha}_F &= \widehat{s}(f^1)f^2 = \alpha^{-1}s(f^1)\alpha f^2 = \alpha^{-1}\alpha_F, \\ \widehat{\beta}_F &= F^1\beta\alpha\widehat{s}(F^2) = F^1\beta s(F^2)\alpha = \beta_F\alpha. \end{aligned}$$

Chapter 4

Appendix

4.1 The Krull Schmidt Theorem

This appendix is devoted to the proof of the Krull Schmidt Theorem, that states that every module that is both Artinian and Noetherian admits a unique decomposition into indecomposable components, up to isomorphism. For an exhaustive treatment, we refer to [AF, Section 12] and [Ja, Section 3.4].

In what follows, A will always denote a (unital) associative ring. By 'module' we will always mean left A-module. Recall that a module M is said to be an *(internal) direct sum* of the submodules $M_1, M_2, \ldots, M_n$ if these satisfies:

$$M = M_1 + M_2 + \cdots M_n \tag{4.1}$$

$$M_s \cap (M_1 + \cdots + M_{s-1} + M_{s+1} + \cdots + M_n) = \{0\} \quad (\forall\, s = 1, \ldots, n) \tag{4.2}$$

and we denote it by

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Observe that (4.1) and (4.2) implies that every element m of M can be written in one and only one way in the form

$$m = m_1 + m_2 + \cdots + m_n$$

for $m_i \in M_i$. Indeed, suppose that $m_1 + m_2 + \cdots + m_n = 0$ for some $m_i \in M_i$, $i = 1, \ldots, n$. Then

$$m_i = -(m_1 + \cdots + m_{i-1} + m_{i+1} + \cdots + m_n) \in M_i \cap (M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n)$$

so that $m_i = 0$ for all $i = 1, \ldots, n$ and this is enough to show that the claim holds.

Hence we have injections $j_s \colon M_s \to M$ for all $s \in \{1, \ldots, n\}$, that are just the usual injections, and projections:

$$\pi_s \colon M \longrightarrow M_s \colon m = m_1 + m_2 + \cdots + m_n \longmapsto m_s.$$

Observe that $\pi_s \circ i_s = \mathrm{Id}_{M_s} =: 1_s$, for all $s \in \{1, \ldots, n\}$.

Let us denote with $\mathrm{End}(M) := \hom_A(M, M)$ and with $1_M := \mathrm{Id}_M \in \mathrm{End}(M)$ or simply 1, if it clear the identity of which object it is. The same for the 0 morphism.

Consider the compositions $e_s := j_s \circ \pi_s$ for each $s = 1, \dots, n$. These are idempotent, pairwise orthogonal, endomorphisms of M. Indeed, we have that

$$e_s^2 = j_s \circ \pi_s \circ j_s \circ \pi_s = j_s \circ \pi_s = e_s.$$

Moreover, if $s \neq t$, then

$$e_s(e_t(m)) = e_s(m_t) = 0,$$

for all $m \in M$. Let us call them *projections* and forget about the previous projections π_s. Note that M_s is the image of e_s for each s. Thus every internal direct summand of M is the image of an idempotent endomorphism of M. Also the converse holds, as the following lemma shows.

Lemma 4.1.1. *Let e be an idempotent in End(M). Then $1 - e$ is an idempotent in End(M), orthogonal to e and such that*

$$M = e(M) \oplus (1 - e)(M).$$

Proof. Actually, it is easy to see that:

$$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$$

and that

$$e(1 - e) = e - e^2 = 0 = (1 - e)e.$$

Moreover, every $m \in M$ can be written as

$$m = e(m) + (1 - e)(m)$$

and if there exists elements $m, n \in M$ such that $e(n) = (1 - e)(m) \in e(M) \cap (1 - e)(M)$, then

$$e(n) = e^2(n) = e((1 - e)(m)) = 0.$$

□

Furthermore, note that since e_s is idempotent for all $s = 1, \dots, n$, then it is the identity of M_s when restricted to it.

Definition 4.1.2. (Decomposable and indecomposable modules)
An A-module $M \neq 0$ is said to be *decomposable* if there exist submodules M_1, M_2 such that $M = M_1 \oplus M_2$ and $M_i \neq 0$ for $i = 1, 2$. Otherwise, it is said to be *indecomposable.*

Proposition 4.1.3. *A module $M \neq 0$ is indecomposable if and only if End(M) contains no idempotent except the trivial ones: $0, 1_M$.*

Proof. Assume that M is decomposable, $M = M_1 \oplus M_2$, and consider the projections e_1 and e_2. These are idempotents in End(M). Moreover, since $M_i \neq 0$, also $e_i \neq 0$ for both $i, j = 1, 2$. Observe that, if $e_1 = 1_M$, then $e_2 = e_2 \circ 1_M = e_2 \circ e_1 = 0$, that is impossible by what we have just said. Thus $e_1 \neq 1_M$ and analogously $e_2 \neq 1_M$.

On the other hand, suppose that End(M) contains an idempotent $e \neq 0, 1$. Then also $1 - e$ is an idempotent different from $0, 1$ and M is decomposable by Lemma 4.1.1. □

Definition 4.1.4. (Local rings)
A ring A is called *local* if the set of non units forms additive group. Equivalently, A is said to be local in case for each pair $a, b \in A$ if $a+b$ is invertible, then either a or b is invertible.

Remark 4.1.5. There's plenty of characterizations of local rings. For example, A is local if and only if for each $a \in A$, either a or $1-a$ is invertible in A, if and only if $\mathrm{Jac}(A) = \{a \in A \mid a \text{ is not invertible}\}$, where $\mathrm{Jac}(A)$ denotes the *Jacobson radical* of A. For the moment, we can simply observe that if A is local, then it admits only the trivial idempotents $0, 1$. Indeed, if e is an idempotent different from $0, 1$, then also $1-e$ is idempotent and neither e nor $1-e$ is invertible. The interested reader may refer to [AF, Section 15]

Observe also that if $\mathrm{End}(M)$ for a module $M \neq 0$ is local, then M is indecomposable by Proposition 4.1.3. This justifies the following definition.

Definition 4.1.6. (Strongly indecomposable modules)
A module M is said to be *strongly indecomposable* if $M \neq 0$ and $\mathrm{End}(M)$ is local.

Lemma 4.1.7. *Let M and N be modules such that N is indecomposable and $M \neq 0$. Let $f\colon M \to N$ and $g\colon N \to M$ be homomorphisms such that $g \circ f$ is an automorphism of M. Then f and g are isomorphisms.*

Proof. Let us simplify the notation by setting $g \circ f = gf$ and let $h\colon M \to M$ be the inverse of gf. Then $hgf = 1_M$. Set $l = hg$. We find that $e = fl\colon N \to N$ is idempotent, since $e^2 = flfl = f1_M l = e$, thus $e = 0, 1$ because N is indecomposable. Since $e = 0$ would imply that $1_M = 1_M^2 = lflf = lef = 0$, we have that $e = 1$ and both f and g are then invertibles. □

Theorem 4.1.8. ([Ja, Theorem 3.6]) *Let*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \tag{4.3}$$
$$N = N_1 \oplus N_2 \oplus \cdots \oplus N_m \tag{4.4}$$

where the M_i are strongly indecomposable for $i = 1, \ldots, n$ and the N_j are indecomposable for $j = 1, \ldots, m$ and suppose $M \cong N$. Then $m = n$ and there is a permutation σ of the indexes j such that $M_i \cong N_{\sigma(i)}$ for $1 \leq i \leq n$.

Proof. We proceed by induction on n. If $n = 1$, then M is indecomposable and N is indecomposable, too. Thus all the N_j are 0 except one, that we may assume to be N_1.

Now assume that $n > 1$. Let $e_1, \ldots, e_n$ be the projections defined by the decomposition (4.3) of M, and let $f_1, \ldots, f_m$ be those determined by the decomposition (4.4) of N. Let also $g\colon M \to N$ be an isomorphism and set

$$h_j := f_j g e_1 \qquad k_j := e_1 g^{-1} f_j, \qquad (1 \leq j \leq m).$$

Observe that $\sum_{j=1}^m f_j = 1_N$ by definition of the projections, thus

$$\sum_{j=1}^m k_j h_j = \sum_{j=1}^m e_1 g^{-1} f_j f_j g e_1 = e_1 g^{-1} \left(\sum_{j=1}^m f_j \right) g e_1 = e_1.$$

Now, the restrictions of e_1 and k_jh_j to M_1 are endomorphisms of M_1. Let us indicate them with e_1' and $(k_jh_j)'$ respectively. We already know that $e_1' = 1_{M_1}$, hence we have that

$$\sum_{j=1}^{m}(k_jh_j)' = 1_{M_1}.$$

However, $\operatorname{End}(M_1)$ is local and so at least one of the $(k_jh_j)'$ is invertible. Reordering the indexes j we can assume that $(k_1h_1)'$ is an automorphism of M_1.

Restricting both h_1 and k_1 to M_1 and N_1 respectively, we find that $h_1' \colon M_1 \to N_1$ and $k_1' \colon N_1 \to M_1$ are morphisms such that $k_1'h_1' = (k_1h_1)'$ is an automorphism of M_1 and N_1 is an indecomposable module. By Lemma 4.1.7 we have that they actually are isomorphisms between M_1 and N_1.

Next, let us prove that

$$M = g^{-1}(N_1) \oplus (M_2 + \cdots + M_n).$$

Let $x \in g^{-1}(N_1) \cap (M_2 + \cdots + M_n)$. Since $x \in M_2 + \cdots + M_n$, we have that $e_1(x) = 0$. On the other hand, $x = g^{-1}(y) = g^{-1}f_1(y)$, so that

$$0 = e_1(x) = e_1g^{-1}f_1(y) = k_1(y) = k_1'(y).$$

Hence $y = 0$ because k_1' is an isomorphism and consequently $x = 0$. Now set

$$M' := g^{-1}(N_1) + M_2 + \cdots + M_n$$

and pick $x \in g^{-1}(N_1)$. We have that x, $e_2(x)$, $e_3(x), \ldots, e_n(x)$ are all in M', but we also have that $x = \sum_{i=1}^{n} e_i(x)$, so that $e_1(x) \in M'$. This implies that

$$M' \supseteq e_1(g^{-1}(N_1)) = e_1g^{-1}f_1(N_1) = k_1(N_1) = k_1'(N_1) = M_1$$

and so $M' \supseteq M$.

Finally, since g is an isomorphism from M to N, it clearly maps $g^{-1}(N_1)$ onto N_1, so that it induces an isomorphism

$$\tilde{g} \colon \frac{M}{g^{-1}(N_1)} \xrightarrow{\sim} \frac{N}{N_1}.$$

This implies that we have isomorphisms:

$$N_2 \oplus N_3 \oplus \cdots \oplus N_m \cong \frac{N}{N_1} \cong \frac{M}{g^{-1}(N_1)} \cong M_2 \oplus M_3 \oplus \cdots \oplus M_n.$$

Now, we conclude by induction. □

Observe that the hypothesis of the previous theorem are quite strong. Recall where we would like to come: we would say that a certain family of modules admits a unique decomposition into indecomposable components (up to isomorphism, of course). The result we have just proven is practically what we need, but we are requesting that at least one decomposition exists and that it is composed by strongly indecomposable submodules. We are going to show now that modules that are both Artinian and Noetherian satisfy all these conditions. However, before going on, we need a lemma.

Lemma 4.1.9. (Fitting's Lemma) *Let M be a module and $f \in End(M)$ be an endomorphism of M. Set $f^n := f \circ f \circ \cdots \circ f$ for n times. Note that we have a descending chain:*

$$M \supset f(M) \supset f^2(M) \supset \cdots$$

Moreover, if $f^n(m) = 0$, then also $f^{n+1}(m) = 0$, so that we have an ascending chain:

$$0 \subset \ker(f) \subset \ker(f^2) \subset \cdots$$

Define $f^\infty(M) = \bigcap_{n=1}^\infty f^n(M)$ and $f^{-\infty}0 = \bigcup_{n=1}^\infty \ker(f^n)$. If M is both Artinian and Noetherian, then we have the Fitting decomposition

$$M = f^\infty(M) \oplus f^{-\infty}0.$$

Moreover, the restriction of f to $f^\infty(M)$ is an automorphism and the restriction of f to $f^{-\infty}0$ is nilpotent.

Proof. Since M is Artinian, there is an integer s such that

$$f^s(M) = f^{s+1}(M) = \cdots = f^\infty(M).$$

Let $m \in M$. Since $f^s(M) = f^{2s}(M)$, there exists an $n \in M$ such that $f^s(m) = f^{2s}(n)$, which implies also that $f^s(m - f^s(n)) = 0$. Therefore, we can write m as

$$m = f^s(n) + (m - f^s(n)) \in f^s(M) + \ker(f^s)$$

and so

$$M = f^s(M) + \ker(f^s).$$

Moreover, since M is Noetherian, there exists an integer t such that

$$\ker(f^t) = \ker(f^{t+1}) = \cdots = f^{-\infty}0.$$

If $m \in f^t(M) \cap \ker(f^t)$, then there exists a n in M such that $m = f^t(n)$ and we have that $0 = f^t(m) = f^{2t}(n)$. Hence, $n \in \ker(f^{2t}) = \ker(f^t)$ and $m = 0$.

Let $r = \max(s,t)$, so that $f^\infty(M) = f^r(M)$ and $f^{-\infty}0 = \ker(f^r)$. By the previous observations we can conclude that

$$M = f^r(M) \oplus \ker(f^r).$$

Now, indicate with f' the restriction of f to $f^\infty(M) = f^r(M)$ and let $p \in f^r(M)$ be such that $f'(p) = 0$. Since $p \in f^r(M)$, there exists $m \in M$ such that $p = f^r(m)$. On the other hand, $p \in \ker(f')$, so that $0 = f'(p) = f(p) = f^{r+1}(m)$ and then $m \in \ker(f^{r+1}) = \ker(f^r)$. This implies that actually $p = 0$ and f' is injective. Moreover, $f'(f^r(M)) = f^{r+1}(M) = f^r(M)$ so that $f' \colon f^r(M) \to f^r(M)$ is an automorphism, as claimed. Finally, f^r restricted to $\ker(f^r)$ is identically 0. That means that f restricted to $f^{-\infty}0$ is nilpotent. □

The following corollary is the analogue of Schur's Lemma for indecomposable modules that are both Artinian and Noetherian.

Corollary 4.1.10. *Let M be an indecomposable module that is both Artinian and Noetherian. Then End(M) is a local ring and M is strongly indecomposable.*

Proof. Let $f \in \operatorname{End}(M)$ be an endomorphism of M. Since M is indecomposable, by Fitting's Lemma we have that $M = f^{\infty}(M)$ or $M = f^{-\infty}0$. In the first case, f is an automorphism (recall that f restricted to $f^{\infty}(M)$ is always an automorphism). In the second case, it is nilpotent. To show that $\operatorname{End}(M)$ is local, it is enough to show that the sum of two non invertible elements is still non invertible. Hence, let $f, g \in \operatorname{End}(M)$ be nilpotents (otherwise they should be invertibles). If h is any other endomorphism, gh is again nilpotent, since it cannot be invertible. Indeed, assume that n is the minimum integer such that $g^n = 0$ and that there exists $k \in \operatorname{End}(M)$ such that $ghk = 1_M$. Then we should have $0 = g^n hk = g^{n-1}$, which contradict our choice of n.

Now, consider $f + g$ and assume, by contradiction, that it is invertible. Hence we have $h \in \operatorname{End}(M)$ such that $1 = (f+g)h = fh + gh$. However, gh is nilpotent. Therefore, there exists an n such that $(gh)^n = 0$ and so f is invertible, since:

$$\begin{aligned}(1 + gh + (gh)^2 + \cdots + (gh)^{n-1})f &= (1 - (gh)^n) = \\ = 1 = (1 - (gh)^n) &= (1 - gh)(1 + gh + (gh)^2 + \cdots + (gh)^{n-1}) = \\ &= f(1 + gh + (gh)^2 + \cdots + (gh)^{n-1}).\end{aligned}$$

Contradiction. □

Proposition 4.1.11. ([AF, Proposition 10.14]) *Let $M \neq 0$ be a module that is either Artinian or Noetherian. Then M is the direct sum*

$$M = M_1 \oplus \cdots \oplus M_n$$

of a finite set of indecomposable submodules.

Proof. If M is indecomposable, then the claim holds and there's nothing to prove. Therefore we can assume that M is decomposable and, by contradiction, that it does not have a finite indecomposable decomposition. Choose a proper decomposition

$$M = M_1 \oplus N_1$$

such that N_1 has no finite indecomposable decomposition. Inductively,

$$N_1 = M_2 \oplus N_2, \quad N_2 = M_3 \oplus N_3, \quad \ldots$$

so that we can construct two infinite chains:

$$M_1 \subset M_1 \oplus M_2 \subset M_1 \oplus M_2 \oplus M_3 \subset \cdots$$

and

$$N_1 \supset N_2 \supset N_3 \supset \cdots$$

contradicting Artinianity and Noetherianity. □

Now, the following theorem is just an immediate consequence of the previous results.

Theorem 4.1.12. (Krull-Schmidt Theorem)
Let M be a module that is both Artinian and Noetherian and let

$$N_1 \oplus N_2 \oplus \cdots \oplus N_m = M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

be two indecomposable decompositions of M. Then $n = m$ and there exists a permutation σ of the indexes i such that

$$M_i \cong N_{\sigma(i)}$$

for $1 \leq i \leq n$.

Proof. By Proposition 4.1.11, M admits a finite indecomposable decomposition. If it admits two,

$$N_1 \oplus N_2 \oplus \cdots \oplus N_m = M = M_1 \oplus M_2 \oplus \cdots \oplus M_n,$$

then we can apply Corollary 4.1.10 to obtain that the M_i are strongly indecomposable and then apply Theorem 4.1.8 to obtain uniqueness, as desired. □

Corollary 4.1.13. *Let M and N be two Artinian and Noetherian modules. Assume that there exists a positive integer n such that $M^n \cong N^n$. Then $M \cong N$.*

Proof. By Proposition 4.1.11 we can write

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_s$$

and

$$N = N_1 \oplus N_2 \oplus \cdots \oplus N_t.$$

Since the n-th power just represents the direct sum of n copies, we have that:

$$M_1^n \oplus M_2^n \oplus \cdots \oplus M_s^n \cong N_1^n \oplus N_2^n \oplus \cdots \oplus N_t^n. \tag{4.5}$$

Therefore Krull-Schmidt Theorem implies that:

- $ns = nt$, so that $s = t$;
- there exists an index j such that $M_1 \cong N_j$. We can assume without loss of generality that $j = 1$;
- killing all the occurrences of M_1 and N_1 from (4.5), we can apply again Krull-Schmidt Theorem to find out that $M_2 \cong N_2$ and so on.

To conclude, we have that $M_i \cong N_i$ for all $i = 1, \ldots, s$ and so $M \cong N$. □

Bibliography

[Ab] E. Abe, *Hopf algebras*. Cambridge University Press, Cambridge, 1980.

[AF] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*. Graduate Texts in Mathematics, **13**. Springer-Verlag, New York, 1973.

[AP1] A. Ardizzoni, A. Pavarin, *Preantipodes for Dual Quasi-Bialgebras*. Israel J. Math. **192** (2012), no. 1, 281-295.

[AP2] A. Ardizzoni, A. Pavarin, *Bosonization for dual quasi-bialgebras and preantipode*. J. Algebra **390** (2013), 126-159.

[BC] D. Bulacu, S. Caenepeel, *Integrals for (dual) quasi-Hopf algebras. Applications*. J. Algebra **266** (2003), no. 2, 552-583.

[BCT] D. Bulacu, S. Caenepeel, B. Torrecillas, *Involutory Quasi-Hopf Algebras*. Algebr. Represent. Theory **12** (2009), no. 2-5, 257-285.

[BW] T. Brzezinski, R. Wisbauer, *Corings and Comodules*. London Mathematical Society Lecture Note Series, **309**. Cambridge University Press, Cambridge, 2003.

[CP] V. Chari, A. Pressley, *A Guide to Quantum Groups*. Cambridge University Press, Cambridge, 1994.

[DPR] R. Dijkgraaf, V. Pasquier, P. Roche, *Quasi-Hopf Algebras, Group Cohomology and Orbifold Models*. Nuclear Phys. B, Proc. Suppl. **18B** (1990), 60-72.

[Dr1] V. G. Drinfel'd, *Quasi-Hopf algebras*. (Russian) Algebra i Analiz **1** (1989), no. 6, 114-148; translation in Leningrad Math. J. **1** (1990), no. 6, 1419-1457.

[Dr2] V. G. Drinfel'd, *Quasi-Hopf Algebras and Knizhnik-Zamolodchikov Equations*. Problems of modern quantum field theory (Alushta, 1989), 1-13, Res. Rep. Phys., Springer 1989.

[EG] P. Etingof, S. Gelaki, *Finite dimensional quasi-Hopf algebras with radical of codimension 2*. Math. Res. Lett.**11** (2004) no. 5-6, 685-696.

[Ge] S. Gelaki, *Basic quasi-Hopf algebras of dimension* n^3. J. Pure Appl. Algebra **198** (2005), no. 1-3, 165-174.

[HN] F. Hausser, F. Nill, *Integral theory for quasi-Hopf algebras*, preprint (arXiv:math/9904164v2).

[Ja] N. Jacobson, *Basic Algebra II.* W. H. Freeman and Co., San Francisco, Calif., 1980.

[Ka] C. Kassel, *Quantum Groups.* Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[LS] R. G. Larson, M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras.* Amer. J. Math. **91** (1969), 75-94.

[ML] S. Mac Lane, *Categories for the Working Mathematician. Second edition.* Graduate Texts in Mathematics, **5**. Springer-Verlag, New York, 1998.

[Ma] S. Majid, *Foundations of Quantum Group Theory.* Cambridge University Press, Cambridge, 1995.

[Ro] J.J. Rotman, *Advanced Modern Algebra.* Prentice-Hall, 2002.

[Sc1] P. Schauenburg, *Two characterizations of finite quasi-Hopf algebras.* J. Algebra **273** (2004), no. 2, 538-550.

[Sc2] P. Schauenburg, *Hopf modules and the double of a quasi-Hopf algebra.* Trans. Amer. Math. Soc. **354** (2002), no. 8, 3349-3378 (electronic).

[Sc3] P. Schauenburg, *Hopf algebra extensions and monoidal categories*, in *New Directions in Hopf Algebras*, 321-381, Math. Sci. Res. Inst. Publ., **43**, Cambridge Univ. Press, Cambridge, 2002.

[Sw] M. E. Sweedler, *Hopf Algebras.* W.A. Benjamin, Inc., New York, 1969.